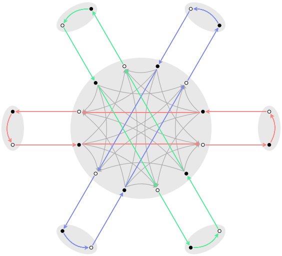
NIELS LINDNER
ZUSE INSTITUTE BERLIN, GERMANY

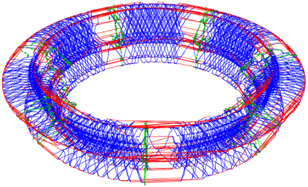
PERIODIC TIMETABLE OPTIMIZATION IN PUBLIC TRANSPORT



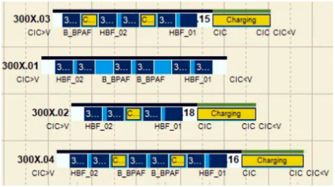
CO@Work 2024, Zuse Institute Berlin, September 26, 2024

MobilityLab @ Zuse Institute Berlin (ZIB)

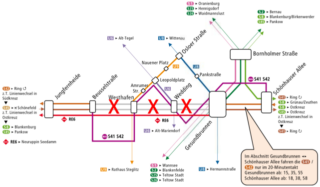
Discrete Optimization in Public Transport



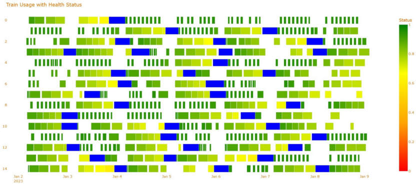
ICE Rolling Stock Rotation Planning
with DB Fernverkehr AG



Electric Bus Scheduling
with IVU Traffic Technologies AG



S-Bahn Construction Site Timetabling
with DB InfraGO AG



Predictive Maintenance
with Ostdeutsche Eisenbahn GmbH (ODEG)

Public Transport...

... is often operated **periodically**

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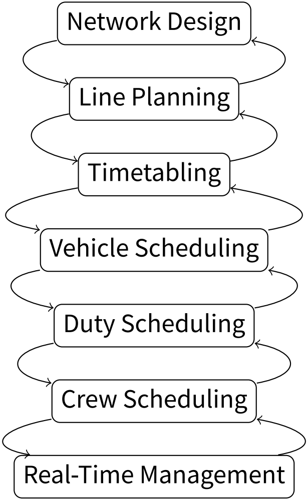
→ Periodic Timetable Optimization

Public Transport Planning Cycle



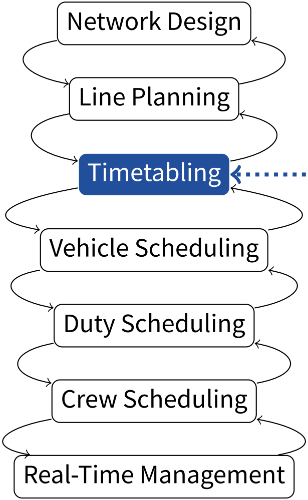
strategic
long-term
many decision makers

↓

operational
short-term
few decision makers

Bussieck et al.: Discrete optimization in public rail transport, 1997
Liebchen: Periodic timetable optimization in public transport, 2006

Public Transport Planning Cycle



Periodic Timetable Optimization

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Timetable Optimization

Informal Definition

A **timetable** is an assignment of arrival and departure times to a given set of trips.

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Feasibility

What makes a *valid* timetable?

- realistic travel times (lower and upper bounds on driving times, ...)
- conflict-freeness (sufficient headway between trains, station capacities, ...)

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Optimization Objectives

What makes a *good* timetable?

- *passenger perspective*: short travel times (in particular: transfers)
- *operator perspective*: efficient resource usage (vehicles, drivers)
- *railway infrastructure manager perspective*: sell all track capacities
- *disposition perspective*: maximize robustness

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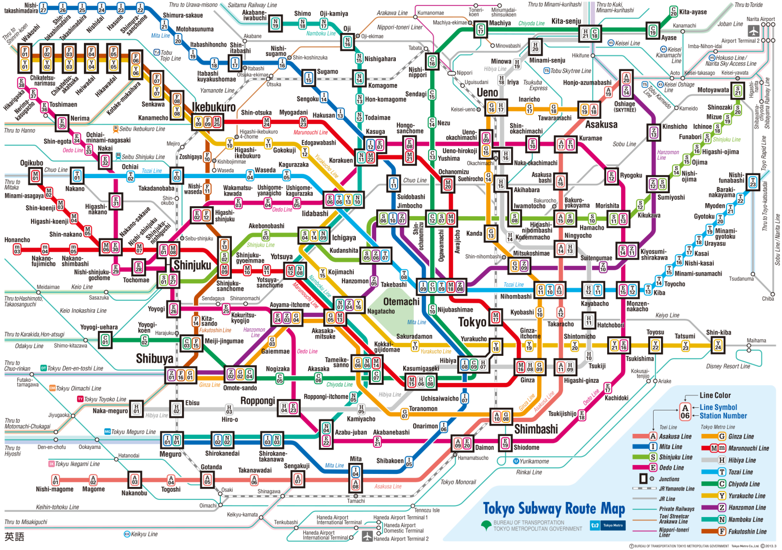
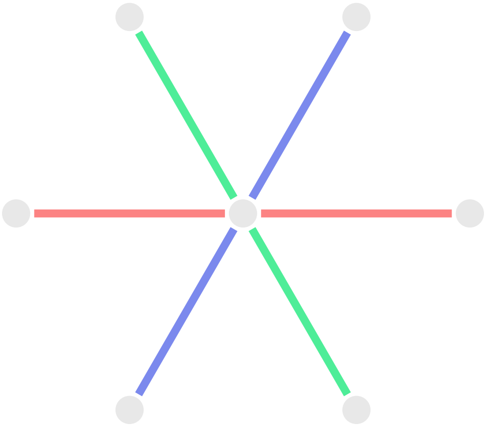
Observation: These goals are partially conflicting!

A Line Network: Tokyo



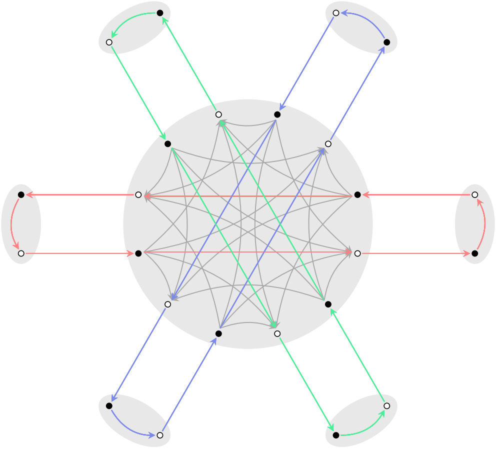
Tokyo Metro Co., Ltd.

