Benders' decomposition: Fundamentals and implementations

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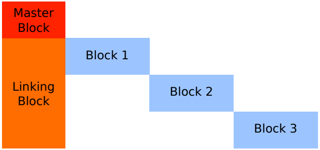
Part 1

Fundamentals

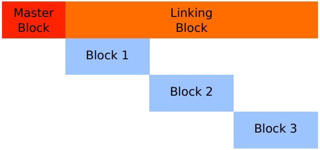
Mixed integer programming

$$\begin{aligned} \min \quad & \bar{c}^\top \bar{x}, \\ \text{subject to} \quad & \bar{A}\bar{x} \geq \bar{b}, \\ & \bar{x} \geq 0, \\ & \bar{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p}, \end{aligned}$$

Decomposition methods for mixed integer programming



Linking variables

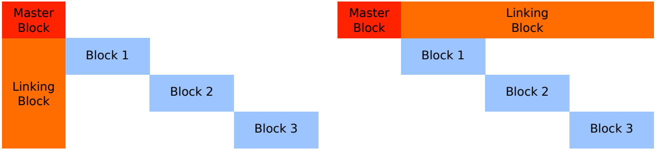


Linking constraints

- **Constraint decomposition**
  - Existence of a set of linking constraints

- **Variable decomposition**
  - Existence of a set of linking variables

Decomposition methods for mixed integer programming



Linking variables　　　　Linking constraints

- **Constraint decomposition**
  - Existence of a set of linking constraints
  - Exploits property of relaxation, i.e. blocks exhibit structure after relaxation, such as network flow or knapsack.
- **Variable decomposition**
  - Existence of a set of linking variables
  - Exploits property of restriction, i.e. blocks are “easy” to solve after fixing variables

Structured mixed integer programming

Basic idea: Minimise a linear objective function over a set of solutions satisfying a structured set of linear constraints.

$$
\begin{aligned}
\text{min} \quad & c^\top x + d^\top y, \\
\text{subject to} \quad & Ax \geq b, \\
& Bx + Dy \geq g, \\
& x \geq 0, \\
& y \geq 0, \\
& x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}, \\
& y \in \mathbb{Z}^{p_2} \times \mathbb{R}^{n_2 - p_2}.
\end{aligned}
$$

Solving structured mixed integer programs - Resources

- D. Bertsimas and J. N. Tsitsiklis. Introduction to Linear Optimization, 1997.
- J. F. Benders. Partitioning procedures for solving mixed-variables programming problems, Numerische Mathematik, 1962, 4, 238-252.
- R. Rahmaniani, T. G. Crainic, M. Gendreau, and W. Rei. The Benders decomposition algorithm: A literature review. European Journal of Operational Research, 2017, 259, 801-817.
- A. Maheo. A Short Introduction to Benders. `https://arthur.maheo.net/a-short-introduction-to-benders/`.

Benders' decomposition

Original problem

$$
\begin{aligned}
\min \quad & c^\top x + d^\top y, \\
\text{subject to} \quad & Ax \geq b, \\
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& x \geq 0, \\
& y \geq 0, \\
& x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}, \\
& y \in \mathbb{R}^{n_2}.
\end{aligned}
$$

Benders' decomposition

$$
\begin{aligned}
\min \quad & c^\top x + f(x), \\
\text{subject to} \quad & Ax \geq b, \\
& x \geq 0, \\
& x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}.
\end{aligned}
$$

where

$$
f(x) = \min_{y \geq 0} \{ d^\top y \mid Bx + Dy \geq g,\ y \in \mathbb{R}^{n_2} \}
$$

Benders' decomposition

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\min \quad & c^\top x + f(x), \\
\text{subject to} \quad & Ax \geq b, \\
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\end{aligned}$$

where

$$f(x) = \min_{y \geq 0} \{ d^\top y \mid Bx + Dy \geq g, \; y \in \mathbb{R}^{n_2} \}$$

The dual formulation of $f(x)$ is important for Benders' decomposition.
Can you write down the dual formulation?

Benders' decomposition

$$\begin{aligned}
\min \quad & c^\top x + f(x), \\
\text{subject to} \quad & Ax \geq b, \\
& x \geq 0, \\
& x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}.
\end{aligned}$$

where

$$f(x) = \min_{y \geq 0} \{ d^\top y \mid Bx + Dy \geq g,\ y \in \mathbb{R}^{n_2} \}$$

equivalently, using the dual formulation we can define

$$f'(x) = \max_{u \geq 0} \{ u^\top (g - Bx) \mid D^\top u \geq d^\top,\ u \in \mathbb{R}^{m_2} \}$$

$$(f'(x) = f(x))$$

Benders' decomposition

Using the dual formulation of $f(x)$, given by

$$f'(x) = \max_{u \geq 0} \{ u^\top (g - Bx) \mid D^\top u \geq d^\top,\ u \in \mathbb{R}^{m_2} \}$$

an equivalent formulation of the original problem is

$$\begin{aligned} \text{min} \quad & c^\top x + \varphi, \\ \text{subject to} \quad & Ax \geq b, \\ & \varphi \geq f'(x) \\ & x \geq 0, \\ & x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}. \end{aligned}$$

Benders' decomposition

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- Note that the feasible region of $f'(x)$ does not depend on x,
  - only the objective function depends on the input value of x
- Thus, we can describe $f'(x)$ as a set of extreme points and extreme rays.
  - Equivalently, we can describe $f(x)$ as a set of *dual* extreme points and *dual* extreme rays

Benders' decomposition

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$$f'(x) = \max_{u \geq 0} \{ u^\top (g - Bx) \mid D^\top u \geq d^\top, \ u \in \mathbb{R}^{m_2} \}$$

let

- $\mathcal{O}$ be the set of all extreme points of $f'(x)$
- $\mathcal{F}$ be the set of all extreme rays of $f'(x)$

Can you write down the expressions for the optimality and feasibility cuts?

