Primal Heuristics

When MIP solvers roll dice

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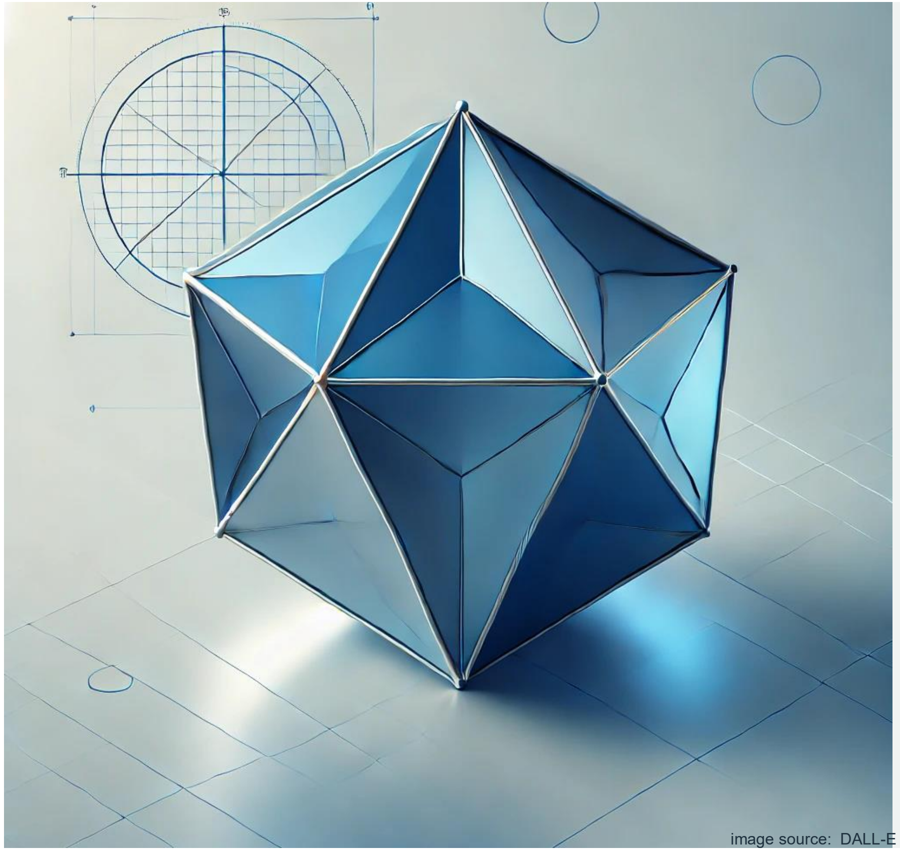
Agenda