Periodic Timetabling in Public Transport



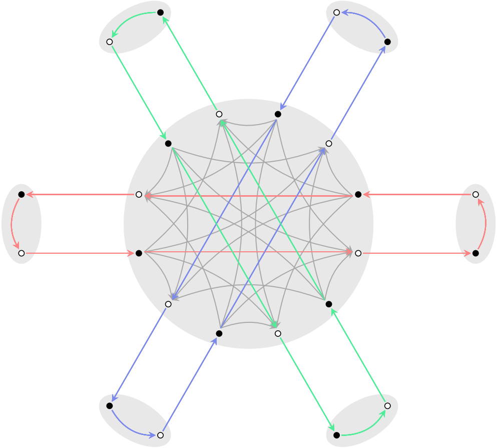
Line Network, 3 bidirectional lines

Periodic Timetabling in Public Transport



Event-Activity Network

Periodic Timetabling in Public Transport

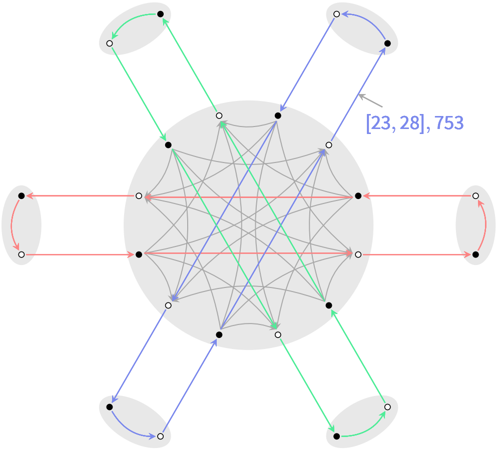


Event-Activity Network

Events:
- ● arrival
- ○ departure

Activities:
- → drive, dwell, turn
- → transfer
- . . .

Periodic Timetabling in Public Transport



Periodic Event Scheduling Instance

Bounds:

- driving times
- minimum transfer times
- maximum dwell times
- minimum headway times
- ...

Weights:

- passenger load
- turnaround penalties
- ...

Period time:

- e.g., $T = 60$ for 1 hour, resolution of 1 minute

Periodic Event Scheduling Problem (PESP)

Given

G	event-activity network,
$T \in \mathbb{N}$	period time,
$\ell \in \mathbb{R}_{\geq 0}^{A(G)}$	lower bounds,
$u \in \mathbb{R}_{\geq 0}^{A(G)}$	upper bounds,
$w \in \mathbb{R}_{\geq 0}^{A(G)}$	weights,

the **Periodic Event Scheduling Problem (PESP)** is to find

$\pi \in [0, T)^{V(G)}$	periodic timetable,
$x \in \mathbb{R}^{A(G)}$	periodic tension

such that

(1) $\pi_j - \pi_i \equiv x_{ij} \mod T$ for all $ij \in A(G)$,

(2) $\ell \leq x \leq u$,

(3) $w^\top x$ is minimum,

or decide that no such (π, x) exists.

(Serafini and Ukovich, 1989)

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We can formulate PESP as the following mixed integer program:

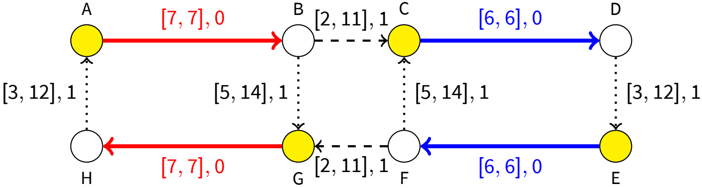
$$\begin{aligned}
\text{Minimize} \quad & \sum_{ij \in A(G)} w_{ij} x_{ij} \\
\text{s.t.} \quad & \pi_j - \pi_i + T p_{ij} = x_{ij}, && ij \in A(G), \\
& \ell_{ij} \leq x_{ij} \leq u_{ij}, && ij \in A(G), \\
& 0 \leq \pi_i < T, && i \in V(G), \\
& p_{ij} \in \mathbb{Z}, && ij \in A(G).
\end{aligned}$$

This uses integer variables p_{ij} (*periodic offsets*) for each edge $ij \in A(G)$ to model the modulo T constraints.

Note 1: This is formally not quite a MIP, as the constraints $\pi_i < T$ should be replaced with $\pi_i \leq T - \varepsilon$ for a suitable $\varepsilon > 0$.

Note 2: Equivalently, one may minimize $w^\top y$, where $y := x - \ell$ is the *periodic slack*.

PESP: Example



PESP instance, period time $T = 10$, arcs labeled with $[\ell, u], w$

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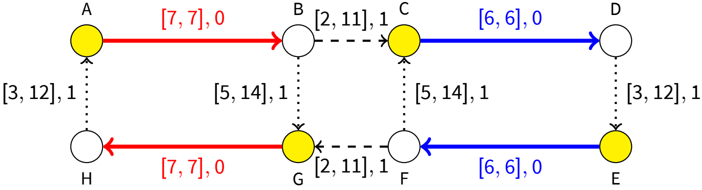

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Let's find an optimal integer-valued timetable for this PESP instance by hand.

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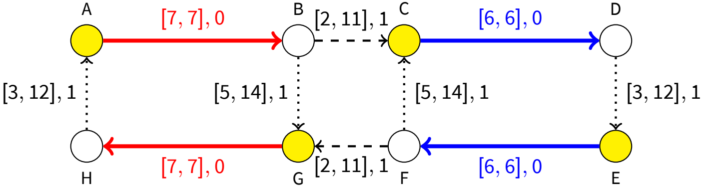

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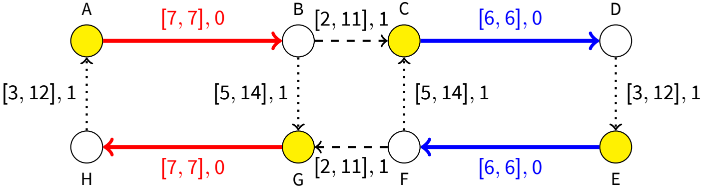
Let's find an optimal integer-valued timetable for this PESP instance by hand.

First observation: If π and x are optimal, then $x_{ij} = [\pi_j - \pi_i - \ell_{ij}]_{10} + \ell_{ij}$ for all edges ij, as $x_{ij} \equiv \pi_j - \pi_i \mod 10$ and x_{ij} is the smallest such number with $x_{ij} \geq \ell_{ij}$. Due to the bounds, we have $x_{AB} = x_{GH} = 7$ and $x_{CD} = x_{EF} = 6$, hence we can substitute π_B by $\pi_A + 7$ modulo 10 and similarly π_D, π_F, π_H.

PESP: Example



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Second observation: Now any assignment of integer values to $\pi_A, \pi_C, \pi_E, \pi_G$ produces a feasible periodic timetable, as all remaining edges have $u_{ij} - \ell_{ij} = T - 1$.

