Linear Programming & Polyhedral Theory

Ambros Gleixner
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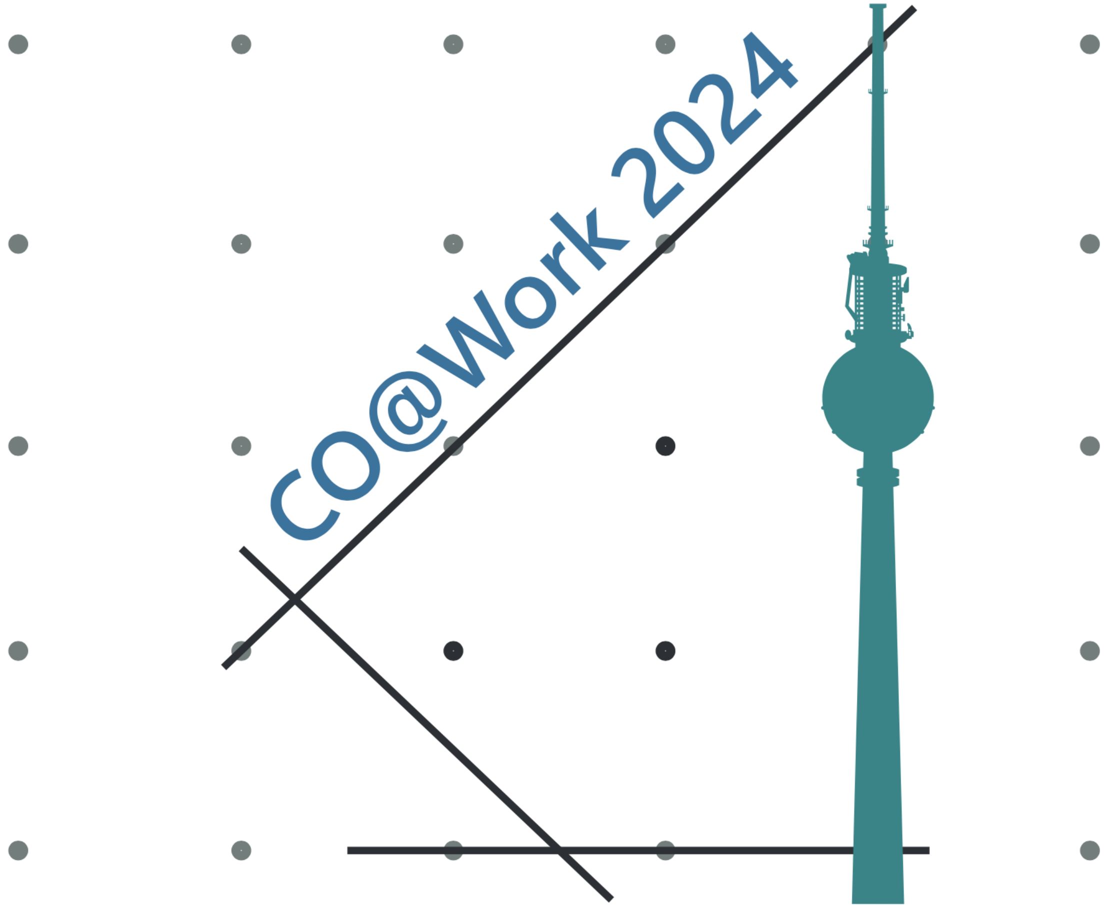


Agenda for this lecture

- Fundamentals about Mathematical Optimization
- Brief LP history
- Polyhedral theory
- Duality theory
- Simplex algorithm
- Glimpse on ellipsoid method, barrier, and crossover

Mathematical Optimization

Optimization Problem

$\rightarrow$ variables $\rightarrow$ solution $\rightarrow$ feasibility
$\rightarrow$ constraints $\rightarrow$ activity $\rightarrow$ feasibility
$\rightarrow$ objective function $\rightarrow$ value $\rightarrow$ quality

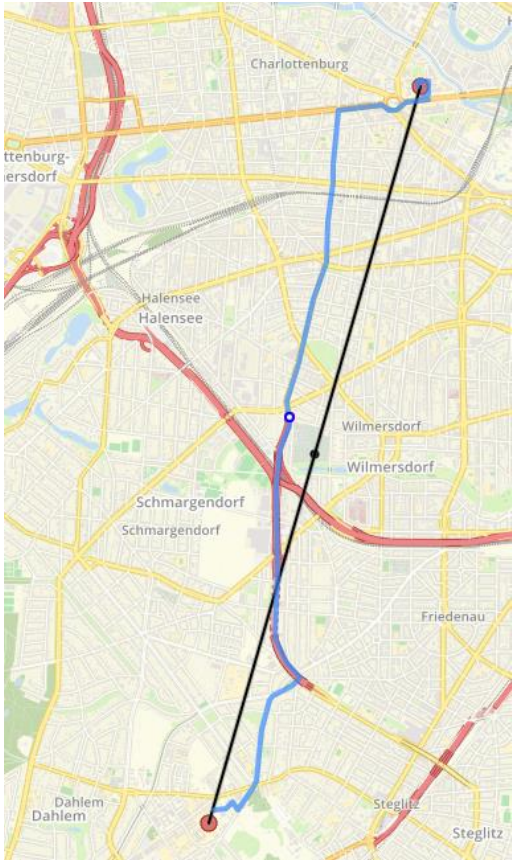
$\Rightarrow$ optimal solution = feasible + best possible objective value

Dual bound $\Leftarrow$ **GAP** $\Rightarrow$ **Primal bound**

If the gap is zero, we have a mathematical proof, that the incumbent solution is optimal.

Example: Primal-Dual Gap

- Example: What is the shortest path from
  - Zuse Institute Berlin
  - TU Berlin
- Primal Bound: 8.07 km driving route
  - Heuristic solution
  - Addressing a different objective
- Dual Bound: Bee line: 6.74 km
  - https://www.distance.to/
- Gap: $\frac{8.07-6.74}{8.07}$ = 16.48%



Linear Programming

Linear Program

Objective function:

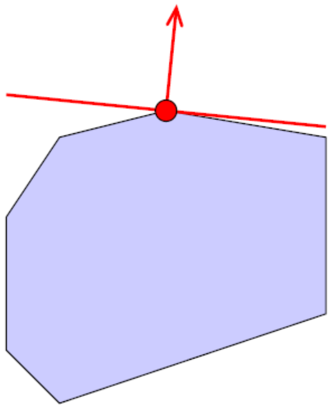
- ▷ linear function

Feasible set:

- ▷ described by linear constraints

Variable domains:

- ▷ real values



$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \in R^n_{\geq 0} \end{aligned}$$

- ▷ convex set
- ▷ "basic" solutions

Various forms, all equivalent

- All representations can be converted into each other:
  - min to max: multiply objective vector c bei -1
  - Equation to inequality: $a_i^T x = b_i \rightarrow a_i^T x \leq b_i, -a_i^T x \leq -b_i$
  - $\leq$-inequality to $\geq$-inequality: multiply by -1
  - Inequality to equation: Introduce slack variable, $a_i^T x \leq b_i \rightarrow a_i^T x + s_i = b_i$
  - Unbounded variable to bounded: $x = x^+ - x^-, x \in \mathbb{R},\ x^+, x^- \in \mathbb{R}_{\geq 0}$
  - Bounded to unbounded: Consider bounds as constraints
- LP literature typically uses the standard form $\min \{c^T x \mid Ax = b, x \geq 0\}$
- MIP literature often uses inequalities for the constraints

Integer Programming

Integer Program

Objective function:

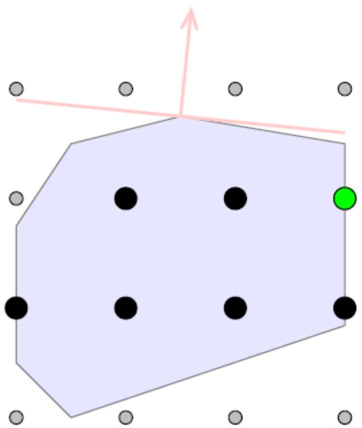
- ▷ linear function

Feasible set:

- ▷ described by linear constraints

Variable domains:

- ▷ integer values



$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \le b \\ & x \in \mathbb{Z}_{\ge 0} \end{aligned}$$

- ▷ not even connected
- ▷ $\mathcal{NP}$-hard problem

Mixed-Integer Programming

Definition: MIP

Objective function:

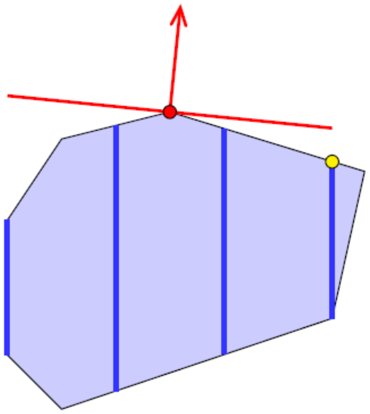
- ▷ linear function

Feasible set:

- ▷ described by linear constraints

Variable domains:

- ▷ integer or real values



$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \leq b \\ & x \in \mathbb{Z}^I \times \mathbb{R}^C \end{aligned}$$

- ▷ not even connected
- ▷ $\mathcal{NP}$-hard problem

LP History

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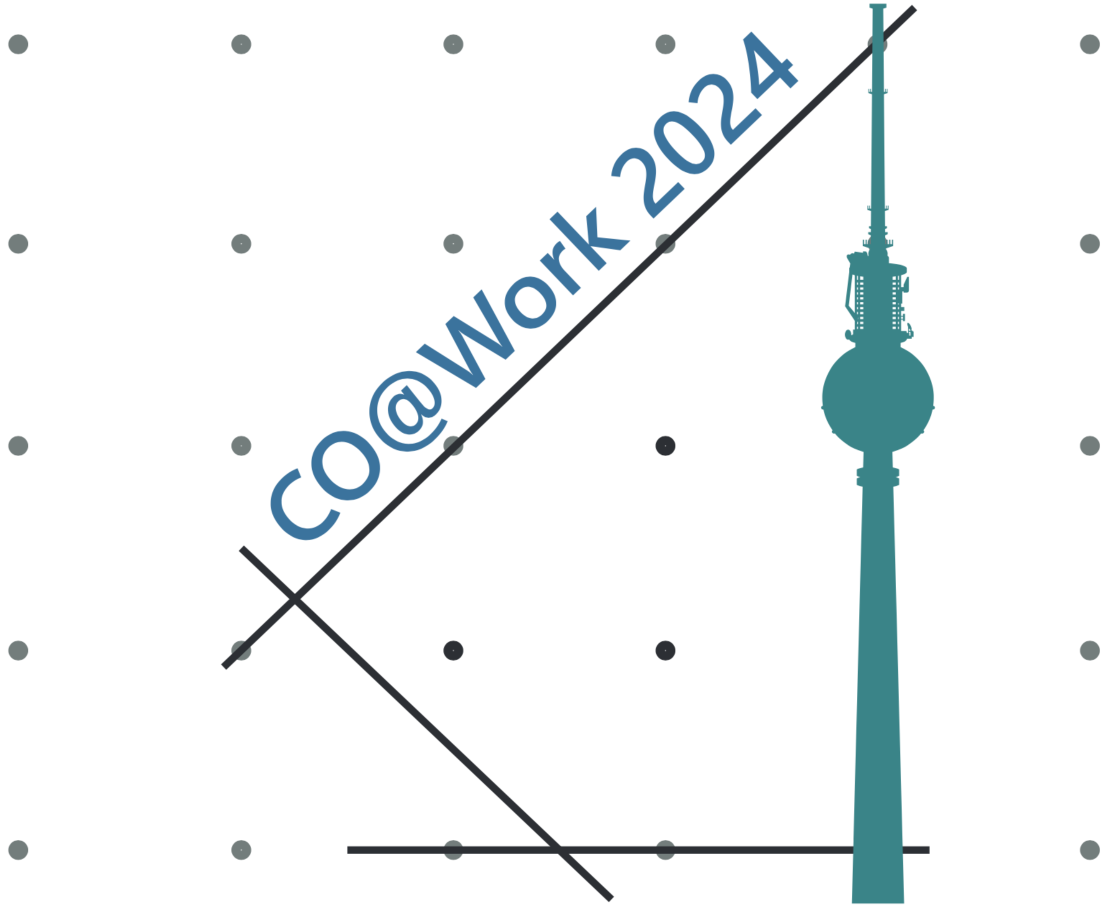

LP History: The first LP

- In 1827 Fourier described a variable elimination method for linear inequalities, today often called Fourier-Motzkin elimination (Motzkin 1936).
  - By adding one variable and one inequality, Fourier-Motzkin elimination can be turned into an LP solver.
- Who formulated the first LP?
  - The usual credit goes to George J. Stigler (1939)

Min $x1 + x2$	costs
$2x1 + x2 \geq 3$	protein
$x1 + 2x2 \geq 3$	carbohydrates
$x1 \geq 0$	potatoes
$x2 \geq 0$	beans

minimizing the cost of food

- Full example:
  - 77 foods, 9 nutritients
  - Stigler's heuristic solution was 0.7% from optimal

LP History: The first LP algorithms

- 1939 L. V. Kantorovitch (1912-1986): Foundations of linear programming
- 1947 G. B. Dantzig (1914-2005): Invention of the (primal) simplex algorithm
- 1954 C.E. Lemke & E.M.L. Beale: Dual simplex algorithm
- 1953 G.B. Dantzig, 1954 W. Orchard Hays, and 1954 G. B. Dantzig & W. Orchard Hays: Revised simplex algorithm

More on this later...

First computational study in 1953

COMPUTATIONAL EXPERIENCE IN SOLVING LINEAR PROGRAMS*

A. HOFFMAN, M. MANNOS, D. SOKOLOWSKY and N. WIEGMANN

1. Introduction. This paper is a discussion of three methods which have been employed to solve problems in linear programming, and a comparison of results which have been yielded by their use on the Standards Eastern Automatic Computer (SEAC) at the National Bureau of Standards.

LP history: commercial implementations

- The first commercial LP-Code was on the market in 1954 and available on an IBM CPC (card programmable calculator)
  - Record: 71 variables, 26 constraints, 8 h running time
- About 1960: LP became commercially viable, used largely by oil companies
- 1972: first commercial IP solver (almost 50 years ago)

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NIC 10424                                        Karl Kelley
NWG/RFC 345                           University of Illinois
                                                May 26, 1972

     INTEREST IN MIXED INTEGER PROGRAMMING (MPSX ON 360/91 AT CCN)