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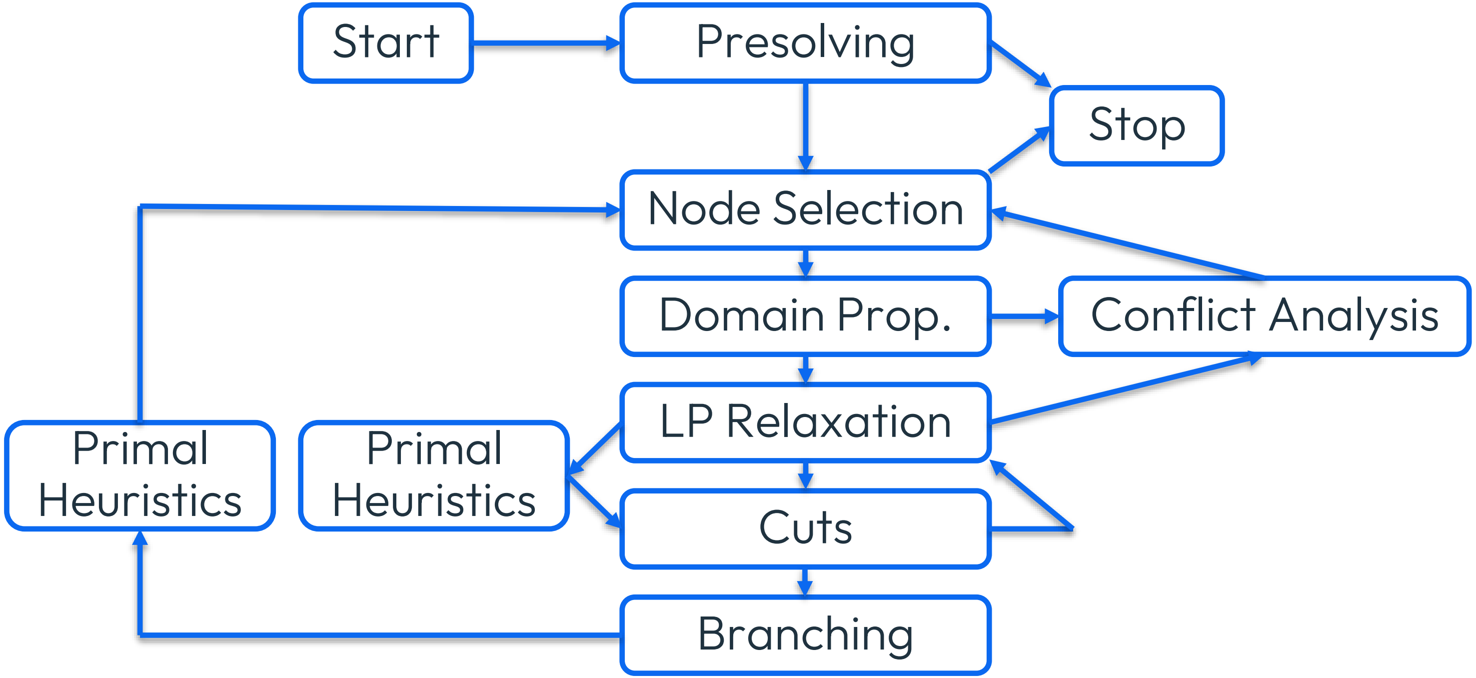
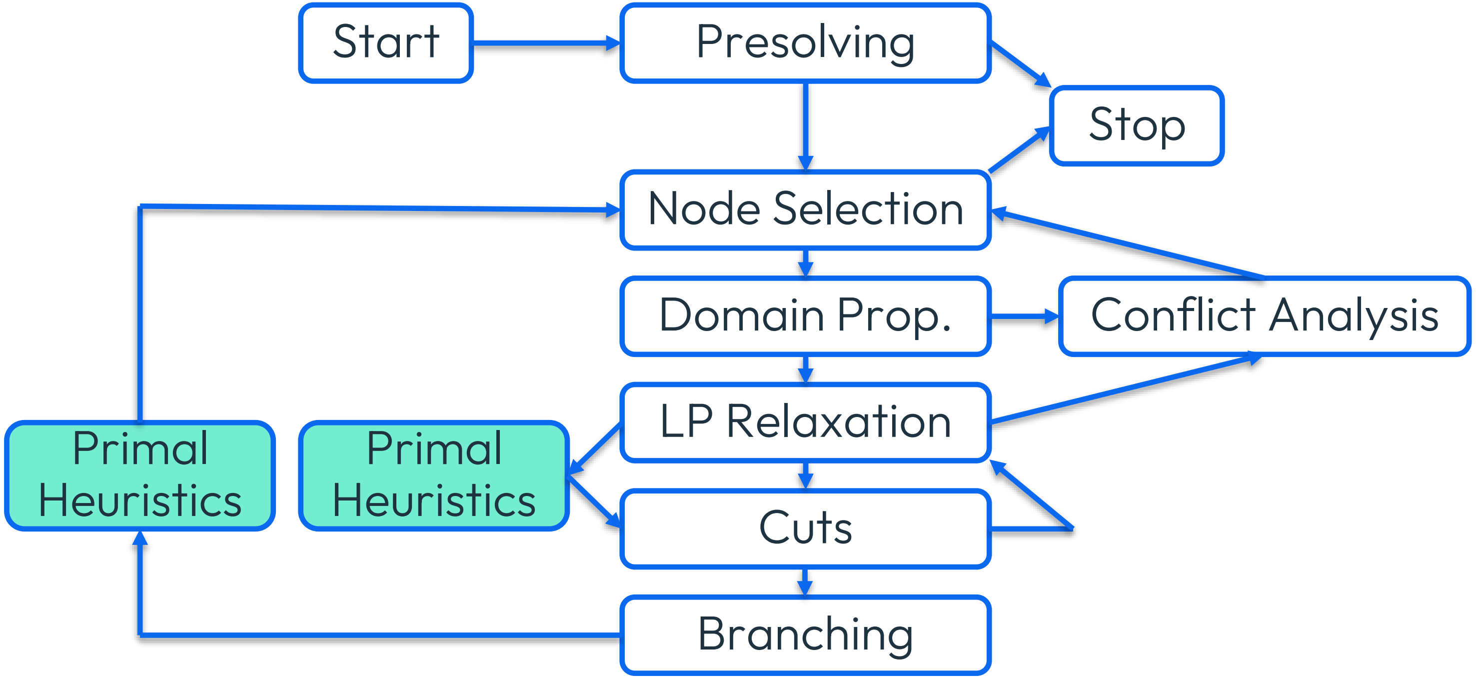
1. Heuristics: Idea, Information
2. Rounding, Diving
3. Feasibility Pump
4. LNS
5. Primal-Dual Integral