PESP: Example

Hence the objective value is

$$\begin{aligned}\sum_{ij \in A(G)} w_{ij} x_{ij} = &[\pi_A - \pi_H - \ell_{HA}]_{10} + [\pi_C - \pi_B - \ell_{BC}]_{10} + [\pi_C - \pi_F - \ell_{FC}]_{10} \\ &+ [\pi_E - \pi_D - \ell_{DE}]_{10} + [\pi_G - \pi_B - \ell_{BG}]_{10} + [\pi_G - \pi_F - \ell_{FG}]_{10} \\ &+ \ell_{HA} + \ell_{BC} + \ell_{FC} + \ell_{DE} + \ell_{BG} + \ell_{FG} \\ = &[\pi_A - \pi_G]_{10} + [\pi_C - \pi_A - 9]_{10} + [\pi_C - \pi_E - 1]_{10} \\ &+ [\pi_E - \pi_C - 9]_{10} + [\pi_G - \pi_A - 2]_{10} + [\pi_G - \pi_E - 8]_{10} + 20\end{aligned}$$

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Observe that

$$\begin{aligned}[\pi_A - \pi_G]_{10} + [\pi_G - \pi_A - 2]_{10} &\in \{8, 18\} \\ [\pi_C - \pi_E - 1]_{10} + [\pi_E - \pi_C - 9]_{10} &\in \{0, 10\} \\ [\pi_C - \pi_A - 9]_{10} + [\pi_G - \pi_E - 8]_{10} &\in [0, 18].\end{aligned}$$

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$$\sum_{ij \in A(G)} w_{ij} x_{ij} = [\pi_A - \pi_H - \ell_{HA}]_{10} + [\pi_C - \pi_B - \ell_{BC}]_{10} + [\pi_C - \pi_F - \ell_{FC}]_{10}$$

$$+ [\pi_E - \pi_D - \ell_{DE}]_{10} + [\pi_G - \pi_B - \ell_{BG}]_{10} + [\pi_G - \pi_F - \ell_{FG}]_{10}$$

$$+ \ell_{HA} + \ell_{BC} + \ell_{FC} + \ell_{DE} + \ell_{BG} + \ell_{FG}$$

$$= [\pi_A - \pi_G]_{10} + [\pi_C - \pi_A - 9]_{10} + [\pi_C - \pi_E - 1]_{10}$$

$$+ [\pi_E - \pi_C - 9]_{10} + [\pi_G - \pi_A - 2]_{10} + [\pi_G - \pi_E - 8]_{10} + 20$$

Observe that

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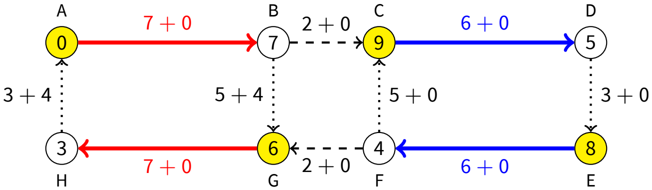
$$[\pi_C - \pi_A - 9]_{10} + [\pi_G - \pi_E - 8]_{10} \in [0, 18].$$

We conclude that the optimal objective value is at least 28. In fact, $(\pi_A, \pi_C, \pi_E, \pi_G) = (0, 9, 8, 6)$ has an objective value of 28 and is hence optimal.

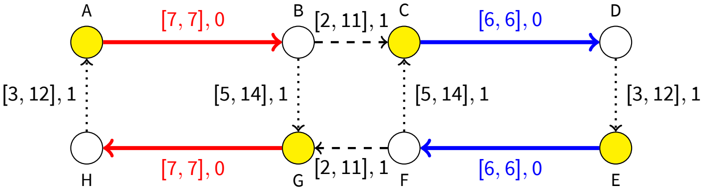
PESP: Example



PESP instance, period time $T = 10$, edges labeled with $[\ell, u], w$

We conclude that $(\pi_A, \pi_B, \pi_C, \pi_D, \pi_E, \pi_F, \pi_G, \pi_H) = (0, 7, 9, 5, 8, 4, 6, 3)$ is an optimal periodic timetable. The weighted periodic tension is 28, and the weighted periodic slack is 8.

PESP: Example



Less painful: Formulate and solve the MIP:

Minimize	$x_{BC} + x_{BG} + x_{DE} + x_{FC} + x_{FG} + x_{HA}$	
s.t.	$\pi_B - \pi_A + 10p_{AB} = x_{AB}$	$0 \leq \pi_A \leq 9$
	$\pi_C - \pi_B + 10p_{BC} = x_{BC}$	$0 \leq \pi_B \leq 9$
	$\vdots$	$\vdots$
	$7 \leq x_{AB} \leq 7$	$p_{AB} \in \mathbb{Z}$
	$2 \leq x_{BC} \leq 11$	$p_{BC} \in \mathbb{Z}$
	$\vdots$	$\vdots$

Complexity of PESP

Theorem

The PESP Feasibility Problem is NP-complete for every fixed $T \geq 3$.

In more words, given (G, T, ℓ, u, w) for a fixed value of $T \geq 3$, the problem whether there exists a periodic timetable π with tension x such that $\ell \leq x \leq u$ is NP-complete.

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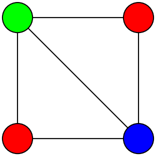
Proof.

The *3-Coloring Problem* is NP-complete (Garey, Johnson, Stockmeyer, 1976): Given an undirected graph H, is there a 3-coloring, i.e., a function $f : V(H) \to \{0, 1, 2\}$ such that $f(i) \neq f(j)$ for all $\{i, j\} \in E(H)$?



a 3-colorable graph, not 2-colorable

□

Complexity of PESP

Proof (cont.)

In particular, the *T-Coloring Problem*, where f is allowed to take values in $\{0, 1, \ldots, T-1\}$, is NP-complete for any fixed $T \geq 3$.

We reduce T-Coloring to PESP Feasibility, the membership of the latter in NP being clear. For a T-Coloring instance H, define G by arbitrarily orienting the edges in H. We keep T and set $\ell_{ij} := 1$, $u_{ij} := T - 1$ for all $ij \in A(G)$. Weights do not influence the feasibility, we can choose them arbitrarily.

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($\Rightarrow$) Let f be a T-coloring of H. Then $\pi := f$ is a feasible periodic timetable, as

$$x_{ij} := [\pi_j - \pi_i]_T = [f(j) - f(i)]_T \in \{1, \dots, T-1\} \quad \text{for all } ij \in A(G)$$

is a feasible periodic tension.

($\Leftarrow$) Conversely, if π is a feasible periodic timetable and x is a tension for π, then $f := \pi$ is a T-coloring of H, as

$$[f(j) - f(i)]_T = [x_{ij}]_T \in [1, T-1] \quad \text{for all } ij \in A(G)$$

implies $f(i) \neq f(j)$ for all $ij \in A(G)$. □

More Complexity

Lemma

If G is a tree, then there is an optimal solution (π, x) *with* $x = \ell$*, and* π *can be computed in linear time, e.g., by depth-first search.*

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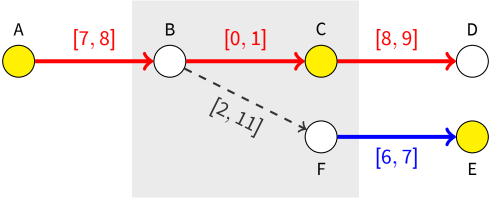

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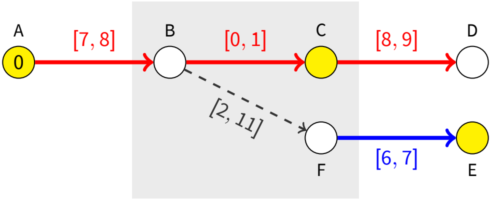

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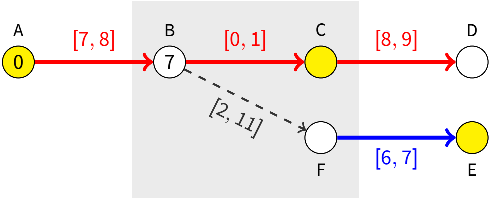