     MPSX is a newer version of the IBM project MPS, used for integer
programming.  From what I've been told, MPSX outperforms the previous
package.  In addition, it has available a feature of mixed integer
programming.
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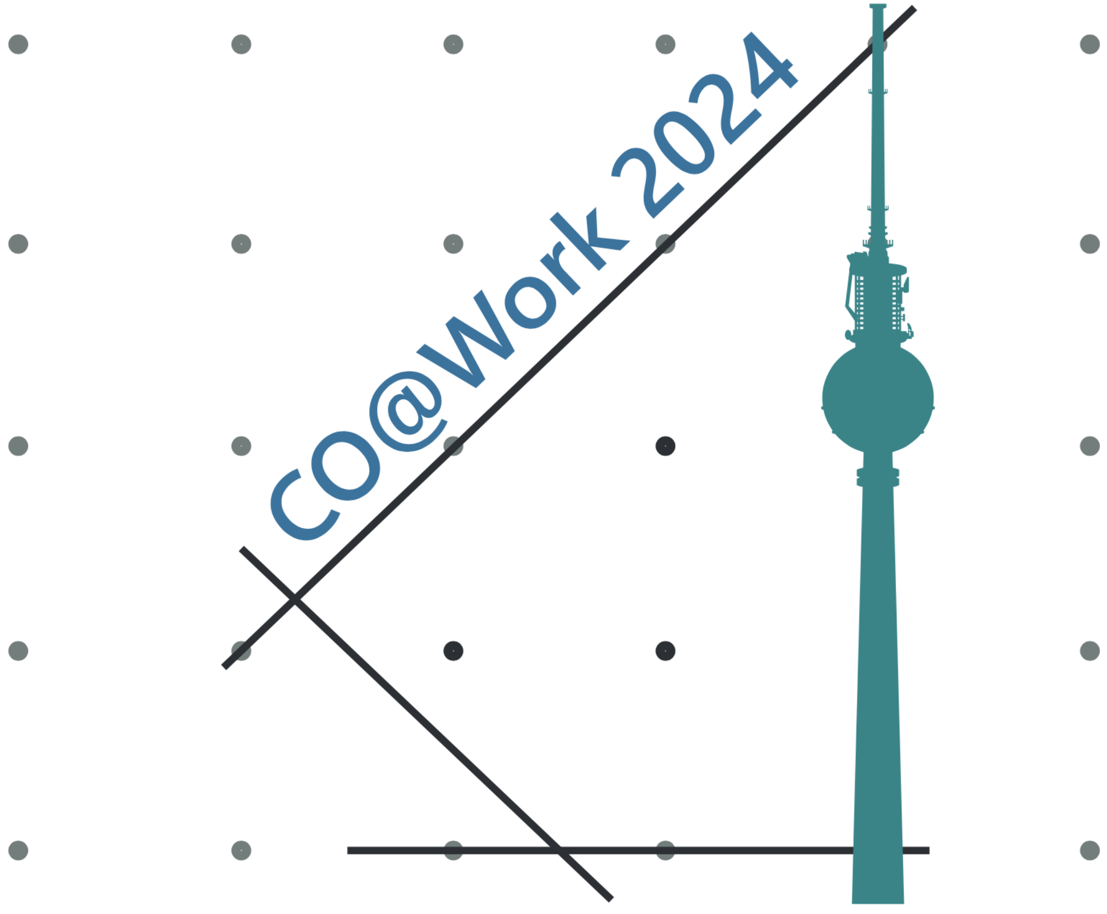

Polyhedra

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LP & Polyhedra

For nice, interactive visualizations of the 120 regular convex polyhedra, check out: https://polyhedra.tessera.li/

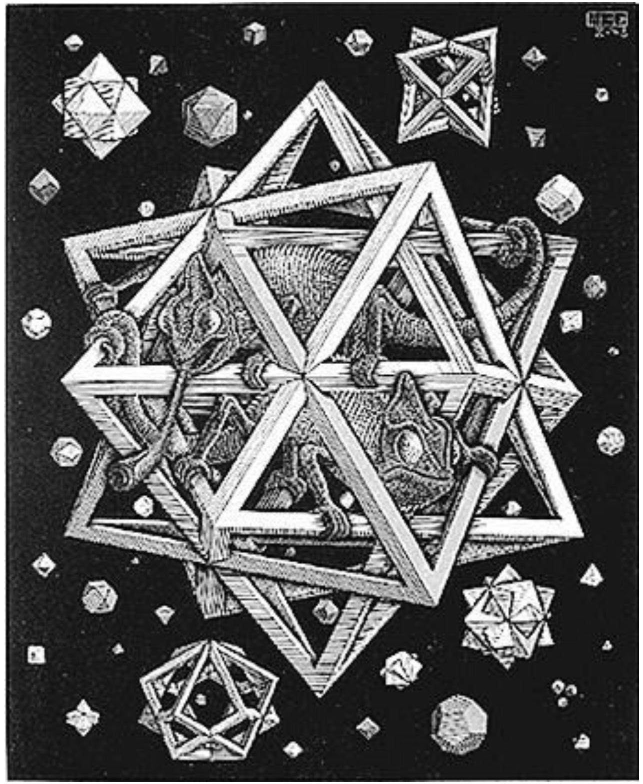


Image source: Wikipedia

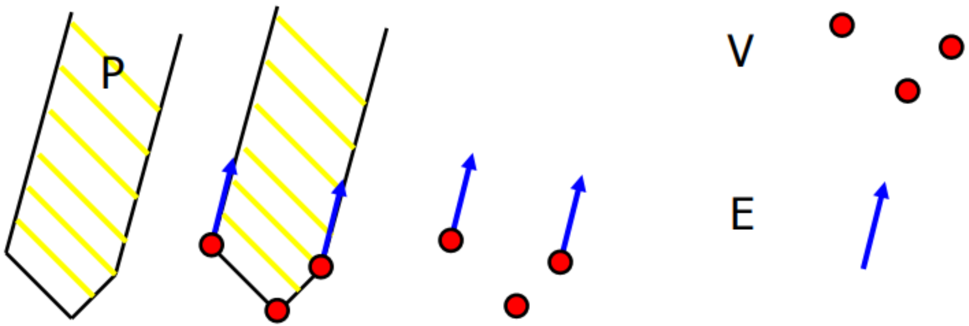
LP & Polyhedra

- Linear programming lives (for our purposes) in the n-dimensional real vector space.

- polyhedron: intersection of finitely many halfspaces
  - $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
- polytope: convex hull of finitely many points
  - $P = conv(V)$, V a finite set in $\mathbb{R}^n$.
- convex polyhedral cone:
  conic combination (i.e., a nonnegative linear combination) of finitely many rays
  - $K = cone(E)$, E a finite set in $\mathbb{R}^n$.

Representation of polyhedra

- Theorem: For a subset $\mathrm{P} \subseteq \mathbb{R}^n$ the following are equivalent:
  1. P is a polyhedron: the intersection of finitely many halfspaces, i.e., there exist a matrix A and a vector b with $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ (outer representation).
  2. P is the sum of a convex polytope and a finitely generated (polyhedral) cone, i.e., there exist finite sets V and E with $P = conv(V) + cone(E)$ (inner representation)



Important special cases

- $P = \ conv(0, e_i) \qquad P = \{x \mid x \geq 0, \sum x_i \leq 1\}$
  - Simplex: n+1 points, n+1 inequalities
- $P = \ conv(-e_i, e_i) \qquad P = \{x \mid a^T x \leq 1, \forall\, a \ \in \{-1,1\}^n\}$
  - Cross polytope: 2n points, 2^n inequalities
- $P = \ conv(\{-1,1\}^n) \qquad P = \{x \mid a^T x \leq 1, \forall\, a \ \in \{-e_i, e_i\}\}$
  - Cube: 2^n points, 2n inequalities