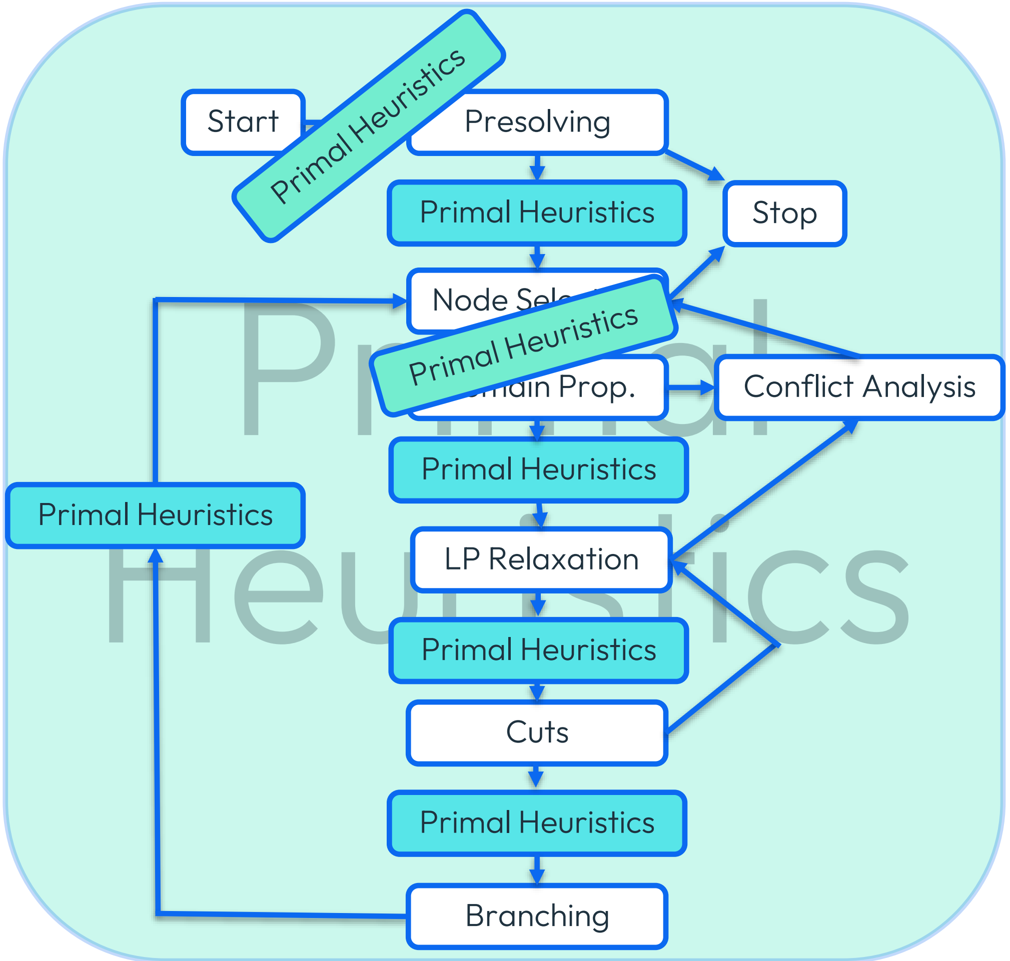
MIP Solver Flowchart



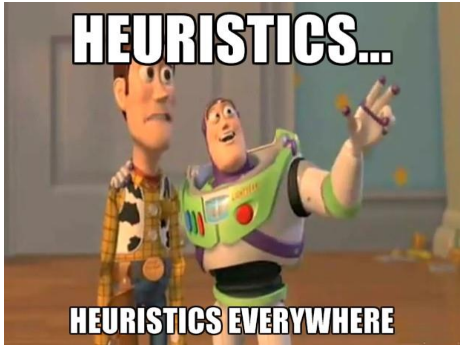
MIP Solver Flowchart



Heuristics Everywhere...



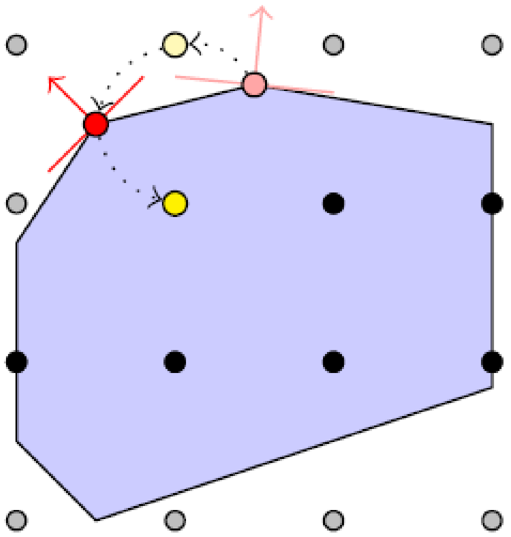
- Heuristics may typically be applied:
  - Before presolving
  - After presolving, before LP
  - After LP, before cut loop
  - During cut loop
  - After node
  - When backtracking
  - In the background!
- Heuristics may trigger other heuristics



What are primal heuristics?

- Primal heuristics . . .
  - are incomplete methods which
  - often find good solutions
  - within a reasonable time
  - without any warranty!

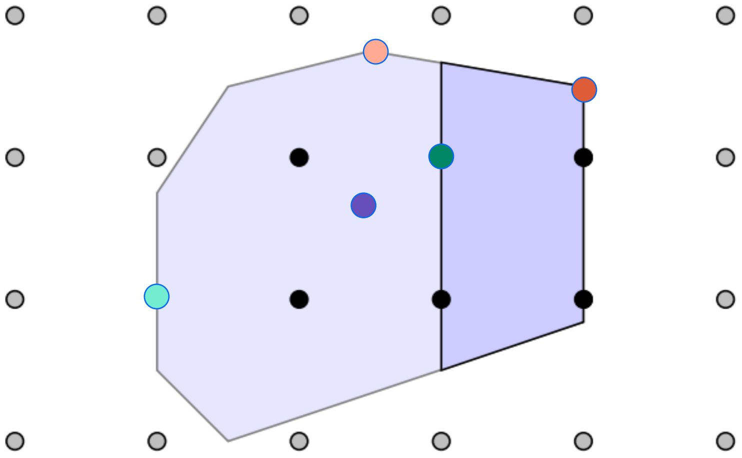
- Why use primal heuristics inside a (complete) MIP solver?
  - to prove the feasibility of the model
  - often nearly optimal solutions suffice in practice
  - feasible solutions guide the remaining search process



Heuristics: General information

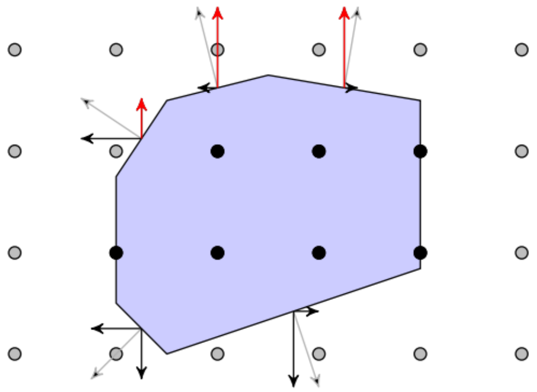
Reference points

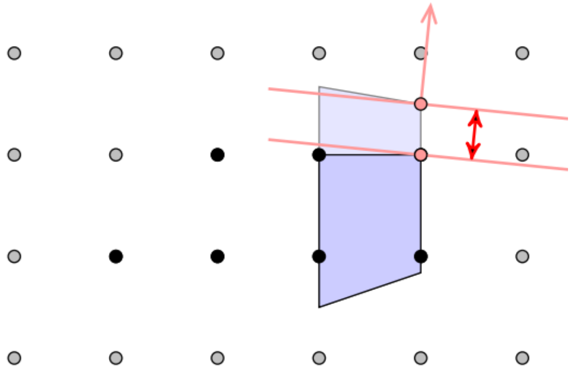
- Current LP optimum
- Current incumbent
- Other feasible solutions
- LP optimum at root node
- Analytic center



Variable statistics

- Locking numbers
  - Number of potentially violated rows
- Pseudo-costs
  - Average objective change
- Conflict Statistics
  - How often has a variable been involved in proving local infeasibility?