Benders' decomposition

Using the dual formulation of $f(x)$, given by

$$f'(x) = \max_{u \geq 0} \{ u^\top (g - Bx) \mid D^\top u \geq d^\top, \, u \in \mathbb{R}^{m_2} \}$$

let

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- $\mathcal{F}$ be the set of all extreme rays of $f'(x)$

an equivalent formulation of the original problem is

$$\begin{aligned}
\min \quad & c^\top x + \varphi, \\
\text{subject to} \quad & Ax \geq b, \\
& \varphi \geq u^\top (g - Bx) \quad \forall u \in \mathcal{O} \\
& 0 \geq u^\top (g - Bx) \quad \forall u \in \mathcal{F} \\
& x \geq 0, \\
& x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}.
\end{aligned}$$

Benders' decomposition

- The sets $\mathcal{O}$ and $\mathcal{F}$ are exponential in size
- The reformulated original problem becomes intractable

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- **Need to use a delayed constraint generation algorithm**

Benders' decomposition

- The sets $\mathcal{O}$ and $\mathcal{F}$ are exponential in size
- The reformulated original problem becomes intractable

- **Need to use a delayed constraint generation algorithm**

Cut generating LP $\Leftrightarrow$ Benders' subproblem

$$
\begin{aligned}
z(\hat{x}) = \min \quad & d^\top y, \\
\text{subject to} \quad & Dy \geq g - B\hat{x}, \\
& y \geq 0, \\
& y \in \mathbb{R}^{n_2}.
\end{aligned}
$$

Benders' decomposition

Benders' master problem

$$
\begin{aligned}
\min \quad & c^\top x + \varphi, \\
\text{subject to} \quad & Ax \geq b, \\
& \varphi \geq u^\top (g - Bx) \quad \forall u \in \mathcal{O}' \\
& 0 \geq u^\top (g - Bx) \quad \forall u \in \mathcal{F}' \\
& x \geq 0, \\
& x \in \mathbb{Z}^{p_1} \times \mathbb{R}^{n_1 - p_1}.
\end{aligned}
$$

- $\mathcal{O}$ is replaced by $\mathcal{O}'$ (which is a subset of $\mathcal{O}$).
- $\mathcal{F}$ is replaced by $\mathcal{F}'$ (which is a subset of $\mathcal{F}$).

Benders' decomposition

Benders' subproblem

$$
\begin{aligned}
z(\hat{x}) = \min \quad & d^\top y, \\
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& y \geq 0, \\
& y \in \mathbb{R}^{n_2}.
\end{aligned}
$$

If $\hat{x}$ induces

- an infeasible instance, then the dual ray u is used to generate a feasibility cut

$$0 \geq u^\top (g - Bx)$$

- a feasible instance, then the dual solution u is used to generate an optimality cut

$$\varphi \geq u^\top (g - Bx)$$

  - the auxiliary variable φ is an underestimator of the optimal subproblem objective value

Benders' decomposition

Benders' subproblem – discrete variables
(Note: master problem must be pure binary)

$$\begin{aligned} z(\hat{x}) = \min \quad & d^\top y, \\ \text{subject to} \quad & Dy \geq g - B\hat{x}, \\ & y \geq 0, \\ & y \in \mathbb{Z}^{p_2} \times \mathbb{R}^{n_2 - p_2}. \end{aligned}$$

Define the index set $\mathcal{B}^+ := \{i | \hat{x}_i = 1\}$. If $\hat{x}$ induces

- an infeasible instance, then the add no-good cut

$$\sum_{i \in \mathcal{B}^+} (1 - x_i) + \sum_{i \notin \mathcal{B}^+} x_i \geq 1$$

- a feasible instance, then add a Laporte and Louveaux optimality cut (L is a valid lower bound of $z(\hat{x})$)

$$\varphi \geq L + \left(\hat{z}(\hat{x}) - L\right) \left(1 - \left(\sum_{i \in \mathcal{B}^+} (1 - x_i) + \sum_{i \notin \mathcal{B}^+} x_i\right)\right)$$

Benders' decomposition

- Exposes an iterative delayed constraint generation algorithm
  1. Find an $\hat{x}$
  2. Solving subproblem using $\hat{x}$ as input
  3. Add optimality/feasibility cut to master problem to eliminate $\hat{x}$.

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Key questions

- When to terminate?
- When to solve the Benders' subproblem to generate cuts?
- What solution $\hat{x}$ should be used?
- How to best used MIP solvers to boost iterative algorithm performance?

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$\rightarrow$ *Algorithm design decisions and enhancement techniques*