Convex Optimization via Cones and MOSEK 9

CO@Work
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Motivation: $x_1^2 + 2x_1x_2 + x_2^2 = (x_1 + x_2)^2$

... made complicated.

Let $Q = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and suppose we have the constraint

$$t \geq x^t Q x = x_1^2 + 2x_1x_2 + x_2^2. \tag{1}$$

Now Q is p.s.d., and $Q = F^t F$ with $F = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Thus, (1) is equivalent to

$$t \geq \langle Fx, Fx \rangle = \|Fx\|_2^2 = \|x_1 + x_2\|_2^2 \qquad \ldots = (x_1 + x_2)^2.$$

$t \geq \|x_1 + x_2\|_2^2$ can be cast as a conic constraint intersected with linear (in-)equalities!

In Convex Optimization, representation can affect both theory and practice (i.e., computational aspects).

(Mixed-Integer) Conic Optimization

We consider problems of the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{K} \cap \left(\mathbb{Z}^p \times \mathbb{R}^{n-p}\right), \end{array}$$

where $\mathcal{K}$ is a convex cone.

- Typically, $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_K$ is a product of lower-dimensional cones.
- How can these so-called conic building blocks look like?

Symmetric cones

- the nonnegative orthant

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_j \geq 0,\ j = 1, \ldots, n\},$$

- the quadratic cone

$$\mathcal{Q}^n = \{x \in \mathbb{R}^n \mid x_1 \geq \left(x_2^2 + \cdots + x_n^2\right)^{1/2} = \|x_{2:n}\|_2\},$$

- the rotated quadratic cone

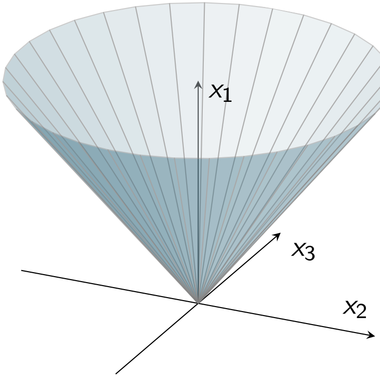
$$\mathcal{Q}_r^n = \{x \in \mathbb{R}^n \mid 2x_1x_2 \geq x_3^2 + \cdots + x_n^2 = \|x_{3:n}\|_2^2,\ x_1, x_2 \geq 0\}.$$

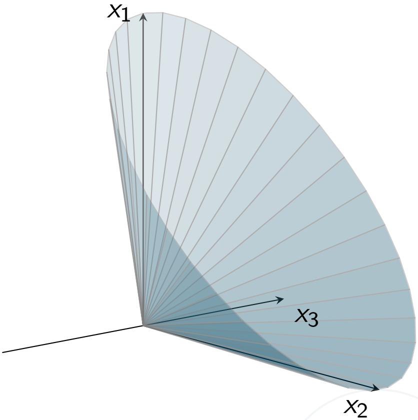
- the semidefinite matrix cone

$$\mathcal{S}^n = \{x \in \mathbb{R}^{n(n+1)/2} \mid z^T \mathbf{mat}(x) z \geq 0,\ \forall z\},$$

with $\mathbf{mat}(x) := \left[\begin{array}{cccc} x_1 & x_2/\sqrt{2} & \ldots & x_n/\sqrt{2} \\ x_2/\sqrt{2} & x_{n+1} & \ldots & x_{2n-1}/\sqrt{2} \\ \vdots & \vdots & & \vdots \\ x_n/\sqrt{2} & x_{2n-1}/\sqrt{2} & \ldots & x_{n(n+1)/2} \end{array}\right].$

Quadratic cones in dimension 3





Quadratic-cone use cases

- Simple quadratics:

$$t \geq (x+y)^2 \iff (0.5, t, x+y) \in \mathcal{Q}_r^3.$$

- Every convex (MI)QCP can be reformulated as a (MI)SOCP:

$$t \geq x^T Q x \text{ with } Q \text{ p.s.d.} \iff (0.5, t, Fx) \in \mathcal{Q}_r^{n+2}$$

with with $Q = F^T F$.

- In some applications, like least-squares regression, a SOC-formulation is more direct than a QP-formulation.

Non-symmetric cones

Symmetric cones are self-dual and homogeneous by definition, and the two cones below lack at least one of these properties.

- the three-dimensional exponential cone

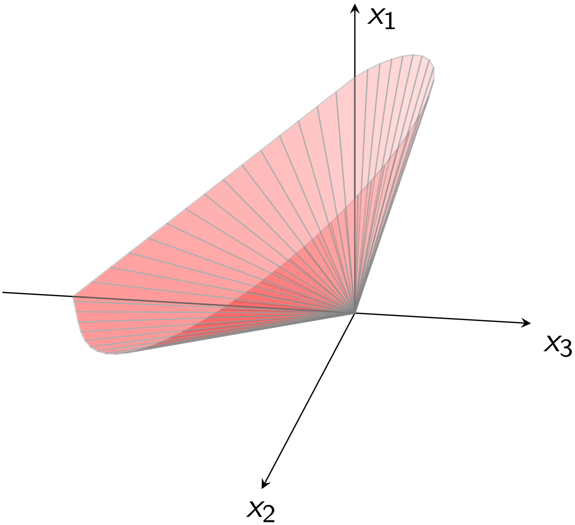
$$\mathcal{K}_{exp} = \mathrm{cl}\{x \in \mathbb{R}^3 \mid x_1 \geq x_2 \exp(x_3/x_2),\ x_2 > 0\}.$$