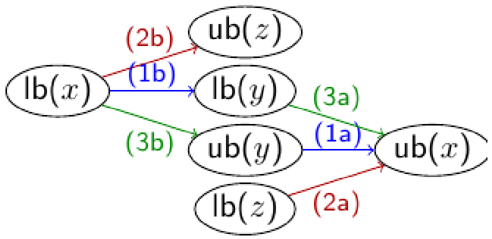


Global structures

- General structures that are automatically detected during presolving
  - Clique table
  - Flow structures
  - Implication graph
  - Variable bound graph
  - Symmetry information

$$
\begin{aligned}
x - 2y &\le 3 \quad (1)\\
x + 2z &\le 2 \quad (2)\\
x + 3y &\le 6 \quad (3)
\end{aligned}
$$

$$
\begin{array}{llll}
x & \le & 2y + 3 & (1a)\\
y & \ge & \frac{1}{2}x - \frac{3}{2} & (1b)\\
x & \le & 2 - 2z & (2a)\\
z & \le & 1 - 0.5x & (2b)\\
x & \le & 6 - 3y & (3a)\\
y & \le & 2 - \frac{1}{3}x & (3b)
\end{array}
$$



Different types of heuristics

- Rounding: Take a (fractional) LP solution, change fractional to integral values without reoptimization
- Diving: simulate a depth-first search (DFS) in the branch-and-bound tree using some special branching rule (i.e. fix variables and reoptimize)
- FP-type: manipulate objective function in order to reduce fractionality, reoptimize
    - FP=Feasibility Pump, sometimes referred to as objective diving
- Large Neighborhood Search: fix variables, add constraints, solve resulting subproblem
- Pivoting: manipulate simplex algorithm
- …

Start vs. Improvement heuristics

- Start heuristics
  - Applied early in the search process
  - Often at root node
  - Typically start from LP optimum
  - Ignore incumbent (if one exists)
- Improvement heuristics
  - Require feasible solution
  - Normally at most once for each incumbent
  - Quick improvement directly after incumbent
  - Heavy improvement only after long time without new incumbent

Rounding and Diving

Rounding heuristics

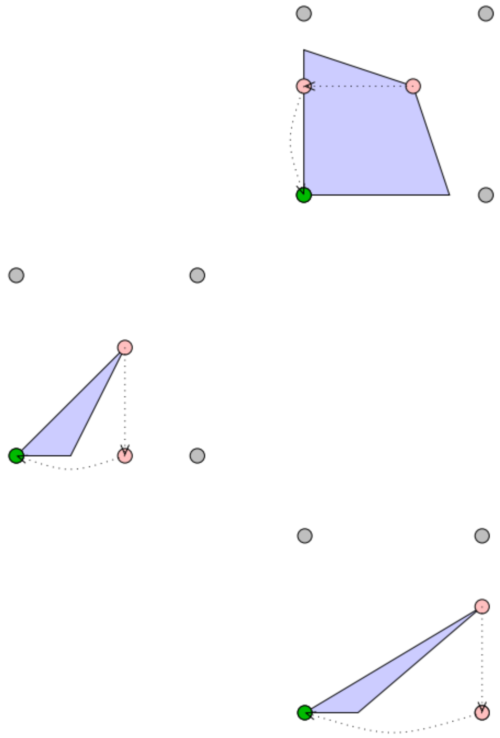
Variable locks as main guide, fast fail strategy

- Simple Rounding: always stays feasible
- Rounding may violate constraints
- Shifting: may unfix integers

Other approaches:

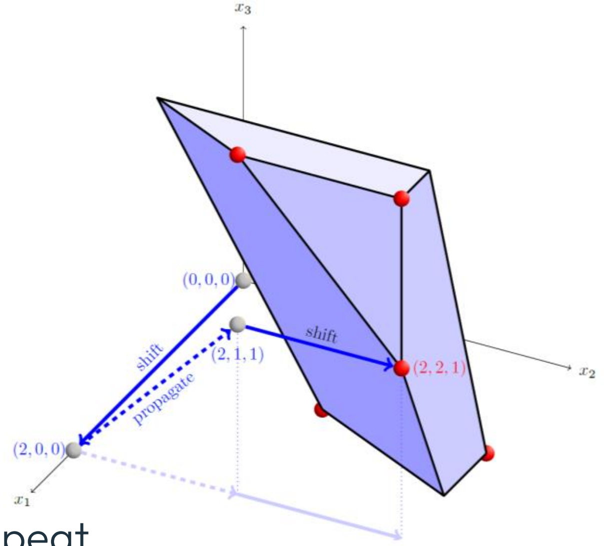
- Random rounding
- Analytic center rounding

For continuous variables: solve final LP



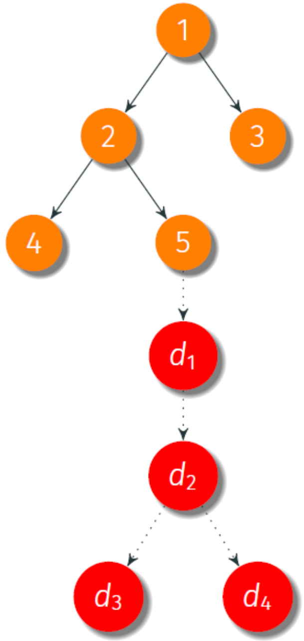
Fix-and-Propagate Heuristics

- Given an integer, but possibly infeasible, reference point.
- Applies several rounds of greedy fixings and bound propagations to improve the solution.
- Columns are typically scored using some combination of objective value and row violations.
- In one round:
  - Select an unfixed column
  - Fix the column to the value that minimizes the score.
  - Propagate bound implications from the fixing.
    - Reverse fixing if infeasible.
  - Continue until all integer variables are fixed.
- Update scoring weights, randomize parts of the solution and repeat.