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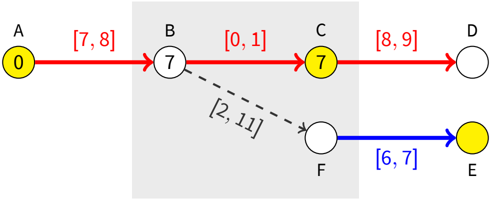

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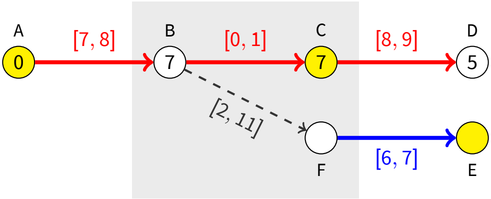

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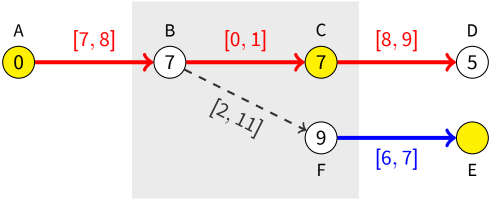

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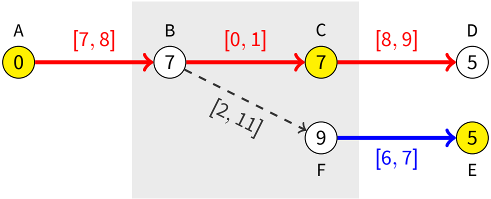

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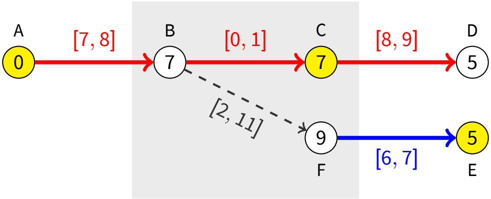


Theorem (Lindner and Reisch, 2022)

For any $k \geq 2$, PESP Feasibility is weakly NP-complete on graphs of treewidth $\leq k$.

- In particular, PESP is NP-hard on planar graphs.
- For event-activity networks of treewidth $\leq k$, there is an $O(|V(G)| \cdot T^k)$ dynamic program that solves PESP (to optimality).

The Role of Cycles

Question

Let $x \in \mathbb{R}^{A(G)}$. When does a periodic timetable $\pi \in [0, T)^{V(G)}$ exist for which x is a periodic tension, i.e., $x_{ij} \equiv \pi_j - \pi_i \mod T$ for all $ij \in A(G)$?

A Necessary Condition

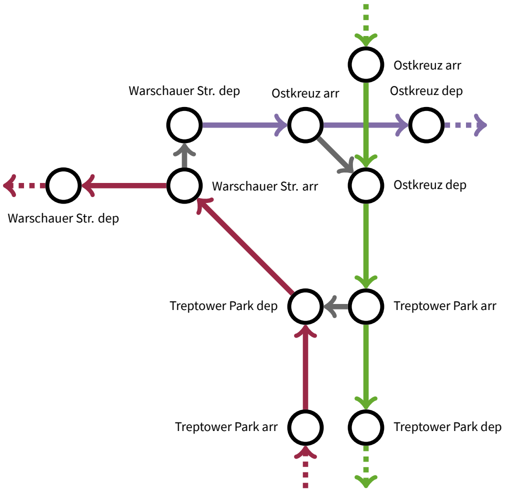
Let C be a directed cycle in G with vertex sequence $(i_0, i_1, \ldots, i_n)$ with $i_0 = i_n$. Let (π, x) be a feasible pair of periodic timetable and tension. Then

$$\sum_{a \in C} x_a = \sum_{k=0}^{n-1} x_{i_k, i_{k+1}} \equiv \sum_{k=0}^{n-1} (\pi_{i_{k+1}} - \pi_{i_k}) = 0 \mod T \quad \text{(telescoping sum)}.$$

Hence feasible periodic tensions must add up to an integer multiple of T along any directed cycle.

Practical Consequence: If there are cycles in G, then it might be infeasible to have travel times at lower bounds.

Transfers Without Waiting?



Excerpt of an event-activity network for three lines (S7, S8, S9) of the S-Bahn Berlin network near Ostkreuz

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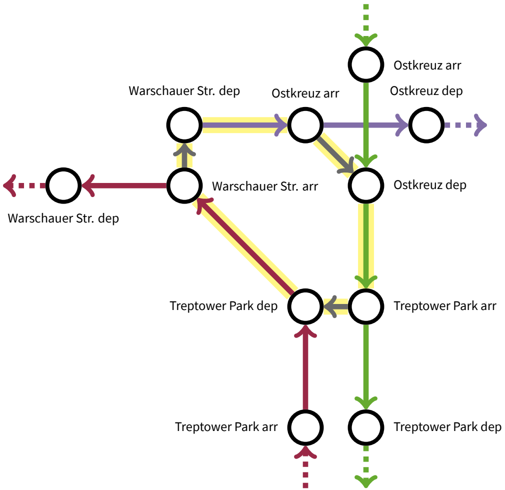

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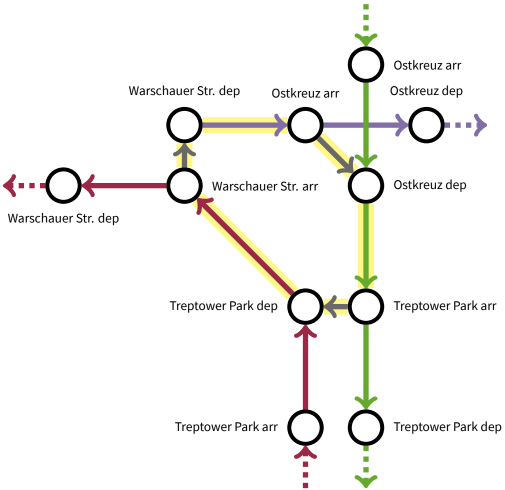

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The necessary condition holds for the yellow cycle. If $T = 10$ min, then the sum of activity durations must be an integer mutliple of 10 minutes.

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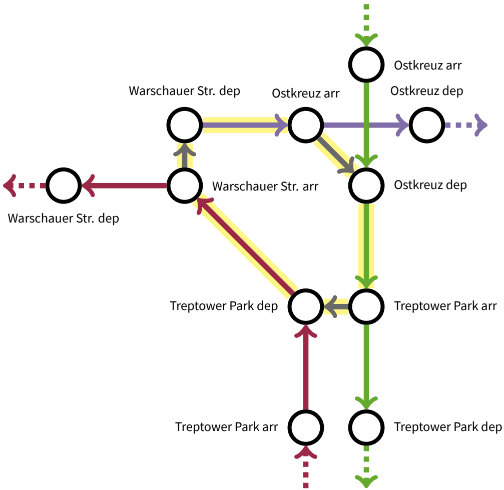

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Suppose each transfer takes at least 2 min. If all three transfer times are supposed to be at this lower bound, then the sum of the three driving times in the cycle must be 4 min modulo 10. In practice, each of those is around 2 min. This does not fit together! We hence *must increase* driving or transfer times to be feasible.

Cycle Space & Cyclomatic Number

Let G be a digraph. Choosing standard bases, the incidence matrix D of G induces a $\mathbb{Z}$-linear map

$$\mathbb{Z}^{A(G)} \xrightarrow{D} \mathbb{Z}^{V(G)}, \quad (x_{ij})_{ij \in A(G)} \mapsto \left(\sum_{ij \in \delta^+(i)} x_{ij} - \sum_{ji \in \delta^-(i)} x_{ji} \right)_{i \in V(G)}.$$

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Definition

$\mathcal{C}(G) := \ker A$ is the **cycle space** of G.