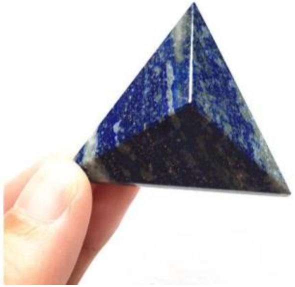


Image source: aliexpress.com

Image source: healingcrystals.com

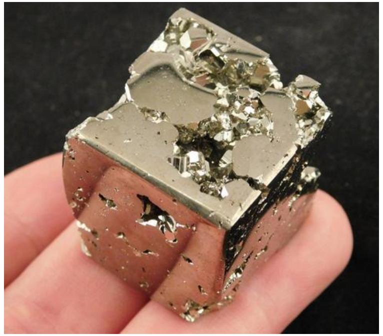


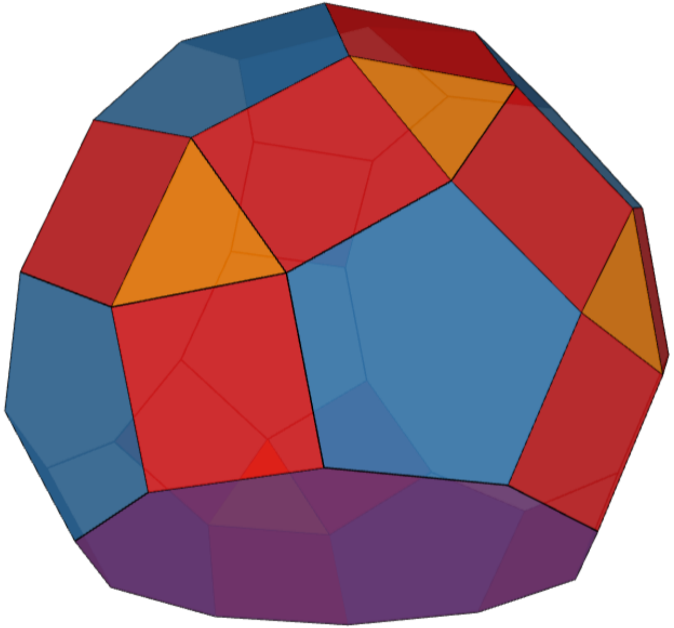
Image source: naturshop.cz

Faces of polyhedra

An n-dimensional polyhedron has the following different faces:

- Vertex (0-dimensional)
- Edge (1-dimensional)
- ...
- Ridge = subfacet (n-2)-dimensional
- Facet (n-1)-dimensional

and each of them is a polyhedron itself!



Source: polyhedra.tessera.li

Question: Is a polyhedron non-empty?

- Given $P = conv(V) + cone(E)$ → Yes, trivially $V \subseteq P$!
- Given $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ → Answered by the Farkas-Lemma (1908):

$$P = \emptyset \;\Leftrightarrow\; \exists y \geq 0\colon\; y^T A = 0^T,\; y^T b < 0^T$$

- i.e., if we can reformulate $Ax \leq b$ to the trivially wrong statement $0 \leq y^T b < 0$ by scaling and adding constraints with dual multipliers y_i.
- Example:

$$\begin{aligned} 2x_1 + 2x_2 &\leq 1 \\ x_1 \qquad &\geq 1 \\ x_2 &\geq 1 \end{aligned} \longrightarrow \left.\begin{aligned} 2x_1 + 2x_2 &\leq 1 \\ -x_1 \qquad &\leq -1 \\ -x_2 &\leq -1 \end{aligned}\right\} \; 0 \leq -3$$

- This Theorem of Alternatives is the foundation of all important higher-level LP theory: duality theorems, complementary slackness, proof of LP optimality, etc.

Duality

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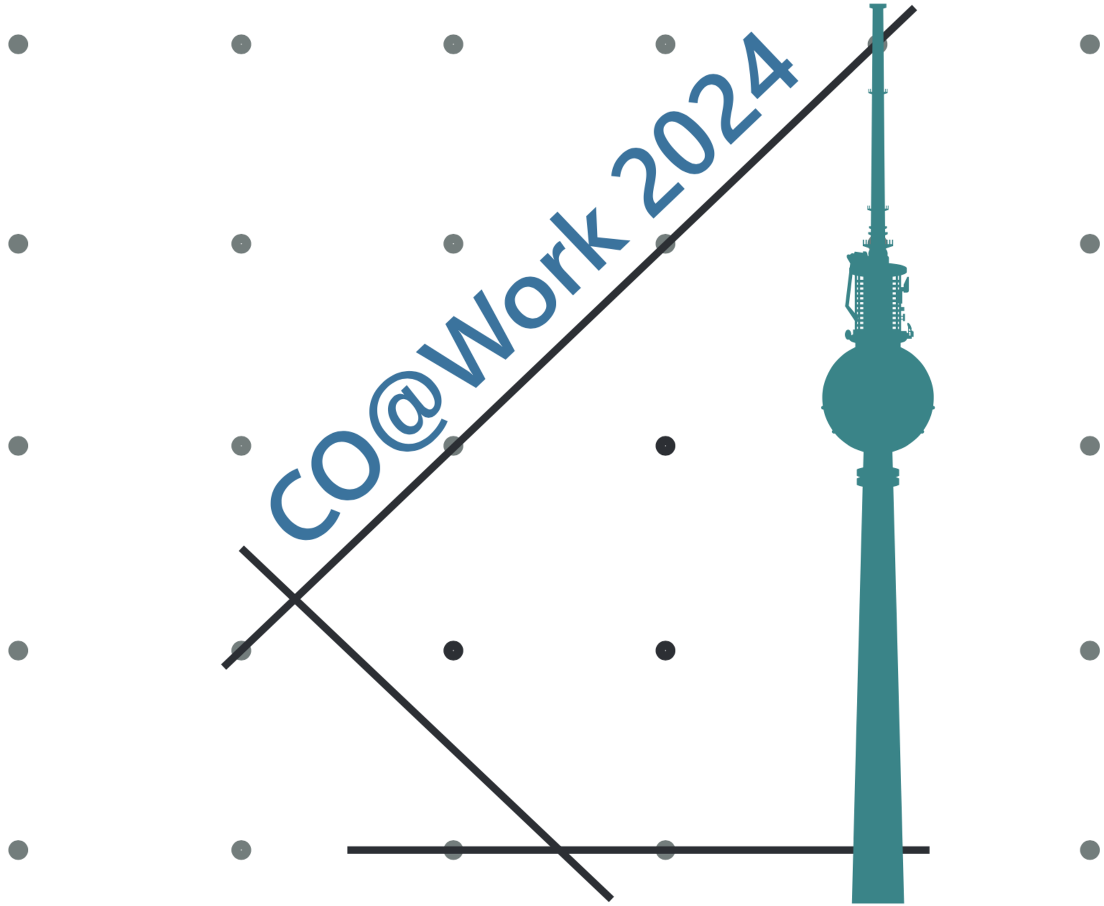
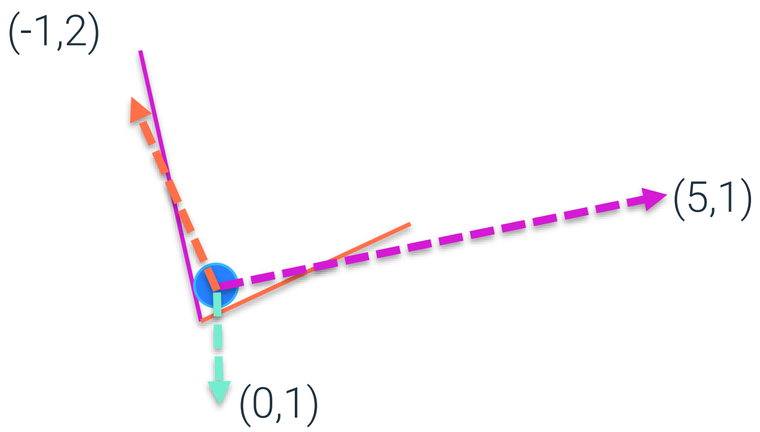


The dual LP: example

- Picture a ball resting between two planes:

$$\begin{aligned} \min \quad & 0x_1 + 1x_2 \\ \text{s.t.} \quad & 5x_1 + 1x_2 \geq 6 \\ & -1x_1 + 2x_2 \geq 1 \end{aligned}$$



- How do we get a proof that (1,1) is indeed the minimal position of the ball?

- Build a conic combination of the constraints to act precisely against the objective function:

$$\begin{aligned} \max \quad & 6y_1 + 1y_2 \\ \text{s.t.} \quad & 5y_1 - 1y_2 = 0 \\ & 1y_1 + 2y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

- This is the dual LP.

The dual LP: general

$$\min\{c^T x \mid Ax \geq b, x \geq 0\}$$

- How do we get a proof of solution quality?
- Easy case: One of our constraints underestimates the objective function
  - E.g., if $c^T x = 2x_1 + x_2$ and one constraint $x_1 + x_2 \geq 3$, then also $\min c^T x \geq 3$
- General case: Conic combinations of constraints yield valid constraints.
  - Observation 1: Constraints are invariant to scaling by positive numbers.
  - Observation 2: The sum of two valid constraints is a valid constraint.
- Dual LP:
  - Find a conic combination of all constraints that is a
  - maximal underestimator for the objective function.