Diving heuristics

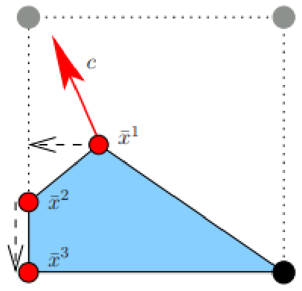
- Simulated tree search
- Pure DFS
  - At most 1-level backtracking
- „One-sided“ Branching rule
- Might fix several variables per branch
- Might skip LP and only propagate at some nodes
- Often applied before MIP solver would backjump in main tree



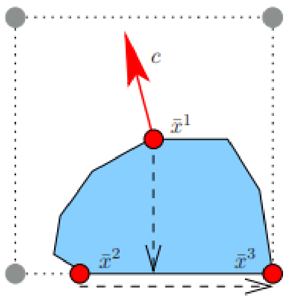
Rule of thumb: Good branching strategies are bad diving strategies

Main difference: Variable fixing strategy

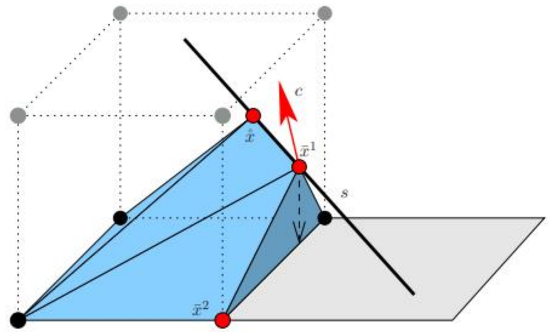
- Fractional diving: Round least fractional
- Guided diving: Round towards reference solution
- Coefficient diving: Round in direction of fewer locks
- Line search diving: Observe development since root LP
- Vector length diving: Fix variables in long constraints



Fractional Diving



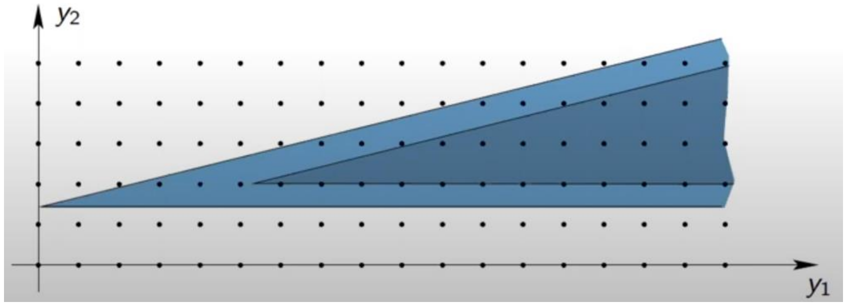
Coefficient Diving



Linesearch Diving

Working with inner parallel sets (Neumann, Stein et al 2018, 2021)

- Inner parallel set: All points in LP relaxation for which a unit box centered at them is completely inside polyhedron
  - Closest rounding(s) lie inside relaxation → integer feasible



- Shrink, optimize, round:
  - Solve $\min c^T x \;\; \text{s.t.} \;\; Ax \leq b - {}^{\|\beta\|}/_{2}$, where $\|\beta\|$ is the vector of row 1-norms
- Enhanced approach: Use smaller shrinking offsets to generate promising inner points
- Can also be used as diving heuristics: Fixing variables grows inner parallel sets

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Quiz time

- Diving heuristics
  a) Simulate a depth-first search
  b) Manipulate the simplex algorithm
  c) Change the objective function
- Fix-and-propagate heuristics typically
  a) Solve a sub-MIP
  b) Solve a series of LPs
  c) Do not solve LPs or auxiliary MIPs
- Primal heuristics
  a) Typically have an approximation ratio
  b) Always improve the primal bound
  c) Might terminate unsuccessfully

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Large Neighborhood Search

Local Search

1. Consider reference solution
2. Unfix small number of variables and explicitly try alternative values
   - Efficient for 1-neighborhood
   - Runtime (naively): $O(n^k)$ for k-neighborhood
     - Already for 2-neighborhood, only a subset of candidates can be considered
     - Exploit structures to reduce runtime
       - Example: Lin Kernighan Heuristic
       - Check https://stemlounge.com/animated-algorithms-for-the-traveling-salesman-problem/
3. Accept new solution if it improves incumbent
4. Stop if locally optimal
   - Might allow some suboptimal moves to break out