- the three-dimensional power cone

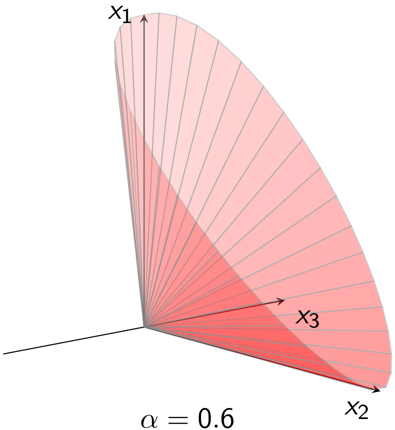
$$\mathcal{P}^{\alpha} = \{x \in \mathbb{R}^3 \mid x_1^{\alpha} x_2^{(1-\alpha)} \geq |x_3|,\ x_1, x_2 \geq 0\},$$

for $0 < \alpha < 1$.

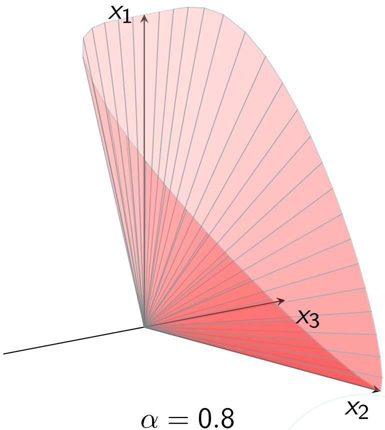
The exponential cone



The power cone



$\alpha = 0.6$



$\alpha = 0.8$

Exponential-cone use cases

Many constraints involving exponentials or logarithms can be formulated using the exponential cone.

- Expontial:

$$e^x \leq t \quad \iff \quad (t, 1, x) \in \mathcal{K}_{exp}.$$

- Entropy:

$$-x \log x \geq t \quad \iff \quad (1, x, t) \in \mathcal{K}_{exp}.$$

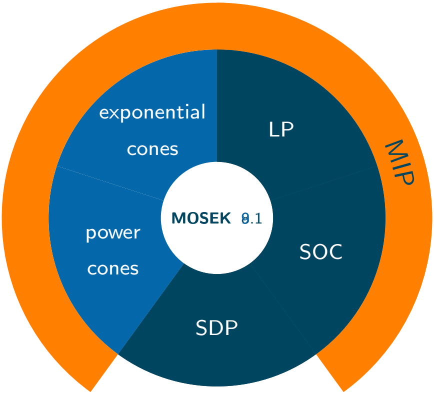
- Softplus function:

$$\log(1+e^x) \leq t \quad \iff \quad (u, 1, x-t), (v, 1, -t) \in \mathcal{K}_{exp},\ u+v \leq 1.$$

- ...

What can you do with **MOSEK** ?

The software package **MOSEK** supports the following conic building blocks:



MOSEK 9.0 released January 2019, 9.2 released February 2020

How general is the **MOSEK** framework?

The 5 cones - linear, quadratic, exponential, power and semidefinite - together are highly versatile for modeling.

Continuous Optimization Folklore

"Almost all convex constraints which arise in practice are representable using these cones."

We call modeling with the aforementioned 5 cones **Extremely Disciplined Convex Programming**.

(*Check the link to CVX in the video description!*)

Other conic solvers

- The leading MIP solvers support SOC modeling these days.
- SCS and ECOS can handle power and/or exponential cones.
- Several software packages for SDP have been around for many years.
- Pajarito is designed for Mixed-Integer Conic Optimization and supports all of the above but the power cone.
- There are recent efforts to building software supporting ever more cones: Coey, Kapelevich, Vielma: *Towards Practical Generic Conic Optimization* (2020).

Check the links in the video description!

The beauty of Conic Optimization

In continuous optimization, conic (re-)formulations have been advocated for quite some time:

- Separation of data and structure:
  - Data: c, A and b - Structure: $\mathcal{K}$.
- Structural convexity.
- No issues with smoothness and differentiability.
- Duality (almost...).quit()

Further reading:

- Ben-Tal, Nemirovski: *Lectures on modern convex optimization* (2001)
- Boyd, Vandenberghe: *Convex Optimization* (2004)
- Nemirovski: *Advances in Convex Optimization: Conic Programming* (2007)

Check the links in the video description!

Cones in Mixed-Integer Optimization

All convex instances (333) from `minlplib.org` can be converted to conic form:

- Lubin et al.: *Extended Formulations in Mixed-integer Convex Programming* (2016)

Exploiting conic structures in the mixed-integer case is an active research area:

- Coey et al.: *Outer approximation with conic certificates for mixed-integer convex problems* (2020)
- Lodi et al.: *Disjunctive cuts for Mixed-Integer Conic Optimization* (2019)
- MISOCP:
  - Andersen, Jensen: *Intersection cuts for mixed integer conic quadratic sets* (2013)
  - Vielma et al.: *Extended Formulations in Mixed Integer Conic Quadratic Programming* (2017)
  - Çay et al.: *The first heuristic specifically for mixed-integer second-order cone optimization* (2018)

Check the links in the video description for more references!

Further information on **MOSEK**

- Documentation at `mosek.com/documentation/`
  - Modeling cook book / cheat sheet.
  - White papers.
  - Manuals for interfaces.
  - Notebook collection.

- Tutorials and more at `github.com/MOSEK/`