The dual LP: formally

$$\min\{c^T x \mid Ax \geq b, x \geq 0\}$$

- Dual multipliers: $y \geq 0$
- Conic combination of all constraints...: $\mathrm{y}^{\mathrm{T}} Ax \geq y^T \mathrm{b}$
- ...that is an (...)underestimator for our objective: $\mathrm{y}^T A \leq c$
- ...and that is a maximal underestimator: $\max \; y^T b$
- This gives the dual LP:

$$\max\{y^T b \mid y^{\mathrm{T}} A \leq c, y \geq 0\}$$

The dual LP: scheme

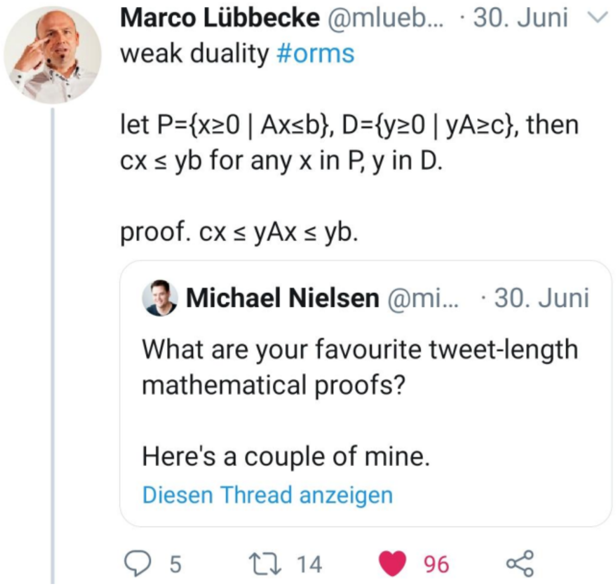
$$\min\{c^T x \mid Ax \geq b, x \geq 0\} \;\leftrightarrow\; \max\{y^T b \mid y^T A \leq c, y \geq 0\}$$

- Each constraint in the primal LP becomes a variable in the dual LP.
- Each variable in the primal LP becomes a constraint in the dual LP.
- The objective direction is inversed.
- Signs are inversed:
  - $\geq$-constraints become $\leq$-bounds,
  - $\leq$-constraints become $\geq$-bounds
  - Consequently, =-constraints become free variables
- By this scheme: Easy to see that the dual of the dual is the primal.

Weak duality theorem

$$\max\{y^T b \mid y^T A \leq c, y \geq 0\} \;\leq\; \min\{c^T x \mid Ax \geq b, x \geq 0\}$$

- Trivial to proof:
- $y^T b \leq y^T(Ax) = (y^T A)x \leq c^T x$
- q.e.d.



Strong duality theorem

- The most important and influential theorem in optimization:
- Primal has a finite optimum if and only if dual has a finite optimum, and

$$\min\{c^T x \mid Ax \geq b, x \geq 0\} = \max\{y^T b \mid y^T A \leq c, y \geq 0\}$$

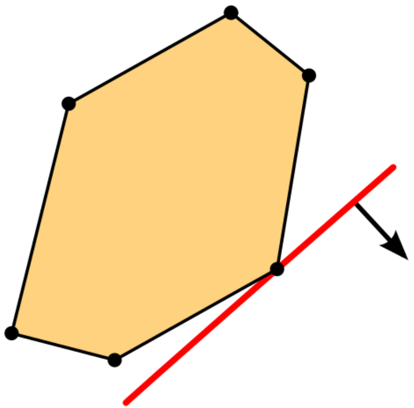
- A relation of this type is called min-max result.
- Proof is not straight-forward, uses weak duality and Farkas lemma.
- Three combinations of primal and dual status are possible:
  - finite (and equal) optima,
  - unbounded and infeasible,
  - both infeasible.

Complementary slackness

- At an optimal solution pair (x,y),
  - for all constraints i : either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij}\, x_j = b_i$ (or both hold);
  - analogously, for all variables j : either $x_j = 0$ or $\sum_{j=1}^{n} a_{ij}\, y_i = c_j$
- Proof: By weak duality, because otherwise

$$y^T b \leq y^T (Ax) = (y^T A)x \leq c^T x$$

  does not hold with equality.

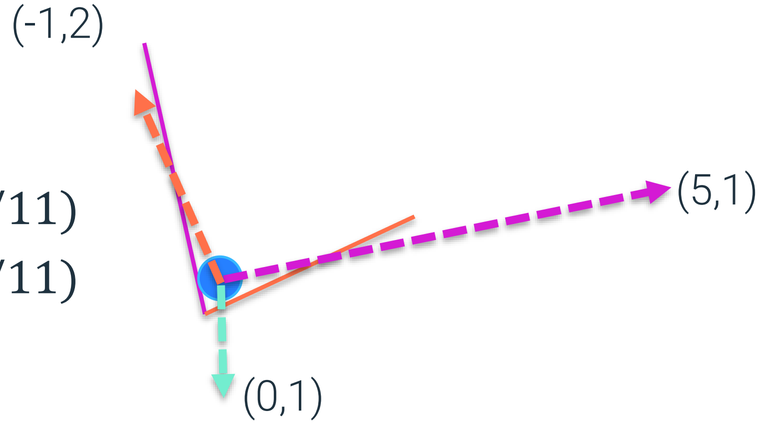


- Consequence: To construct an optimality proof, we can only use constraints that are tight at the optimal point.
- Strong complementary slackness: There is a solution, s.t. $y_i > 0 \Leftrightarrow \sum_{j=1}^{n} a_{ij}\, x_j = b_i$.
- Sensitivity: Interpretation of dual variables as shadow prices: How much would the objective increase, if we released the constraint?

The dual LP: example

- Picture a ball resting between two planes:

$$\begin{aligned} \min \quad & 0x_1 + 1x_2 \\ \text{s.t.} \quad & 5x_1 + 1x_2 \geq 6 \mid \cdot y_1 \ (1/11) \\ & -1x_1 + 2x_2 \geq 1 \mid \cdot y_2 \ (5/11) \end{aligned}$$



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- This is the dual LP.

Quiz time

- If a dual variable has a nonzero value in an optimal primal-dual solution:
  - the corresponding primal variable is nonzero as well
  - the corresponding primal constraint is nonzero as well
  - the corresponding primal constraint is tight
- Describing a polyhedron as $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is called the
  - Standard form
  - Outer representation
  - Inner representation
- The dual of $\min\{c^T x \mid Ax = b, x \geq 0\}$ is
  - $\min\{y^T b \mid \mathrm{y}^\mathrm{T} A = c, y \leq 0\}$
  - $\max\{y^T b \mid \mathrm{y}^\mathrm{T} A \leq c, y \geq 0\}$
  - $\max\{y^T b \mid \mathrm{y}^\mathrm{T} A \leq c, y \text{ free}\}$

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Algorithms

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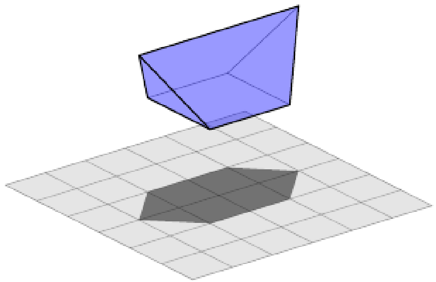
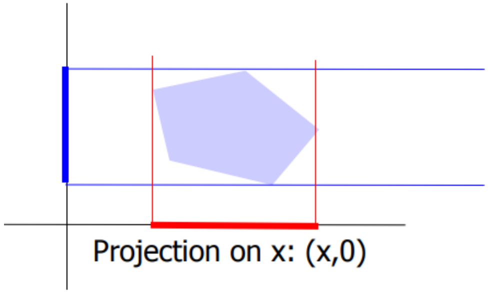
Fourier-Motzkin Elimination

- Fourier, 1827, rediscovered by Motzkin, 1936
- Method: successive projection of a polyhedron in n-dimensional space into a vector space of dimension n-1 by elimination of one variable.