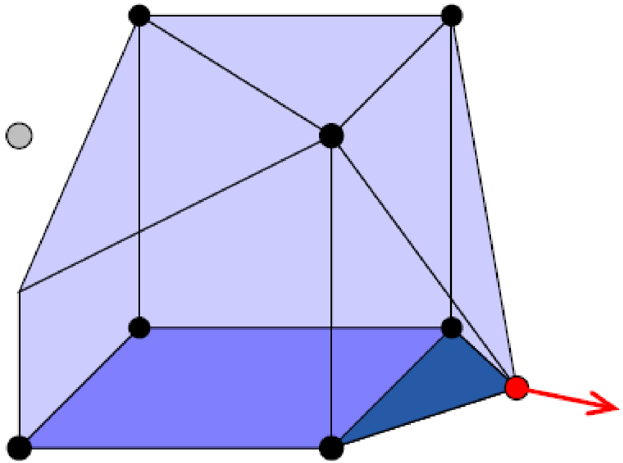
Large Neighborhood Search

1. Consider one or several reference solutions
2. Create an auxiliary problem that is too hard to solve by enumeration („large“)
   - Feasible set is a subset of original
3. According to some neighborhood definition:
   - Fix variables
   - Add constraints
   - Change the objective
4. Perform a partial solve
   - Might use LNS inside LNS

RENS (Berthold 2014)

Idea: Search vicinity of a relaxation solution

1. Reference point:
   $\bar{x} \leftarrow$ LP optimum
2. Fix all integral variables:
   $x_i \leftarrow \bar{x}_i \quad \forall i: \bar{x}_i \in \mathbb{Z}$
3. Reduce domain of fractional variables:
   $x_i \in \{\lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil\}$
4. Solve the resulting sub-MIP



Start heuristic, does not need a feasible solution!

RINS (Danna et al 2005)

Idea: Search common vicinity relaxation solution and incumbent

- Close to LP optimum: high quality
- Close to incumbent: feasible

1. Reference points:

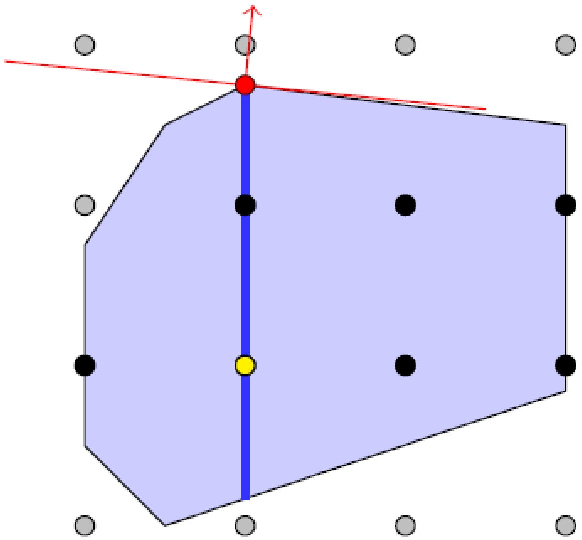
   $\bar{x} \leftarrow$ LP optimum

   $\tilde{x} \leftarrow$ incumbent

2. Fix coinciding variables:

   $x_i \leftarrow \bar{x}_i \quad \forall i: \bar{x}_i = \tilde{x}_i$

3. Solve the resulting sub-MIP



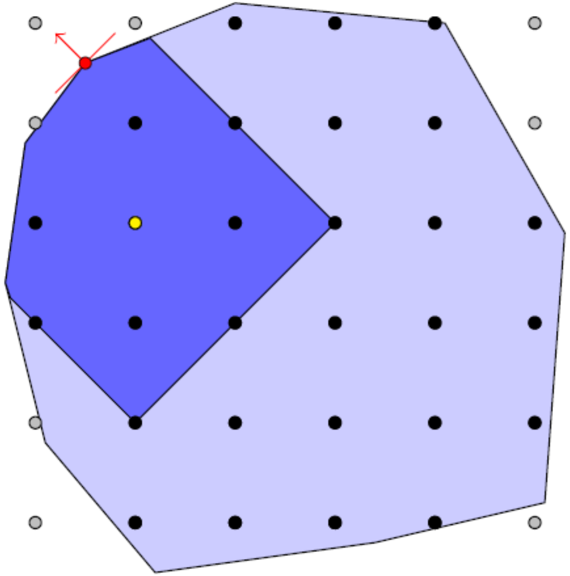
Most common sub-MIP heuristic?

Local Branching (Fischetti&Lodi 2003)

Idea: Search vicinity, induced by 1-norm, of incumbent

- Soft rounding
- Might require auxiliary variables

1. Reference points:
   $\tilde{x} \leftarrow$ incumbent
2. Impose Local Branching Constraint:
   $$\Delta(x, \tilde{x}) = \sum \left| x_j - \tilde{x}_j \right| \leq k$$
3. Solve the resulting sub-MIP



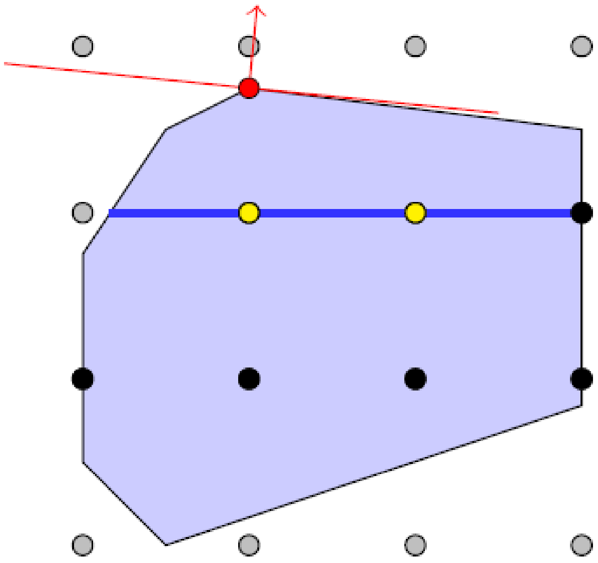
Originally suggested as a branching strategy

Crossover (Rothberg 2007)