In other words, $\mathcal{C}(G)$ is the space of all $\mathbb{Z}$-linear combinations of integral *circulations*, i.e., integral flows (with arbitrary signs) where flow conservation holds everywhere.

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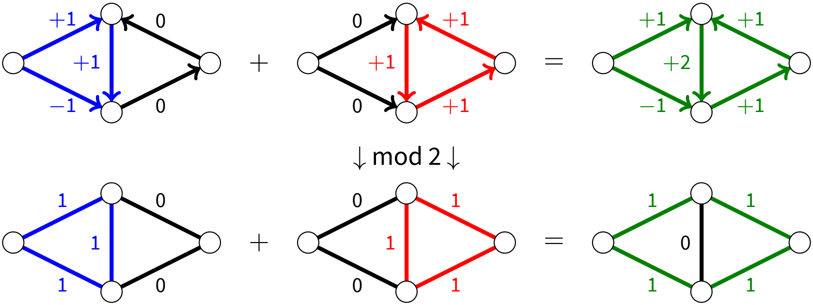
Definition

The **cyclomatic number** $\mu(G)$ of G is defined as the rank of the cycle space $\mathcal{C}(G)$, i.e., the dimension of the $\mathbb{Q}$-vector space $\mathcal{C}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Oriented, Directed and Undirected Cycles

Definition

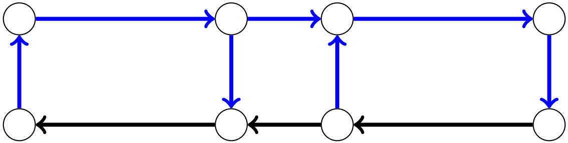
- An **oriented cycle** in G is a vector $\gamma \in \{-1, 0, 1\}^{A(G)} \cap \mathcal{C}(G)$.
- A **directed cycle** in G is an oriented cycle γ with $\gamma \geq 0$.
- An **undirected cycle** in G is a vector $\gamma \in \mathcal{C}(G) \otimes_{\mathbb{Z}} \mathbb{F}_2$.



Observations: The definitions of (un)directed cycles agree with the previous ones. Addition of undirected cycles in $\mathcal{C}(G) \otimes_{\mathbb{Z}} \mathbb{F}_2$ is given by the symmetric difference.

Fundamental Cycle Bases

Assume that G is weakly connected, so that G has a spanning tree S. We call each arc in $A(G) \setminus A(S)$ a **co-tree arc**. Adding an arbitrary co-tree arc ij to S produces a simple oriented cycle γ in G consisting of ij as forward arc, i.e., $\gamma_{ij} = 1$, and the arcs of the unique j-i-path in S. This is the **fundamental cycle** of ij.



blue: spanning tree

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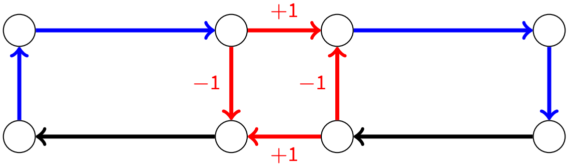

red: fundamental cycle of the middle co-tree arc

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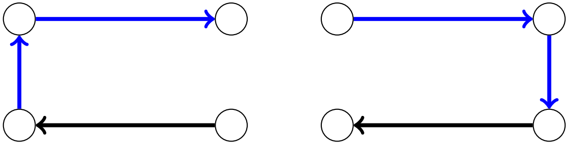

Theorem

Let S be an arbitrary spanning tree of a weakly connected digraph G. Then the set of fundamental cycles of the co-tree arcs w.r.t. S form a $\mathbb{Z}$-basis of the cycle space $\mathcal{C}(G)$. In particular, $\mu(G) = |A(G)| - |V(G)| + 1$.

Fundamental Cycle Bases

Proof.

Since $|A(S)| = |V(G)| - 1$, we obtain $\mu := |A(G)| - |V(G)| + 1$ co-tree arcs $a_1, \dots, a_\mu$ and hence μ fundamental cycles $\gamma^{a_1}, \dots, \gamma^{a_\mu}$. Linear independence is clear: Any fundamental cycle has precisely one co-tree arc, and γ^{a_i} is the only fundamental cycle containing a_i. More formally, if $\sum_{i=1}^{\mu} \lambda_i \gamma^{a_i} = 0$ for some $\lambda_i \in \mathbb{Z}$, then comparing the i-th entries yields $\lambda_i = 0$ for all i.

Now let $\zeta \in \mathcal{C}(G)$ be arbitrary and consider $\zeta' := \zeta - \sum_{i=1}^{\mu} \zeta_{a_i} \gamma^{a_i}$. The entry of ζ' at any co-tree arc a_i vanishes, so that $\{a \in A(G) \mid \zeta'_a \neq 0\}$ is contained in $A(S)$. But S contains no circulation and hence $\zeta' = 0$. □

We call a basis for $\mathcal{C}(G)$ consisting of the fundamental cycles of some spanning tree (forest) a **fundamental cycle basis**.

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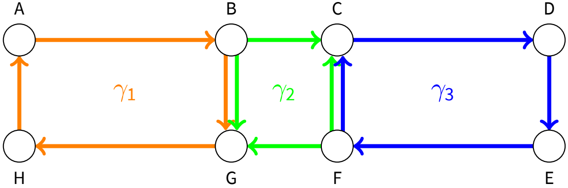
Corollary

If G has c weakly connected components, then $\mu(G) = |A(G)| - |V(G)| + c$.

We call a basis for $\mathcal{C}(G)$ consisting of the fundamental cycles of some spanning tree (forest) a **fundamental cycle basis**.

Fundamental Cycle Basis Example

In the example from Slide 18, this is the corresponding fundamental cycle basis:



The cycles γ_1 and γ_3 use only forward arcs, whereas γ_2 uses two backward arcs.

The cycle space $\mathcal{C}(G)$ is generated by the rows of the following *cycle matrix*:

	HA	AB	BG	BC	FC	CD	DE	GH	FG	EF
γ_1	1	1	1	0	0	0	0	1	0	0
γ_2	0	0	-1	1	-1	0	0	0	1	0
γ_3	0	0	0	0	1	1	1	0	0	1

Observe that the submatrix on the last three columns – the ones corresponding to the co-tree arcs – is the identity matrix.

More Cycle Bases

Let G be a digraph, B a set of $\mu(G)$ oriented cycles.

Definition

- B is an **integral cycle basis** if B is a $\mathbb{Z}$-basis of $\mathcal{C}(G)$.
- B is an **undirected cycle basis** if B reduces to an $\mathbb{F}_2$-basis of $\mathcal{C}(G) \otimes_{\mathbb{Z}} \mathbb{F}_2$.

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We already proved that every fundamental cycle basis is integral. Taking any integral cycle basis modulo 2 generates $\mathcal{C}(G) \otimes_{\mathbb{Z}} \mathbb{F}_2$, and the latter space has dimension $\mu(G)$, as, e.g., our proof also shows that fundamental cycle bases are undirected. In particular we have the implications

$$\text{fundamental} \Rightarrow \text{integral} \Rightarrow \text{undirected}.$$

The reverse implications do not hold in general.

