# Solving Linear Programs: The Dual Simplex Algorithm

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# Outline

- LP basics
- Primal and dual simplex algorithms
- Implementing the dual simplex algorithm

# Some Basic Theory

# Linear Program – Definition

A **linear program (LP)** in **standard form** is an optimization problem of the form

$$\begin{array}{lr} \textit{Minimize} & c^T x \\ \textit{Subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \text{(P)}$$

Where $c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathbf{R}^{m \times n}$, and $x$ is a vector of $n$ **variables.** $c^T x$ is known as the **objective function**, $Ax=b$ as the **constraints**, and $x \geq 0$ as the **nonnegativity** conditions. $b$ is called the **right-hand side**.

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# Dual Linear Program – Definition

The **dual (or adjoint) linear program** corresponding to (P) is the optimization problem

$$\begin{aligned} \text{Maximize} \quad & b^T\pi \\ \text{Subject to} \quad & A^T\pi \le c \\ & \pi \ \text{free} \end{aligned} \qquad \text{(D)}$$

In this context, (P) is referred to as the **primal linear program.**

**Primal**

$$\left( \begin{aligned} \text{Minimize} \quad & c^T x \\ \text{Subject to} \quad & Ax = b \\ & x \ge 0 \end{aligned} \right)$$

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# Weak Duality Theorem

(von Neumann 1947)

Let $x$ be feasible for (P) and $\pi$ feasible for (D). Then

$$\boxed{\text{Maximize}} \quad b^T\pi \le c^T x \quad \boxed{\text{Minimize}}$$

If $b^T\pi = c^T x$, then $x$ is optimal for (P) and $\pi$ is optimal for (D); moreover, if either (P) or (D) is **unbounded**, then the other problem is **infeasible.**

## Proof:

$$\pi^T b \underbrace{=}_{\boxed{Ax = b}} \pi^T A x \underbrace{\le}_{\boxed{\pi^T A \le c^T \ \& \ x \ge 0}} c^T x \quad \blacksquare$$

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# Solving Linear Programs

- Three types of algorithms are available
  - Primal simplex algorithms (Dantzig 1947)
  - Dual simplex algorithms (Lemke 1954)
    - Developed in context of game theory
  - Primal-dual log barrier algorithms
    - Interior-point algorithms (Karmarkar 1989)
    - Reference: Primal-Dual Interior Point Methods, S. Wright, 1997, SIAM

Primary focus: Dual simplex algorithms

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# Basic Solutions – Definition

Let $B$ be an ordered set of $m$ distinct indices $(B_1, \ldots, B_m)$ taken from $\{1,\ldots,n\}$. $B$ is called a **basis** for (P) if $A_B$ is nonsingular. The variables $x_B$ are known as the **basic variables** and the variables $x_N$ as the **non-basic** variables, where $N = \{1,\ldots,n\} \setminus B$. The corresponding **basic solution** $X \in \mathbb{R}^n$ is given by $X_N = 0$ and $X_B = A_B^{-1}\, b$. B is called (**primal**) **feasible** if $X_B \geq 0$.

Note: $AX = b \Rightarrow A_B X_B + A_N X_N = b \Rightarrow A_B X_B = b \Rightarrow X_B = A_B^{-1} b$

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# Primal Simplex Algorithm
## (Dantzig, 1947)

Input: A feasible basis $B$ and vectors

$$X_B = A_B^{-1} b \quad \text{and} \quad D_N = c_N - A_N^T A_B^{-T} c_B.$$

- Step 1: (Pricing) If $D_N \geq 0$, stop, $B$ is optimal; else let
  $$j = \mathrm{argmin}\{D_k : k \in N\}.$$
- Step 2: (FTRAN) Solve $A_B y = A_j$.
- Step 3: (Ratio test) If $y \leq 0$, stop, (P) is unbounded; else, let
  $$i = \mathrm{argmin}\{X_{Bk}/y_k : y_k > 0\}.$$
- Step 4: (BTRAN) Solve $A_B^T z = e_i$.
- Step 5: (Update) Compute $\alpha_N = -A_N^T z$. Let $B_i = j$. Update $X_B$ (using $y$) and $D_N$ (using $\alpha_N$)

**Note:** $x_j$ is called the **entering** variable and $x_{Bi}$ the **leaving** variable. The $D_N$ values are known as **reduced costs** – like partial derivatives of the objective function relative to the nonbasic variables.