Idea: Search vicinity of several feasible solutions

- Detect implicit conditions for feasibility
- Might be self-fulfilling prophecy

1. Reference points:
   $\tilde{X} \leftarrow$ set of solutions
2. Fix all agreeing variables:
   $x_i \leftarrow \tilde{x}_i^1 \;\; \forall i: \tilde{x}_i^j = \tilde{x}_i^k \;\; \forall \tilde{x}^j, \tilde{x}^k \in \tilde{X}$
3. Solve the resulting sub-MIP



Originally part of a genetic algorithm, with a randomized Mutation LNS heuristic

Changing the objective

- Drop it
  - Zero Objective or Hail Mary Heuristic
  - Might allow for many additional fixings
- Inverse it
- Use Local Branching constraint as objective (Fischetti&Monaci 2016)
  - Try to optimize towards a reference point
  - Reference point can be an „almost“ feasible solution
- Use Analytic Center as objective (Berthold et al 2018)
  - Indicates the direction into which a variable is likely to move toward feasibility
  - Particularly interesting for variables that are likely to be 1 in a binary problem

Graph Induced Neighborhood Search

- Fix all variables outside the „constraint neighborhoods“ of one or several central variables
- Consider Variable-constraint graph:

$$G_A := (V = \{v_1, \ldots, v_n\}, W = \{w_1, \ldots, w_m\}, E = \{(v_i, w_j) \in V \times W : a_{ij} \neq 0\})$$

- k-neighborhood of a variable s:

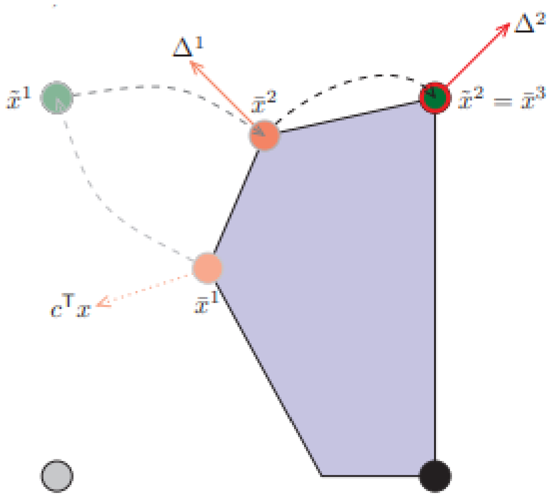
$$N_k(s) := \{t \in V : d(s,t) \leq 2k\}$$

- Fix all variables outside the k-neighborhood
- Choose maximum k to stay above a minimum fixing rate
- Alternatively, can be applied on top of other fixing rules to reach a target fixing rate
  - Fix all variables INSIDE k-neighborhood

Feasibility Pump

Basic Feasibility Pump (Fischetti et al 2005)

1. Solve original LP
2. Round LP optimum
3. If feasible:
   - Stop!
4. If cycle:
   - Perturb
5. Change objective: $\Delta(x, \tilde{x}) = \sum |x_j - \tilde{x}_j|$
6. Solve LP (project)
7. Goto 2



Main variants

- Improved feasibility pump (Bertacco et al 2007)
  - Uses auxiliary variables to model distance function on general integers

    $$\Delta(x,\tilde{x}) := \sum_{j \in \mathcal{I}:\, \tilde{x}_j = l_j} (x_j - l_j) + \sum_{j:\, \tilde{x}_j = u_j} (u_j - x_j) + \sum_{j \in \mathcal{I}:\, l_j < \tilde{x}_j < u_j} d_j \quad \text{s.t.} \quad \mathrm{d}_j \geq x_j - \tilde{x}_j, \quad \mathrm{d}_j \geq \tilde{x}_j - x_j$$

- Objective feasibility pump (Achterberg & Berthold 2007)
  - Convex combination of original objective and distance function
    - $\widetilde{\Delta} = (1-\alpha)\Delta(\mathrm{x}) + \alpha c^T x$, with $\alpha \in [0,1]$, typically $\alpha \in [0.95, 0.99]$
  - Algorithm can recover from cycles
- Feasibility Pump 2.0 (Fischetti & Salvagnin 2009)
  - Applies propagation after each rounding
  - Uses specific propagators for special linear constraints
  - fewer rounding steps, “more feasible”

Measuring the impact of primal heuristics

Performance measures

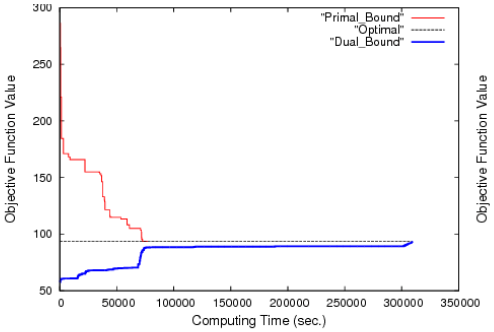
How to measure the added value of a primal heuristic?

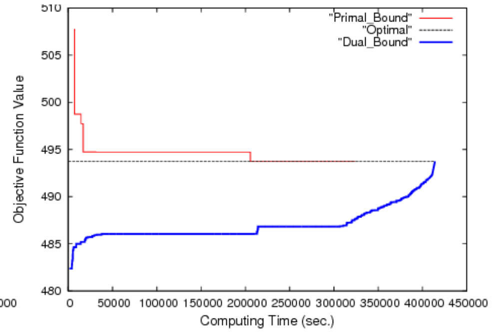
- time to optimality, number of branch-and-bound nodes
  - very much depends on dual bound
- time to first solution
  - disregards solution quality
- time to best solution
  - nearly optimal solution might be found long before
- Some combination of all of those?