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The reverse implications do not hold in general.

Undirected Graphs

$\mathcal{C}(G) \otimes_{\mathbb{Z}} \mathbb{F}_2$ is the natural cycle space for an undirected graph G, its dimension over $\mathbb{F}_2$ equals $\mu(G)$. The cycle spaces of any two orientations of G are isomorphic.

Cycle Matrices and Determinants

Cycle Matrix

If B is a cycle basis, we can consider the **cycle matrix** $\Gamma \in \{-1, 0, 1\}^{B \times A(G)}$ having the oriented cycles in B as rows.

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Determinant of a Cycle Basis

For a spanning forest S of G, define Γ_S as the square submatrix obtained from the columns belonging to the co-forest arcs $A(G) \setminus A(S)$. The **determinant** of B is defined as $\det(B) := |\det(\Gamma_S)|$.

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Theorem (Liebchen, Rizzi, 2007)

(1) *The determinant of a cycle basis is well-defined and positive.*

(2) *B is an undirected cycle basis if and only if* $\det B$ *is odd.*

(3) *B is an integral cycle basis if and only if* $\det B = 1$.

(4) *B is a fundamental cycle basis if and only if Γ can be permuted in such a way that it contains the $\mu(G) \times \mu(G)$ identity matrix in its last $\mu(G)$ columns.*

Cycle Periodicity Property

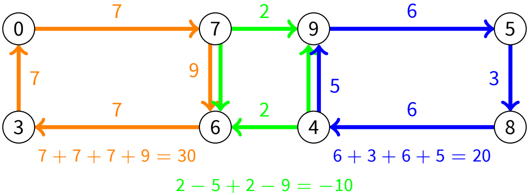
Theorem (Cycle periodicity property – Liebchen, Peeters, 2009)

Let G be a digraph, $x \in \mathbb{R}^{A(G)}$. Then the following are equivalent:

(1) *there is a $\pi \in [0, T)^{V(G)}$ such that for all $ij \in A(G)$ holds $x_{ij} \equiv \pi_j - \pi_i \mod T$,*

(2) *$\gamma^\top x \equiv 0 \mod T$ for all oriented cycles γ in G,*

(3) *$\Gamma x \equiv 0 \mod T$ for any cycle matrix Γ of an integral cycle basis for G.*



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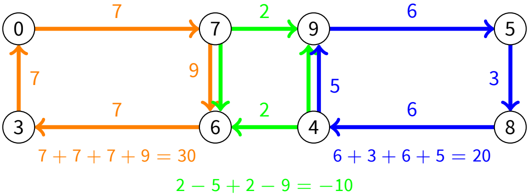

Remark: We can rephrase (1) as: There is a π such that $x \equiv D^\top \pi \mod T$, where D is the incidence matrix of G. This allows for a conceptually simple algebraic proof in terms of *graph homology*: The following sequence of abelian groups is exact:

$$\mathbb{Z}^{V(G)} \xrightarrow{D^\top} \mathbb{Z}^{A(G)} \xrightarrow{\Gamma} \mathcal{C}(G) \to 0.$$

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Proof.

(1) $\Rightarrow$ (2): Telescoping sum as for directed cycles, but now with signs.
(2) $\Rightarrow$ (3): The rows of Γ consist of oriented cycles.
(3) $\Rightarrow$ (2): Any oriented cycle γ is an integer linear combination of the rows of Γ ($\mathbb{Z}$-basis property).
(2) $\Rightarrow$ (1): W.l.o.g. G is weakly connected. Let S be a spanning tree. We construct a periodic timetable π by depth-first search along S, so that $x_{ij} \equiv \pi_j - \pi_i \mod T$. For a co-tree arc $ij \in A(G) \setminus A(S)$, let p be the unique j-i-path in S and γ the fundamental cycle associated to ij. Set $x_{ij} := [-p^\top x]_T$. Then

$$\pi_j - \pi_i \equiv \pi_j - \pi_i + p^\top x - p^\top x \equiv \gamma^\top x + x_{ij} \equiv x_{ij} \mod T. \qquad \square$$

Cycle-based MIP Formulation

The cycle periodicity property allows a cycle-based mixed-integer programming formulation for PESP:

$$
\begin{aligned}
\text{Minimize} \quad & w^\top x \\
\text{s.t.} \quad & \Gamma x = Tz, \\
& \ell \leq x \leq u, \\
& z \in \mathbb{Z}^B.
\end{aligned}
$$

Here, (G, T, ℓ, u, w) is a PESP instance, and B is an integral cycle basis for G with cycle matrix Γ. The z-variables model the modulo T constraints (*cycle offsets*).

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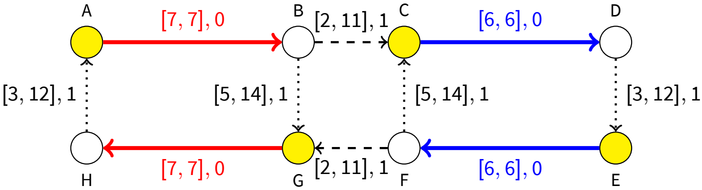
Remarks

- The timetable-based MIP formulation has $|A(G)|$ integral variables, the cycle-based one has only $\mu(G)$ variables. However, the range of the integer variables is typically larger in the cycle-based formulation.
- Periodic timetables are only implicit in the cycle-based MIP formulation. They can be reconstructed by depth-first search as in the proof of the cycle periodicity property.

PESP: Example with Cycle-Based MIP



PESP instance, period time $T = 10$, arcs labeled with $[\ell, u], w$

$$
\begin{array}{lll}
\text{Minimize} & x_{BC} + x_{BG} + x_{DE} + x_{FC} + x_{FG} + x_{HA} & \\
\text{s.t.} & x_{AB} + x_{BG} + x_{GH} + x_{HA} = 10z_1 & z_1 \in \mathbb{Z} \\
& x_{BC} - x_{FC} + x_{FG} - x_{BG} = 10z_2 & z_2 \in \mathbb{Z} \\
& x_{CD} + x_{DE} + x_{EF} + x_{FC} = 10z_3 & z_3 \in \mathbb{Z} \\
& 7 \leq x_{AB} \leq 7 & \\
& 2 \leq x_{BC} \leq 11 & \\
& \vdots &
\end{array}
$$

LP Relaxations

Lemma

For both MIP formulations, $x = \ell$ is an optimal solution to the natural LP relaxation.

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For the incidence-based formulation, set

$$\pi := 0, x := \ell, p := \ell / T.$$

Then certainly

$$\pi_j - \pi_i + Tp_{ij} = \ell = x.$$

For the cycle-based formulation, set

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Then clearly $\Gamma x = Tz$. □

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Upshot: Dropping integrality constraints makes the LP worthless. This is disastrous for branch-and-cut!

Odijk's Cycle Inequalities

Theorem (Cycle inequalities – Odijk, 1994)

Let (G, T, ℓ, u, w) *be a PESP instance and let* γ *be an oriented cycle in G. Then the following* cycle inequality *is valid for all feasible periodic tensions x:*

$$\left\lceil \frac{\gamma_+^\top \ell - \gamma_-^\top u}{T} \right\rceil \leq \frac{\gamma^\top x}{T} \leq \left\lfloor \frac{\gamma_+^\top u - \gamma_-^\top \ell}{T} \right\rfloor,$$

where $\gamma_+ := \max(0, \gamma) \in \{0,1\}^{A(G)}$ *and* $\gamma_- := \max(0, -\gamma) \in \{0,1\}^{A(G)}$ *are the positive and negative parts of* γ*, respectively.*

Proof.