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# Primal Simplex Example

# The Primal Simplex Algorithm

**Consider the following simple LP:**

$$
\begin{array}{llcl}
\text{Maximize} & 3x_1 + 2x_2 + 2x_3 & & \\
\text{Subject to} & x_1 + \phantom{2x_2 +} x_3 & \le & 8 \\
 & x_1 + x_2 & \le & 7 \\
 & x_1 + 2x_2 & \le & 12 \\
 & x_1, x_2, x_3 \ge 0 & &
\end{array}
$$

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# The Primal Simplex Algorithm

$$\text{Maximize} \quad z = 3x_1 + 2x_2 + 2x_3$$

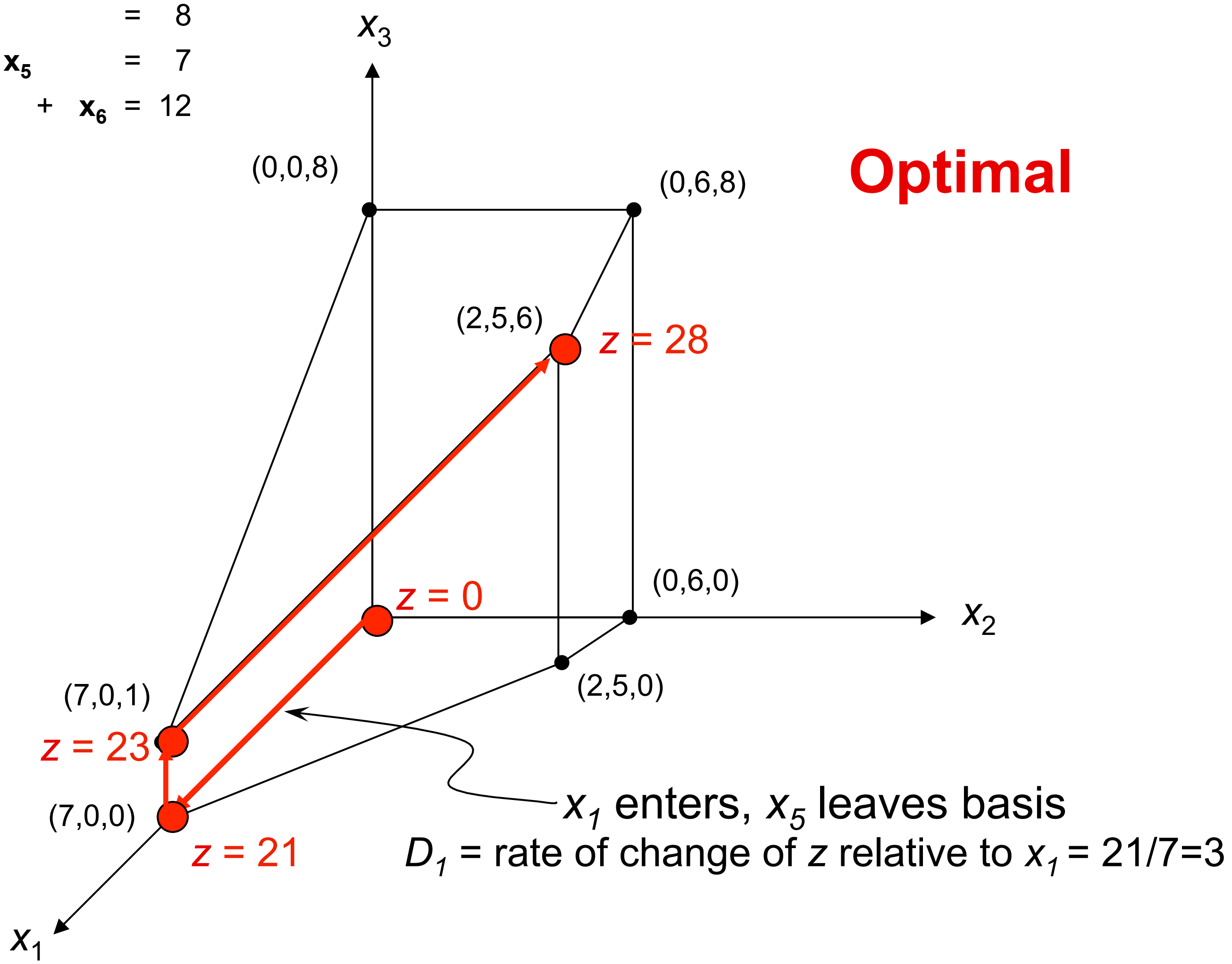
**Add slacks: Initial basis B = (4,5,6)**