Primal-Dual Integral (Berthold 2013)

$$\gamma^p(\tilde{x}) := \begin{cases} 0, & \text{if } c^\mathsf{T} x^\star = c^\mathsf{T}\tilde{x} = 0, \\ 1, & \text{if } c^\mathsf{T} x^\star \cdot c^\mathsf{T}\tilde{x} < 0, \\ \frac{|c^\mathsf{T} x^\star - c^\mathsf{T}\tilde{x}|}{\max\{|c^\mathsf{T} x^\star|, \ |c^\mathsf{T}\tilde{x}|\}}, & \text{otherwise.} \end{cases}$$

$$P(T) := \int_{t=0}^{T} \gamma^p(t)\, dt = \sum_{i=1}^{I} \gamma^p(t_{i-1}) \cdot (t_i - t_{i-1})$$



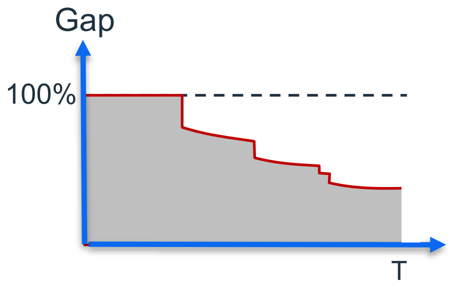


Primal-Dual Integral (PDI):

- favors finding good solutions quickly
- considers each update of incumbent (and the best bound)
- gives you expected solution quality assuming unknown termination time

Heuristic Emphasis

- Textbook: The beauty of MIP is that we can solve problems to proven optimality
- Practice: Often, this takes unbearably long
  - Still, MIP solvers provide us with a proof of quality, the gap
  - Often, solving models to a small gap is good enough
    - And the only viable option within restrictive time limits



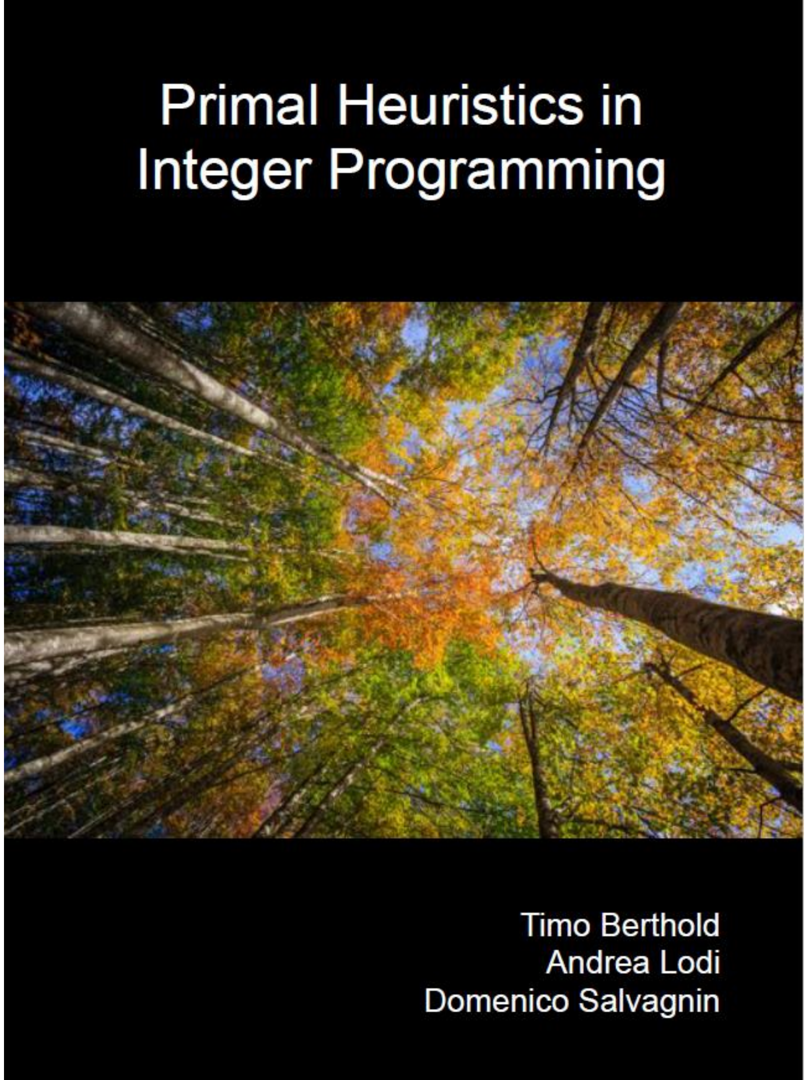
- Heuristic Emphasis mode (HEUREMPHASIS)
  - Focus on reducing the gap in the beginning of a MIP search
    - More precisely: minimizes the Primal-Dual Integral (PDI).
  - Typically at the expense of a longer time to optimality
  - Targets problems out of scope for solving to optimality, but when finding good solutions early matters.

Heuristic Emphasis: Technology

- Many neighborhood search heuristics at the root
- Additional non-default heuristics
- More frequent heuristics in the tree
- Deactivate some techniques specifically tailored for easy problems
- Background heuristics in parallel to root-node search

Coming soon…

- Cambridge University Press
- scheduled for the first half of 2025
- Presents heuristic approaches as part of the MIP-solving process
- Tackles practical concerns by examining trade-offs
- Comprehensive overview of different classes of heuristics



Quiz time

- LNS stands for
  a) Large neighborhood search
  b) Local neighborhood search
  c) Local native search
- What is the idea of Crossover?
  a) Fix variables that coincide in incumbent and LP optimum
  b) Add one constraint for each fractional variable
  c) Fix variables that coincide in all integer solutions
- The primal dual integral
  a) Considers each update of the incumbent
  b) Depends on the number of processed nodes
  c) Measures the time to find a first solution

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