- Can
  - check whether a polyhedron is nonempty.
  - be used to prove Farkas Lemma.
  - be used for Linear Programming.

From inner to outer representation

- Each elimination step might square the number of rows, hence in total $O(m^{2^n})$.
- Fourier-Motzkin essentially the best-known method for polyhedral transformations:
  - Let a polyhedron be given as $P = conv(V) + cone(E)$
  - Goal: Find a representation of P in the form $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
  - Idea: Write

$$P = \{x, y, z \in \mathbb{R}^d \mid x = Vy + Ez, \sum y_i = 1, y \geq 0, z \geq 0\}$$

  - and eliminate y and z
- With some tricks, FME can be reduced to single-exponential running time, which is already best possible for cube/cross polytope.

Simplex idea

- Start at a random vertex
- Among neighboring vertices, choose on which improves the objective, if none: optimal
  - Special case: An unbounded ray along which the objective improves → then, the LP is unbounded.
- Move along edges towards an optimal vertex.
- Why does this work?
  1. Show that an optimal boundary solution exists (convexity)
  2. Narrow down further an optimal vertex solution exists (linearity)
  3. Local optimality is global optimality (convexity)

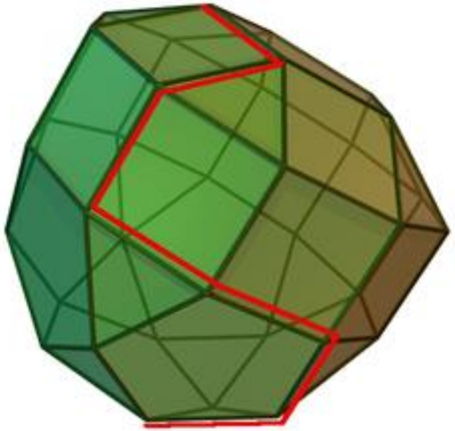


image source: WIkipedia

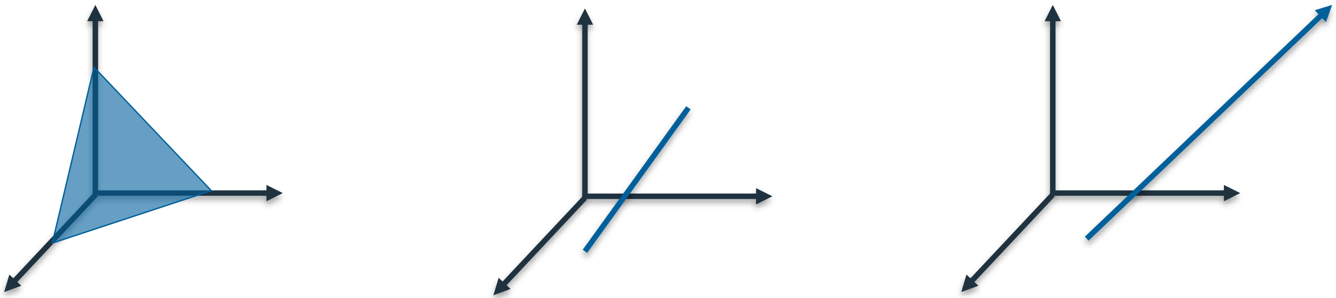
Computational standard form

- By adding slack variables etc., every LP can be equivalently transformed to

$$\min\{c^T x \mid Ax = b, x \geq 0\} = \max\{y^T b \mid y^T A \leq c, y \text{ free}\}$$

(primal LP) (dual LP)

- Nonempty LPs in standard form always have vertices, and are the
- intersection of a cone (nonnegative orthant) and an affine subspace, e.g.



Basic solution: primal

- W.l.o.g. let $rank(A) = m < n$ and consider the computational form

$$\min\{c^T x \mid Ax = b, x \geq 0\}$$

- For every vertex there is a non-singular $m \times m$ sub-matrix B of A

$$A = \begin{bmatrix} B & N \end{bmatrix}$$

- and the corresponding basic solution is given by

$$x_B = B^{-1}b, \qquad x_N = 0$$

- LP is a discrete optimization task: $\binom{n}{m}$ possible bases.
- Basic solution = vertex solution $\Leftrightarrow$?

Basic solution: primal-dual

- W.l.o.g. let $rank(A) = m < n$ and consider the computational form

$$\min\{c^T x \mid Ax = b, x \geq 0\} = \max\{y^T b \mid y^T A \leq c, y \text{ free }\}$$

- For every vertex there is a non-singular $m \times m$ sub-matrix B of A

$$A = \begin{bmatrix} B & N \end{bmatrix}$$

- and the corresponding basic solution is given by

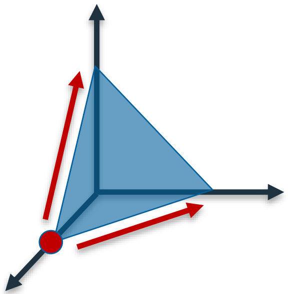
$$x_B = B^{-1} b, \qquad x_N = 0, \qquad y^T = c_B^T B^{-1}$$

- Basic solution = vertex solution $\Leftrightarrow$ x primal feasible $\Leftrightarrow$ $x \geq 0$ $\Leftrightarrow$ $x_B \geq 0$
- x optimal $\Leftrightarrow$?

One step of the primal simplex algorithm (Dantzig 1947)

1. Check optimality = dual feasibility: $\hat{c}_j \leq 0$ for all columns j?
2. If not dual feasible, choose pivot column: $\hat{c}_j > 0$ and increase x_j from 0 to $\delta > 0$.
3. Check boundedness (ratio test):

$$Bx_B + A_{*j}\delta = b \quad \rightsquigarrow \quad x_B = B^{-1}\left(b - A_{*j}\delta\right) \geq 0 \quad \Leftrightarrow \quad \delta \leq \hat{b}_i / \hat{a}_{i,j}$$



for $\hat{b} = B^{-1}b,\ \hat{A} = B^{-1}A,$ and all rows i with $\hat{a}_{i,j} > 0$.

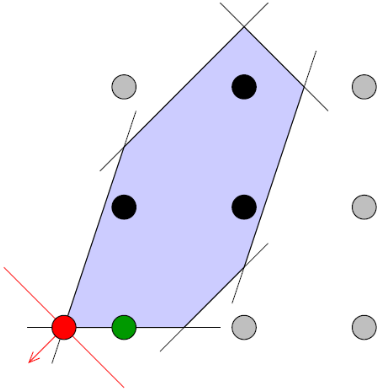
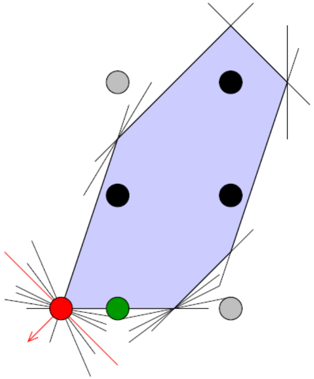
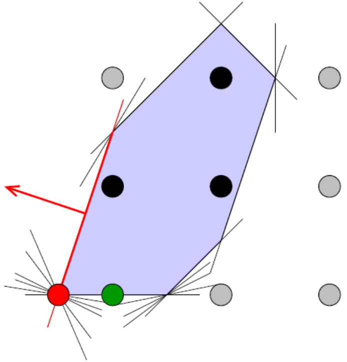
4. Pick pivot row with minimal $\hat{b}_i / \hat{a}_{i,j}$ and exchange:
   - variable x_j becomes basic: in B
   - variable x_i becomes nonbasic: in N, i.e., $x_i = 0$
5. Update all data structures $\hat{A}, \hat{b}, \hat{c}$ and B^{-1}.