Maximize $3x_1 + 2x_2 + 2x_3 + 0\mathbf{x_4} + 0\mathbf{x_5} + 0\mathbf{x_6}$

Subject to

$$\begin{array}{llllllll} x_1 + & & x_3 + & \mathbf{x_4} & & & = & 8 \\ x_1 + & x_2 & & & + \ \mathbf{x_5} & & = & 7 \\ x_1 + & 2x_2 & & & & + \ \mathbf{x_6} & = & 12 \end{array}$$

$$x_1, x_2, x_3, \mathbf{x_4}, \mathbf{x_5}, \mathbf{x_6} \geq 0$$



# Dual Simple Algorithm – Setup

Simplex algorithms apply to problems with constraints in equality form. We convert (D) to this form by adding the dual **slacks** *d*:

$$\begin{aligned} \text{Maximize} \quad & b^T\pi \\ \text{Subject to} \quad & A^T\pi + d = c \\ & \pi \ \text{free},\ d \geq 0 \end{aligned} \qquad \boxed{\Leftrightarrow A^T\pi \leq c}$$

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# Dual Simple Algorithm – Setup

$$\begin{aligned} \textit{Maximize} \quad & b^T\pi \\ \textit{Subject to} \quad & A^T\pi + d = c \\ & \pi \text{ free}, d \geq 0 \end{aligned} \longleftarrow \begin{pmatrix} A_B^T & I_B & 0 \\ A_N^T & 0 & I_N \end{pmatrix} \begin{pmatrix} \pi \\ d_B \\ d_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

Given a basis *B*, the corresponding **dual basic variables** are $\pi$ and $d_N$. $d_B$ are the **nonbasic variables**. The corresponding **dual basic solution** $\Pi, D$ is determined as follows:

$$D_B = 0 \;\Rightarrow\; \Pi = A_B^{-T} c_B \;\Rightarrow\; D_N = c_N - A_N^T \Pi$$

*B* is **dual feasible** if $D_N \geq 0$.

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# Dual Simple Algorithm – Setup

$$\begin{aligned} \textit{Maximize} \quad & b^T\pi \\ \textit{Subject to} \quad & A^T\pi + d = c \\ & \pi \ \textit{free}, \ d \geq 0 \end{aligned} \quad \longleftarrow \quad \begin{pmatrix} A_B^T & I_B & 0 \\ A_N^T & 0 & I_N \end{pmatrix} \begin{pmatrix} \pi \\ d_B \\ d_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

**Observation:** We may assume that every dual basis has the above form.

**Proof:** Assuming that the primal has a basis is equivalent to assuming that *rank(A)=m* (# of rows), and this implies that all $\pi$ variables can be assumed to be basic.

This observation establishes a 1-1 correspondence between primal and dual bases. ∎

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# An Important Fact

If $X$ and $\Pi, D$ are corresponding primal and dual basic solutions determined by a basis $B$, then

$$\Pi^T b = c^T X.$$

Hence, by weak duality, if $B$ is both primal and dual feasible, then $X$ is optimal for (P) and $\Pi$ is optimal for (D).

**Proof:**

$$\begin{aligned} c^T X &= c_B^T X_B && (\text{since } X_N = 0) \\ &= \Pi^T A_B X_B && (\text{since } \Pi = A_B^{-T} c_B) \\ &= \Pi^T b && (\text{since } A_B X_B = b) \quad \blacksquare \end{aligned}$$

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# Dual Simplex Algorithm
## (Lemke, 1954)