Since $\gamma = \gamma^+ - \gamma^-$, we have

$$\gamma_+^\top \ell - \gamma_-^\top u \leq \gamma^\top x \leq \gamma_+^\top u - \gamma_-^\top \ell.$$

It remains to note that $\gamma^\top x / T$ is integer by the cycle periodicity property. □

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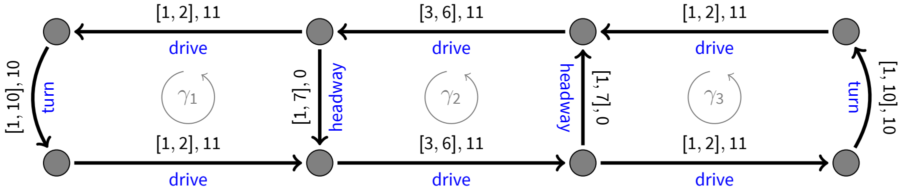
Note: Odijk's cycle inequalities provide bounds on the cycle offset variables z.

Example: Cycle Inequalities as Cutting Planes

PESP instance with period time $T = 10$:



courtesy: Christian Liebchen and Berenike Masing

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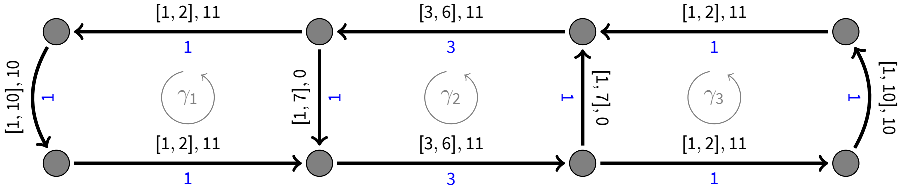
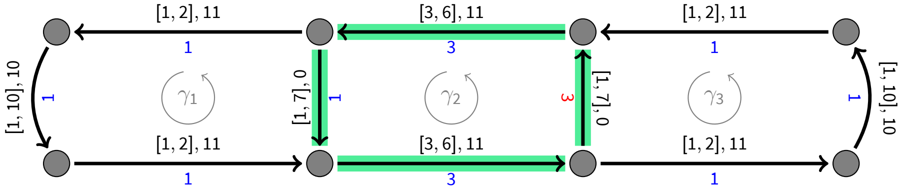


Relaxation	Optimal weighted slack
LP relaxation	0

Example: Cycle Inequalities as Cutting Planes

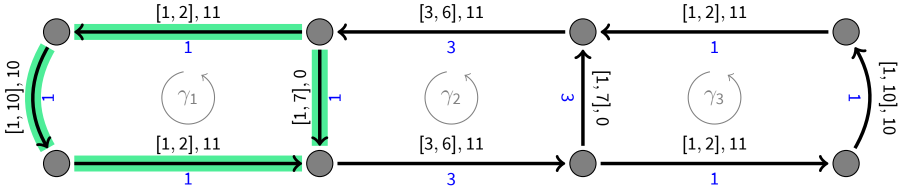
PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\gamma_2^\top x \geq 10 \left\lceil \frac{3+1+3+1}{10} \right\rceil = 10$	$\gamma_2^\top x = 3 + 1 + 3 + 3 = 10$	0

Example: Cycle Inequalities as Cutting Planes

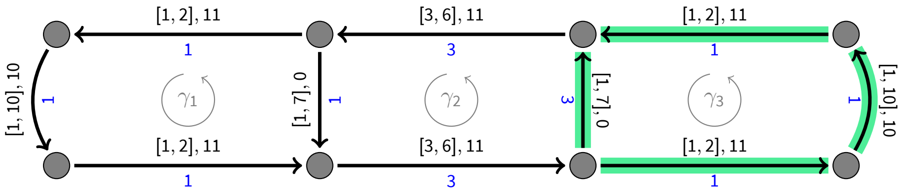
PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\gamma_2^\top x \geq 10 \left\lceil \frac{3+1+3+1}{10} \right\rceil = 10$	$\gamma_2^\top x = 3+1+3+3 = 10$	0
$+\gamma_1^\top x \geq 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_1^\top x = 1+1+1-1 = 2$	0

Example: Cycle Inequalities as Cutting Planes

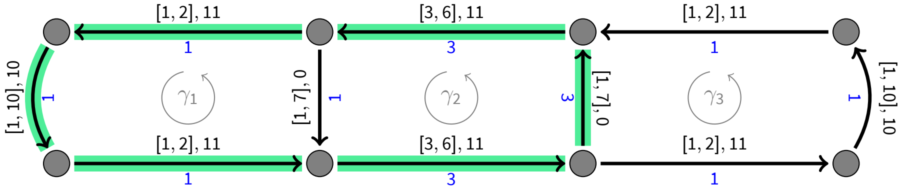
PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\gamma_2^\top x \ge 10 \left\lceil \frac{3+1+3+1}{10} \right\rceil = 10$	$\gamma_2^\top x = 3+1+3+3 = 10$	0
$+\gamma_1^\top x \ge 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_1^\top x = 1+1+1-1 = 2$	0
$+\gamma_3^\top x \ge 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_3^\top x = 1+1+1-3 = 0$	0

Example: Cycle Inequalities as Cutting Planes

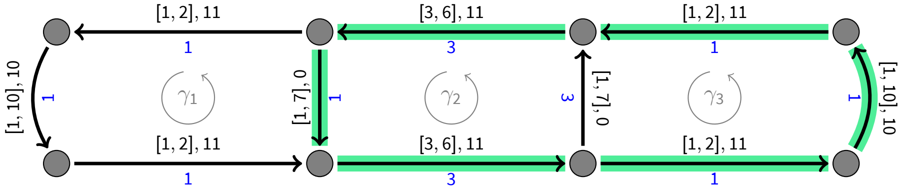
PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\gamma_2^\top x \geq 10 \left\lceil \frac{3+1+3+1}{10} \right\rceil = 10$	$\gamma_2^\top x = 3+1+3+3 = 10$	0
$+\gamma_1^\top x \geq 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_1^\top x = 1+1+1-1 = 2$	0
$+\gamma_3^\top x \geq 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_3^\top x = 1+1+1-3 = 0$	0
$+(\gamma_1+\gamma_2)^\top x \geq 10 \left\lceil \frac{10}{10} \right\rceil = 10$	$(\gamma_1+\gamma_2)^\top x = 12$	0