Correctness of simplex algorihm

- If all vertices non-degenerate, simplex algorithm finds solution in finite time
  - Each step new, strictly improving basis
- Otherwise, cycling can be resolved by lexicographic rules or by slight pertubation
  - But: major numerical issue in practice
- Phase I: an initial feasible solution can be found by auxiliary LP, which itself has a trivial initial feasible solution
  - Ensure $b \geq 0$, then $(A \;\; I)\binom{x}{s} = b$ has trivial solution $x = 0, y = b$ow minimize Σy
  - Phase I LP is bounded and $rank(A) = m < n$
  - Final step: Pivot all auxiliary columns out of the basis

LP degeneracy

- Primal degeneracy:
  - Naïve thought: one optimal solution in $\mathbb{R}^n$ is uniquely determined by n constraints.
  - Second thought: one optimal solution can have more than n tight constraints
- Dual degeneracy:
  - The set of optimal solutions is a higher-dimensional face.
- Can be exploited in some ways, but is mostly a burden.

More general: pivoting through infeasible bases

- Primal simplex: pivot through primal feasible bases until dual feasible → optimal.

- Dual simplex: pivot through dual feasible bases until primal feasible → optimal.
  - Theoretically equivalent to the primal simplex on the dual LP
  - But practically much more efficient to implement

Re-optimization after changing your LP

Suppose you have reached an optimal basic solution and you …

- change the objective
  - Basis stays primal feasible, warm-start primal simplex
- add a column
  - Basis stays primal feasible (add to nonbasis), warm-start primal simplex
- add a row
  - Basis stays dual feasible (add slack to basis), warm-start dual simplex
- change a right hand side / variable bound
  - Basis stays dual feasible, warm-start dual simplex

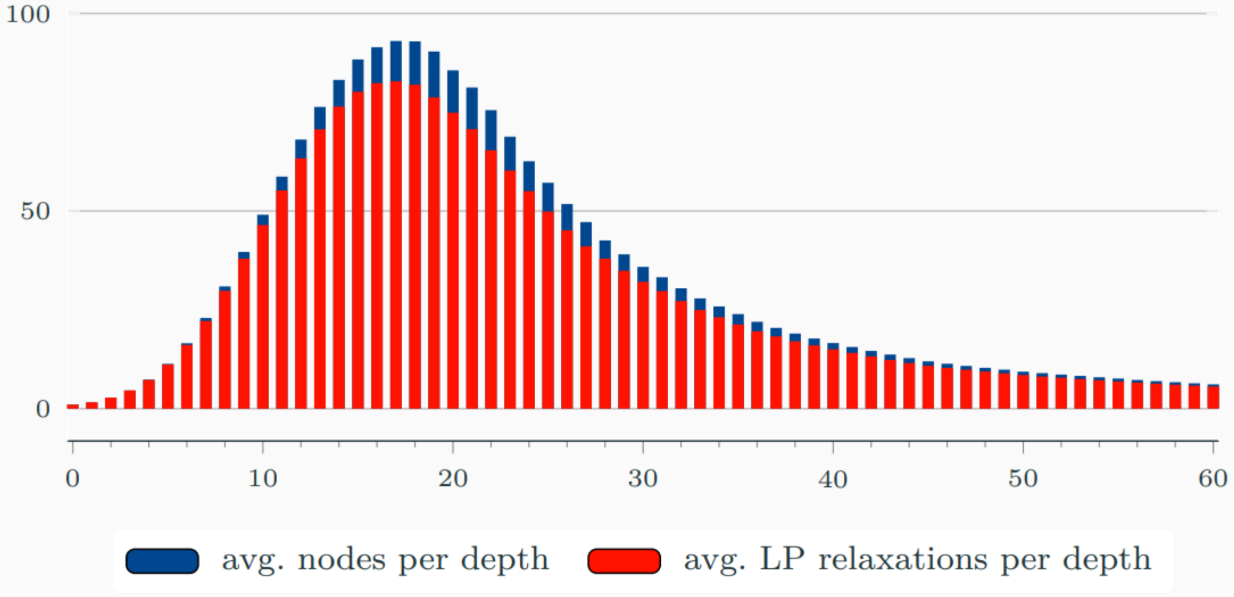
This makes the simplex a very efficient method in branch-and-bound (→ Wednesday) and other algorithms where similar LPs need to be solved after small modifications.

Simplex warmstarting during branch-and-bound

depth	avg. iters
Wurzel	**383.7**
1	17.0
2	14.9
3	12.2
4	10.3
5	9.0
6	8.2
⋮	⋮
13	4.2
14	4.1
15	3.8
16	3.4
17	3.3
18	3.1
19	2.8
20	2.7
21	2.5

$383.7/3.3 \approx 116\text{x speedup}$

330 MIPs “1sec–1hour”



Khachiyan's algorithm – the first polynomial time LP solver

Доклады Академии наук СССР
1979. Том 244, № 5

УДК 519.95 МАТЕМАТИКА

Л. Г. ХАЧИЯН

ПОЛИНОМИАЛЬНЫЙ АЛГОРИТМ В ЛИНЕЙНОМ ПРОГРАММИРОВАНИИ

(Представлено академиком А. А. Дородницыным 4 X 1978)

Рассмотрим систему из $m \geqslant 2$ линейных неравенств относительно $n \geqslant 2$ вещественных переменных $x_1, \ldots, x_j, \ldots, x_n$

$$a_{i1}x_1 + \ldots + a_{in}x_n \leqslant b_i, \quad i = 1, 2, \ldots, m, \tag{1}$$

с целыми коэффициентами a_{ij}, b_i. Пусть

$$L = \left[\sum_{i,j=1}^{m,n} \log_2(|a_{ij}|+1) + \sum_{i=1}^{m} \log_2(|b_i|+1) + \log_2 nm \right] + 1 \tag{2}$$

есть длина входа системы, т. е. число символов 0 и 1, необходимых для записи (1) в двоичной системе счисления.

Mathematical Programming Study 14 (1981) 61–68.
North-Holland Publishing Company

KHACHIYAN'S ALGORITHM FOR LINEAR PROGRAMMING*

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Received 10 October 1979

L.G. Khachiyan's algorithm to check the solvability of a system of linear inequalities with integral coefficients is described. The running time of the algorithm is polynomial in the number of digits of the coefficients. It can be applied to solve linear programs in polynomial time.

Key Words: Linear Programming, Inequalities, Complexity, Polynomial Algorithms.

0. Introduction

L.G. Khachiyan [1, cf. also 2, 3] published a polynomial-bounded algorithm to solve linear programming. These are some notes on this paper. We have ignored his considerations which concern the precision of real computations in order to make the underlying idea clearer; on the other hand, proofs which are missing from his paper are given in Section 2. Let

$$a_i x < b_i \quad (i = 1, \ldots, m,\ a_i \in Z^n,\ b_i \in Z) \tag{1}$$

be a system of *strict* linear inequalities with integral coefficients. We present an algorithm which decides whether or not (1) is solvable, and yields a solution if it is. Define

$$L = \sum_{i,j} \log(|a_{ij}|+1) + \sum_i \log(|b_i|+1) + \log nm + 1.$$

L is a lower bound on the space needed to state the problem.