Input: A dual feasible basis *B* and vectors

$$X_B = A_B^{-1} b \quad \text{and} \quad D_N = c_N - A_N^T B^{-T} c_B.$$

- Step 1: (Pricing) If $X_B \geq 0$, stop, *B* is optimal; else let
$$i = argmin\{X_{Bk} : k \in \{1,\ldots,m\}\}.$$
- Step 2: (BTRAN) Solve $A_B^T z = e_i$. Compute $\alpha_N = -A_N^T z$.
- Step 3: (Ratio test) If $\alpha_N \leq 0$, stop, (D) is unbounded; else, let
$$j = argmin\{D_k/\alpha_k : \alpha_k > 0\}.$$
- Step 4: (FTRAN) Solve $A_B y = A_j$.
- Step 5: (Update) Set $B_i = j$. Update $X_B$ (using $y$) and $D_N$ (using $\alpha_N$)

**Note:** $d_{Bi}$ is the **entering** variable and $d_j$ is the **leaving** variable. (Expressed in terms of the primal: $x_{Bi}$ is the leaving variable and $x_j$ is the entering variable)

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# Simplex Algorithms

**Input:** A primal feasible basis $B$ and vectors

$$X_B=A_B^{-1}b \ \ \& \ \ D_N=c_N - A_N^T A_B^{-T} c_B.$$

- Step 1: (Pricing) If $D_N \geq 0$, stop, $B$ is optimal; else, let
  $j = argmin\{D_k : k \in N\}.$
- Step 2: (FTRAN) Solve $A_B y = A_j$.
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# Summary:
## What we have done and what we have to do

- Done
  - Defined primal and dual linear programs
  - Proved the weak duality theorem
  - Introduced the concept of a basis
  - Stated primal and dual simplex algorithms
- To do (for dual simplex algorithm)
  - Show correctness
  - Describe key implementation ideas

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# Correctness:
# Dual Simplex Algorithm

- Termination criteria
  - Optimality (DONE – by "An Important Fact")
  - Unboundedness
- Other issues
  - Finding starting dual feasible basis, or showing that no feasible solution exists
  - Input conditions are preserved (i.e., that B is still a feasible basis)
  - Finiteness

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# Dual Unboundedness
## ($\Rightarrow$ primal infeasible)

- We carry out a key calculation
- As noted earlier, in an iteration of the dual

$d_{Bi}$ enters basis
$d_j$ leaves basis

in

$$\begin{aligned} \text{Maximize} \quad & b^T\pi \\ \text{Subject to} \quad & A^T\pi + d = c \\ & \pi \text{ free}, \ d \geq 0 \end{aligned}$$

- The idea: Currently $d_{Bi} = 0$, and $X_{Bi} < 0$ has motivated us to increase $d_{Bi}$ to $\theta > 0$, leaving the other components of $d_B$ at $0$ (the object being to increase the objective). Letting $\underline{d},\underline{\pi}$ be the corresponding dual solution as a function of $\theta$, we obtain

$$\underline{d}_B = \theta e_i \qquad \underline{\pi} = \Pi - \theta z \qquad \underline{d}_N = D_N - \theta \alpha_N$$

where $\alpha_N$ and $z$ are as computed in the algorithm.

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# (Dual Unboundedness – cont.)

- Letting $\underline{d}, \underline{\pi}$ be the corresponding dual solution as a function of $\theta$. Using $\alpha_N$ and $z$ from dual algorithm,

$$\underline{d}_B = \theta\, e_i \qquad \underline{d}_N = D_N - \theta\, \alpha_N \qquad \underline{\pi} = \pi - \theta\, z.$$

- Using $\theta > 0$ and $X_{Bi} < 0$ yields

$$\begin{aligned} new\_objective = \underline{\pi}^T b &= (\Pi - \theta\, z)^T b \\ &= \Pi^T b - \theta\, e_i^T A_B^{-1} b = \Pi^T b - \theta\, e_i^T X_B \\ &= old\_objective - \theta\, X_{Bi} > old\_objective \end{aligned}$$

- **Conclusion 1:** If $\alpha_N \leq 0$, then $\underline{d}_N \geq 0\ \forall\ \theta > 0 \Rightarrow$ (D) is unbounded.

- **Conclusion 2:** If $\alpha_N$ not $\leq 0$, then

$$\begin{aligned} \underline{d}_N \geq 0 &\Rightarrow \theta \leq D_j / \alpha_j \ \ \forall \ \ \alpha_j > 0 \\ &\Rightarrow \theta_{max} = min\{D_j / \alpha_j : \alpha_j > 0\} \end{aligned}$$

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# (Dual Unboundedness – cont.)