Example: Cycle Inequalities as Cutting Planes

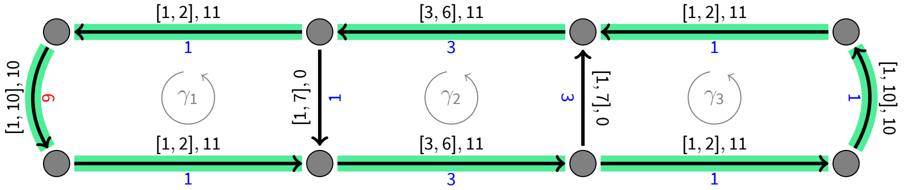
PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\gamma_2^\top x \geq 10\left\lceil\frac{3+1+3+1}{10}\right\rceil = 10$	$\gamma_2^\top x = 3+1+3+3 = 10$	0
$+\gamma_1^\top x \geq 10\left\lceil\frac{1+1+1-7}{10}\right\rceil = 0$	$\gamma_1^\top x = 1+1+1-1 = 2$	0
$+\gamma_3^\top x \geq 10\left\lceil\frac{1+1+1-7}{10}\right\rceil = 0$	$\gamma_3^\top x = 1+1+1-3 = 0$	0
$+(\gamma_1+\gamma_2)^\top x \geq 10\left\lceil\frac{10}{10}\right\rceil = 10$	$(\gamma_1+\gamma_2)^\top x = 12$	0
$+(\gamma_2+\gamma_3)^\top x \geq 10\left\lceil\frac{10}{10}\right\rceil = 10$	$(\gamma_2+\gamma_3)^\top x = 10$	0

Example: Cycle Inequalities as Cutting Planes

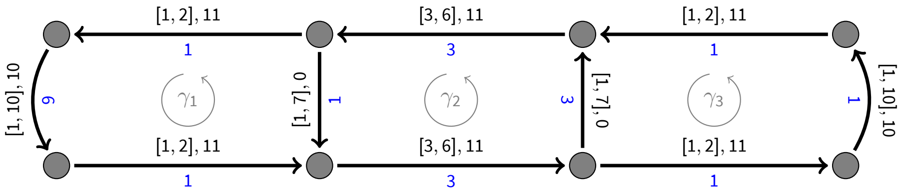
PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\ \gamma_2^\top x \geq 10 \left\lceil \frac{3+1+3+1}{10} \right\rceil = 10$	$\gamma_2^\top x = 3 + 1 + 3 + 3 = 10$	0
$+\ \gamma_1^\top x \geq 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_1^\top x = 1 + 9 + 1 - 1 = 10$	0
$+\ \gamma_3^\top x \geq 10 \left\lceil \frac{1+1+1-7}{10} \right\rceil = 0$	$\gamma_3^\top x = 1 + 1 + 1 - 3 = 0$	0
$+\ (\gamma_1 + \gamma_2)^\top x \geq 10 \left\lceil \frac{10}{10} \right\rceil = 10$	$(\gamma_1 + \gamma_2)^\top x = 20$	0
$+\ (\gamma_2 + \gamma_3)^\top x \geq 10 \left\lceil \frac{10}{10} \right\rceil = 10$	$(\gamma_2 + \gamma_3)^\top x = 10$	0
$+\ (\gamma_1 + \gamma_2 + \gamma_3)^\top x \geq 10 \left\lceil \frac{12}{10} \right\rceil = 20$	$(\gamma_1 + \gamma_2 + \gamma_3)^\top x = 20$	80

Example: Cycle Inequalities as Cutting Planes

PESP instance with period time $T = 10$:



Relaxation		Optimal weighted slack
LP relaxation		0
$+\gamma_2^\top x \geq 10\left\lceil\frac{3+1+3+1}{10}\right\rceil = 10$	$\gamma_2^\top x = 3+1+3+3 = 10$	0
$+\gamma_1^\top x \geq 10\left\lceil\frac{1+1+1-7}{10}\right\rceil = 0$	$\gamma_1^\top x = 1+9+1-1 = 10$	0
$+\gamma_3^\top x \geq 10\left\lceil\frac{1+1+1-7}{10}\right\rceil = 0$	$\gamma_3^\top x = 1+1+1-3 = 0$	0
$+(\gamma_1+\gamma_2)^\top x \geq 10\left\lceil\frac{10}{10}\right\rceil = 10$	$(\gamma_1+\gamma_2)^\top x = 20$	0
$+(\gamma_2+\gamma_3)^\top x \geq 10\left\lceil\frac{10}{10}\right\rceil = 10$	$(\gamma_2+\gamma_3)^\top x = 10$	0
$+(\gamma_1+\gamma_2+\gamma_3)^\top x \geq 10\left\lceil\frac{12}{10}\right\rceil = 20$	$(\gamma_1+\gamma_2+\gamma_3)^\top x = 20$	80
PESP MIP		80

Final Remarks on PESP

Remarks on MIP

- PESP is computationally very hard. Even medium-sized instances can be very challenging. Benchmarking library: `timpasslib.aalto.fi/pesplib.html`
- Nevertheless, PESP-based MIP models are applied in practice.
- The timetable-based MIP formulation is empirically better for finding feasible solutions more quickly.
- The cycle-based MIP formulation, in particular with cutting plane techniques, produces empirically smaller branch-and-bound trees and better dual bounds.

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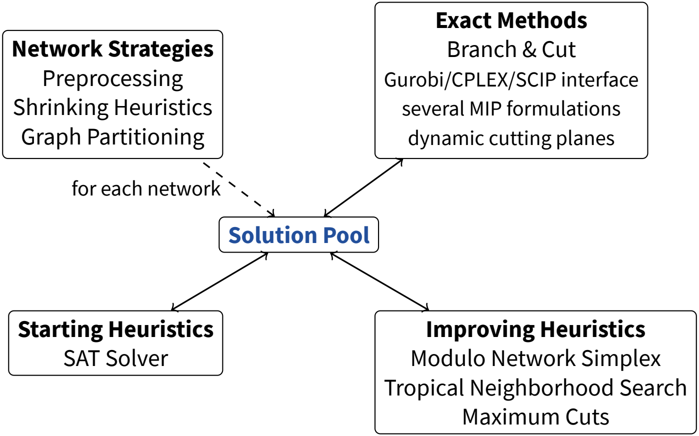
More to Discover

- combinatorics of optimal solutions: spanning tree structures, modulo network simplex (Nachtigall and Opitz, 2008)
- geometry of timetables: tropical neighborhood search (Bortoletto, Lindner, Masing 2022, 2024)
- model extensions: line planning, track allocation, vehicle scheduling (Masing, Lindner, Liebchen, 2023a, 2023b)

ConcurrentPESP

Concurrent Framework for Periodic Timetable Optimization



...trades off by far more than just concurrency
...holds primal and dual records for *all* 22 PESPlib instances

(Borndörfer, Lindner, Roth, 2020)

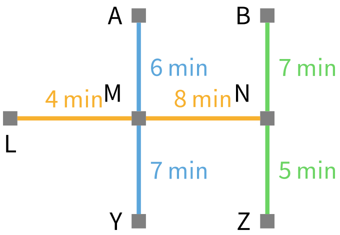
Try It Yourself

Can you find an optimal periodic timetable? ➡

`https://www.zib.de/lindner/tdm22/pesp.html`

The rules are as follows:

- The period time is $T = 10$ minutes.
- Transfers take at least 2 minutes.
- Driving times need to be exactly as indicated in the line network.
- Dwelling at stations must be between 1 and 5 minutes.
- Turnarounds must be between 3 and 5 minutes.



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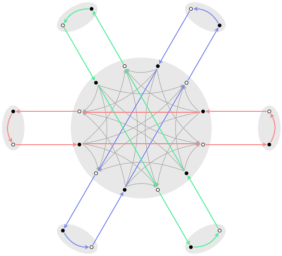
PERIODIC TIMETABLE OPTIMIZATION IN PUBLIC TRANSPORT



CO@Work 2024, Zuse Institute Berlin, September 26, 2024