Ellipsoid method

- Idea: From coefficients in A and b, we can determine largest possible solution value for x and minimum size of polyhedron
- Find large ball, which must contain a feasible solution, if one exist
- Check whether center point is feasible
- Cut ball/ellipsoid in (less than) half, determine smallest ellipsoid that contains half ellipsoid
- Repeat until ellipsoid is so small that polytope must be contained in ellipsoid (or is empty)
- Good online lecture: https://www.coursera.org/lecture/advanced-algorithms-and-complexity/optional-the-ellipsoid-algorithm-N9rzA

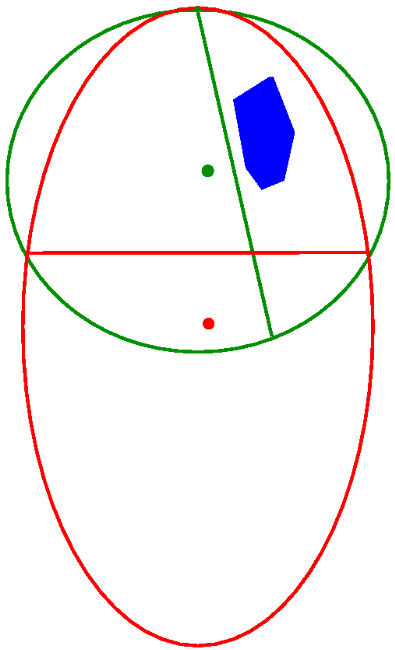

image source: WIkipedia

Karmarkar's algorithm, poly-time with practical impact

[11] **Patent Number: 4,744,028**

[45] **Date of Patent: May 10, 1988**

[54] METHODS AND APPARATUS FOR EFFICIENT RESOURCE ALLOCATION

[75] Inventor: Narendra K. Karmarkar, Somerset, N.J.

[73] Assignee: American Telephone and Telegraph Company, AT&T Bell Laboratories, Murray Hill, N.J.

[21] Appl. No.: 725,342

[22] Filed: Apr. 19, 1985

[51] Int. Cl.⁴ G06F 15/20; H04Q 3/66; H04M 7/00

[52] U.S. Cl. 364/402

[58] Field of Search 364/402; 379/113, 221; 340/524

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Karmarkar at Bell Labs: an equation to find a new way through the maze

Folding the Perfect Corner

A young Bell scientist makes a major math breakthrough

Every day 1,200 American Airlines jets crisscross the U.S., Mexico, Canada and the Caribbean, stopping in 110 cities and bearing over 80,000 passengers. More than 4,000 pilots, copilots, flight personnel, maintenance workers and baggage carriers are shuffled among the flights; a total of 3.6 million gal. of high-octane fuel is burned. Nuts, bolts, altimeters, landing gears and the like must be checked at each destination. And while performing these scheduling gymnastics, the company must keep a close eye on costs, projected revenue and profits.

Like American Airlines, thousands of companies must routinely untangle the myriad variables that complicate the efficient distribution of their resources. Solving such monstrous problems requires the use of an abstruse branch of mathematics known as linear programming. It is the kind of math that has frustrated theoreticians for years, and even the fastest and most powerful computers have had great difficulty juggling the bits and pieces of data. Now Narendra Karmarkar, a 28-year-old Indian-born mathematician at Bell Laboratories in Murray Hill, N.J., after only a years' work has cracked the puzzle of linear programming by devising a new algorithm, a step-by-step mathematical formula. He has translated the procedure into a program that should allow computers to track a greater combination of tasks than ever before and in a fraction of the time.

Unlike most advances in theoretical mathematics, Karmarkar's work will have an immediate and major impact on the real world. "Breakthrough is one of the most abused words in science," says Ronald Graham, director of mathematical sciences at Bell Labs. "But this is one situation where it is truly appropriate."

Before the Karmarkar method, linear equations could be solved only in a cumbersome fashion, ironically known as the simplex method, devised by Mathematician George Dantzig in 1947. Problems are conceived of as giant geodesic domes with thousands of sides. Each corner of a facet on the dome

image source: M. Grötschel

Barrier method

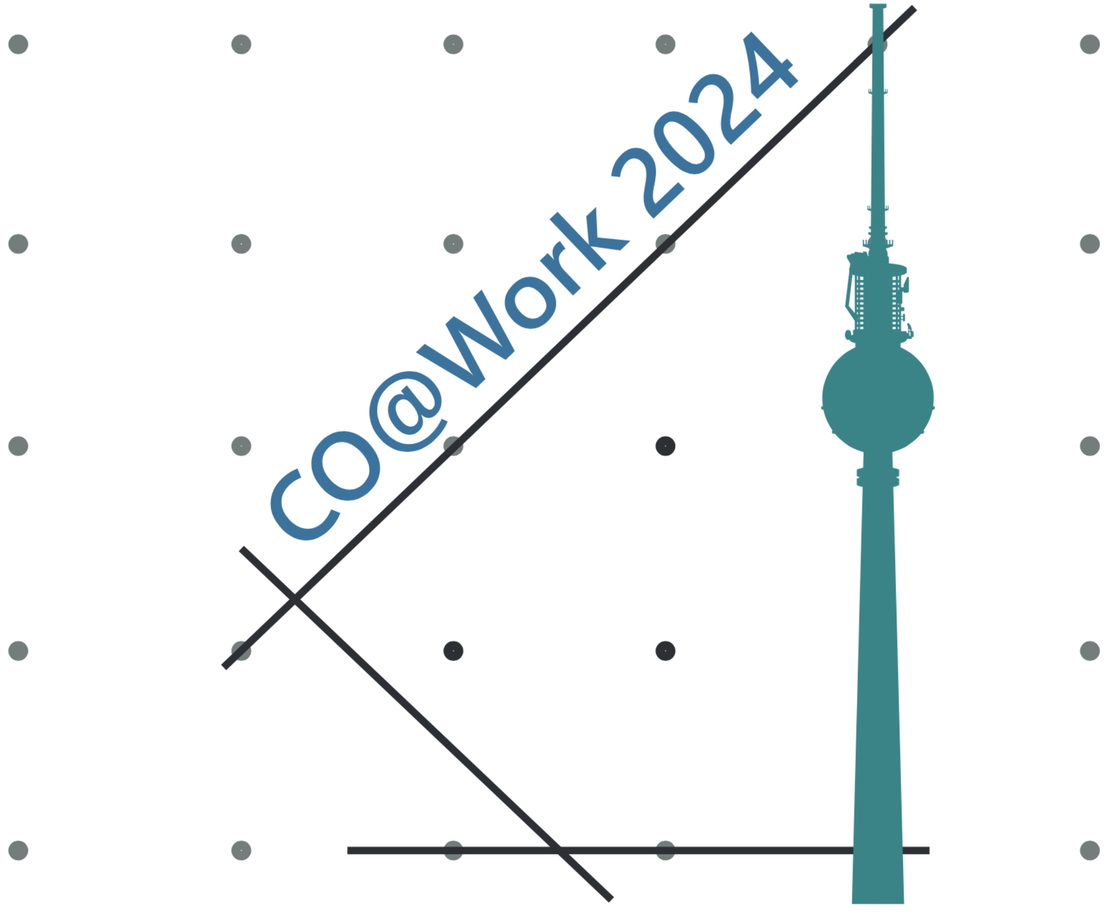
- Instead of $\min\{c^T x \mid Ax = b, x \geq 0\}$ solve $\min\{c^T x - \mu \sum \ln x_i \mid Ax = b, x \geq 0\}$
  - Strictly convex problem, has a single unique solution (when origi… …ble)
- For small x, -ln x becomes large, hence solution is an int…
  - Converges to optimum of original LP when…
- Integrate primal and dual LP into th… …near(!) equation system:
  - $Ax = b$ primal
  - $yA + s = c$ d…
  - xs … …entary slackness
  - x, s …
- This can be solved by a Newton method

More on this later by Joachim Dahl ...

Crossover

- Barrier solutions are not basic
  - Typically only few columns at their bound
  - Can neither be used for simplex warmstart, nor for Gomory cuts
  - Numerically slightly off
- Crossover: similar to simplex
  - Creates a basic vertex solution from a nonbasic interior point solution
  - Guesses initial basis (crash) and nonbasis (note: some columns might be at their bounds), maintains set of superbasic columns (not at their bound and not basic) and tries to push those to zero or to push a basic column to zero and a superbasic into the basis
  - Primal and dual crossover
  - Polynomial-time algorithm

Thank you! Questions?



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