- **Feasibility preserved: follows from the ratio test.**
- **Nonsingularity preserved:** follows from (also yields update)
  - new $A_B = A_B (I + (y - e_i) e_i^T)$
  - new $A_B^{-1} = (I - (1/y_i) (y - e_i) e_i^T) A_B^{-1}$
- **Finiteness:** If $D_B > 0$ for all dual feasible bases $B$, then the dual simplex algorithm is finite: The dual objective strictly increases at each iteration $\Rightarrow$ no basis repeats, and there are a finite number of bases.
- There are various approaches to guaranteeing finiteness in general:
  - **Bland's Rules**: Purely combinatorial, bad in practice.
  - **Gurobi**: A perturbation is added to "guarantee" $D_B > 0$.

# Implementing the Dual Simplex Algorithm

# Some Motivation

- Dual simplex vs. primal: Dual > 2x faster
- Dual is the best algorithm for MIP
- There isn't much in books about implementing the dual.

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# Dual Simplex Algorithm

## (Lemke, 1954: Commercial codes ~1990)

Input: A dual feasible basis *B* and vectors

$$X_B = A_B^{-1} b \quad \text{and} \quad D_N = c_N - A_N^T B^{-T} c_B.$$

- Step 1: (Pricing) If $X_B \geq 0$, stop, *B* is optimal; else let
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# Implementation Issues for Dual Simplex

1. Finding an initial feasible basis or concluding that there is none: Phase I of the simplex algorithm.

2. Pricing: dual steepest edge

3. Solving the linear systems
   - LU factorization and factorization update
   - BTRAN and FTRAN – exploiting sparsity

4. Numerically stable ratio test: Bound shifting and perturbation

5. Bound flipping: Exploiting "boxed" variables to combine many iterations into one.

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# Issue 0
# Preparation: Bounds on Variables

In practice, simplex algorithms need to accept LPs in the following form:

$$\begin{array}{ll} \textit{Minimize} & c^T x \\ \textit{Subject to} & Ax = b \qquad\qquad (\mathrm{P_{BD}}) \\ & l \leq x \leq u \end{array}$$

where $l$ is an n-vector of **lower bounds** and $u$ an n-vector of **upper bounds**. $l$ is allowed to have $-\infty$ entries and u is allowed to have $+\infty$ entries. (Note that ($\mathrm{P_{BD}}$) is in standard form if $l_j = 0$, $u_j = +\infty\ \forall\ j$.)

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## (Issue 0 – Bounds on variables)
# Basic Solution

A **basis** for ($P_{BD}$) is a triple $(B,L,U)$ where $B$ is an ordered $m$–element subset of $\{1,\ldots,n\}$ (just as before), $(B,L,U)$ is a partition of $\{1,\ldots,n\}$, $l_j > -\infty \ \forall\ j \in L$, and $u_j < +\infty \ \forall\ j \in U$. $N = L \cup U$ is the set of **nonbasic** variables. The associated (**primal**) **basic solution** $X$ is given by $X_L = l_L$, $X_U = u_U$ and

$$X_B = A_B^{-1}(b - A_L l_L - A_U u_U).$$

This solution is **feasible** if

$$l_B \leq X_B \leq u_B.$$

The associated **dual basic solution** is defined exactly as before: $D_B = 0$, $\Pi^T A_B = c_B^T$, $D_N = c_N - A_N^T \Pi$. It is **dual feasible** if

$$D_L \geq 0 \ \text{ and } \ D_U \leq 0.$$

# (Issue 0 – Bounds on variables) The Full Story

- Modify simplex algorithm
  - Only the “Pricing” and “Ratio Test” steps must be changed substantially.
  - The complicated part is the ratio test
- Reference: See Chvátal for the primal

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# Issue 1
# The Initial Feasible Basis – Phase I

- Two parts to the solution
  1. Finding some initial basis (probably not feasible)
  2. Modified simplex algorithm to find a feasible basis

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## (Issue 1 – Initial feasible basis)

# Initial Basis

- Primal and dual bases are the same. We begin in the context of the primal. Consider

$$\begin{array}{rl} \textit{Minimize} & c^T x \\ \textit{Subject to} & Ax = b \qquad (P_{BD}) \\ & l \le x \le u \end{array}$$

- Assumption: Every variable has some finite bound.
- Trick: Add artificial variables $x_{n+1},\ldots,x_{n+m}$:

$$Ax + I \begin{pmatrix} x_{n+1} \\ \cdot \\ \cdot \\ x_{n+m} \end{pmatrix} = b$$

  where $l_j = u_j = 0$ for $j = n+1,\ldots,n+m$.

- Initial basis: $B = (n+1,\ldots,n+m)$ and for each $j \notin B$, pick some finite bound and place $j$ in $L$ or $U$, as appropriate.
- Free-Variable Refinement: Make free variables non-basic at value 0. This leads to a notion of a *superbasis*, where non-basic variables can be between their bounds.

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(Issue 1 – Initial feasible basis)

# Solving the Phase I

- If the initial basis is not dual feasible, we consider the problem:

$$\begin{aligned}&\textit{Maximize} \ \ \Sigma\,(d_j : d_j < 0)\\&\textit{Subject to} \ \ A^T\pi + d = c\end{aligned}$$

- This problem is "locally linear": Define $\kappa \in R^n$ by $\kappa_j = 1$ if $D_j < 0$, and $0$ otherwise. Let

$$K = \{j\colon D_j < 0\} \ \ \textit{and} \ \ \underline{K} = \{j\colon D_j \geq 0\}$$

Then our "local" problem becomes

$$\begin{aligned}&\textit{Maximize} \ \ \kappa^T d\\&\textit{Subject to} \ \ A^T\pi + d = c\\&\qquad\qquad\quad d_K \leq 0,\ d_{\underline{K}} \geq 0\end{aligned}$$

- Apply dual simplex, and whenever $d_j$ for $j \in K$ becomes $0$, move it to $\underline{K}$.

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## Solving Phase I:
# An Interesting Computation

- Suppose $d_{Bi}$ is the entering variable. Then $X_{Bi} < 0$ where $X_B$ is obtained using the following formula:

$$X_B = A_B^{-1} A_N \kappa$$

- Suppose now that $d_j$ is determined to be the leaving variable. Then in terms of the phase I objective, this means $\kappa_j$ is replace by $\kappa_j + \varepsilon\, e_j$, where $\varepsilon \in \{0,+1,-1\}$. It can then be shown that

$$\underline{x}_{Bi} = X_{Bi} + \varepsilon\, \alpha_j$$

- Conclusion: If $x_{Bi} < 0$, then the current iteration can continue without the necessity of changing the basis.

- Advantages
  - Multiple iterations are combined into one.
  - $x_{Bi}$ will tend not to change sign precisely when $\alpha_j$ is small. Thus this procedure tends to avoid unstable pivots.

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# Issue 2
# Pricing

- The textbook rule is TERRIBLE: For a problem in standard form, select the entering variable using the formula

$$j = \mathrm{argmin}\{X_{Bi} : i = 1,\ldots,m\}$$

- Geometry is wrong: Maximizes rate of change relative to axis; better to do relative to edge.

- Goldfard and Forrest 1992 suggested the following steepest-edge alternative

$$j = \mathrm{argmin}\{X_{Bi} / \eta_i : i = 1,\ldots,m\}$$

where $\eta_i = ||e_i^T A_B^{-1}||_2$, and gave an efficient update.

- Note that there are two ingredients in the success of Dual SE:
  - *Significantly reduced iteration counts*
  - *The fact that there is a very efficient update for $\eta$*

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# Example: Pricing
## Model: dfl001

### Pricing: Greatest infeasibility

```
Solved in 281829 iterations and 118.68 seconds
Optimal objective  1.126639304e+07
```

### Pricing: Goldfarb-Forrest steepest-edge

```
Solved in 18412 iterations and 5.36 seconds
Optimal objective  1.126639304e+07
```

# Issue 3
# Solving FTRAN, BTRAN

- Computing LU factorization: See Suhl & Suhl (1990). "Computing sparse LU factorization for large-scale linear programming basis", ORSA Journal on Computing 2, 325-335.

- Updating the Factorization: Forrest-Tomlin update is the method of choice. See Chvátal Chapter 24.
  - **There are multiple, individually relatively minor tweaks that collectively have a significant effect on update efficiency.**

- Further exploiting sparsity: This is the main recent development.ds

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# (Issue 3 – Solving FTRAN & BTRAN)

We must solve two linear systems per iteration:

| FTRAN | BTRAN |
|---|---|
| $A_B y = A_j$ | $A_B{}^T z = e_i$ |

where

| | | | |
|---|---|---|---|
| $A_B$ | = | basis matrix | (very sparse) |
| $A_j$ | = | entering column | (very sparse) |
| $e_i$ | = | unit vector | (very sparse) |

$\Rightarrow$ $y$ an $z$ are typically very sparse

**Example:** Model pla85900 (from TSP)

| | |
|---|---|
| Constraints | 85900 |
| Variables | 144185 |
| Average \|y\| | 15.5 |

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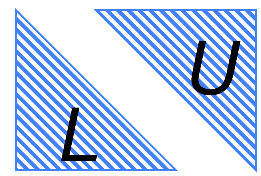
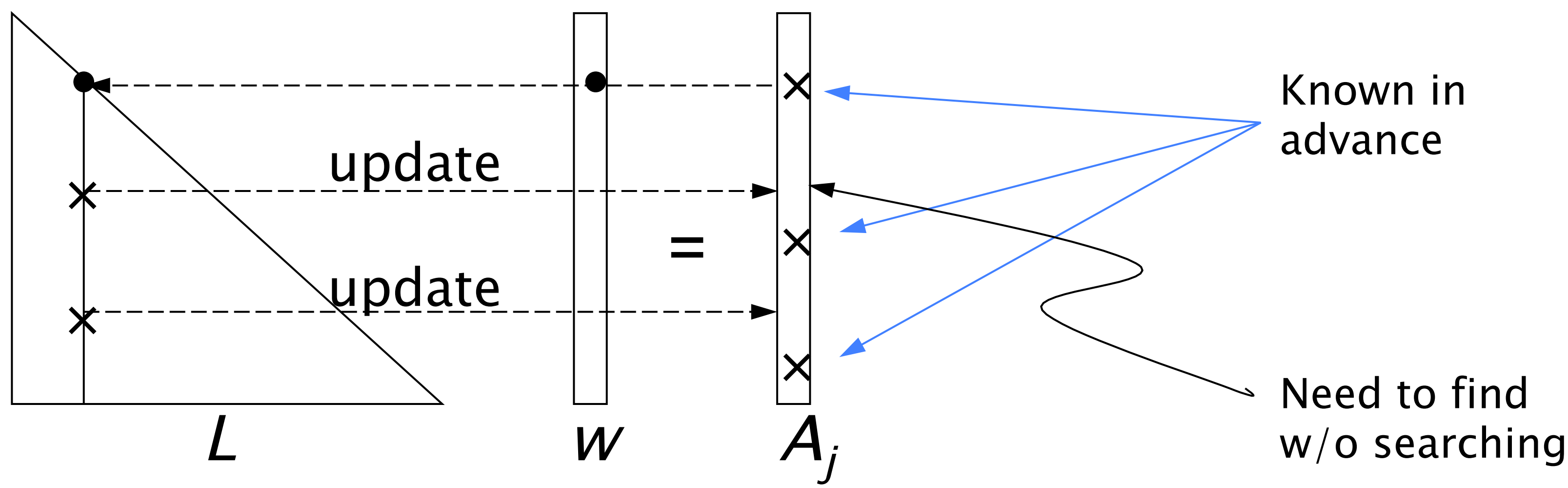
$A_B =$ 


Triangular solve: $Lw=A_j$ $\quad (A_B y = L\underbrace{(Uy)}_{w} = A_j)$




**Graph structure:** Define an acyclic digraph $D = (\{1,\ldots,m\}, E)$ where $(i,j)\in E \Leftrightarrow l_{ij} \neq 0$ and $i \neq j$.

**Solving using $D$:** Let $X = \{i \in V\colon L_{ij} \neq 0\}$. Compute

$$\underline{X} = \{i \in V\colon \exists \text{ a directed path from } i \text{ to } X\}.$$

$\underline{X}$ can be computed in time linear in $|E(\underline{X})|+|\underline{X}|$.

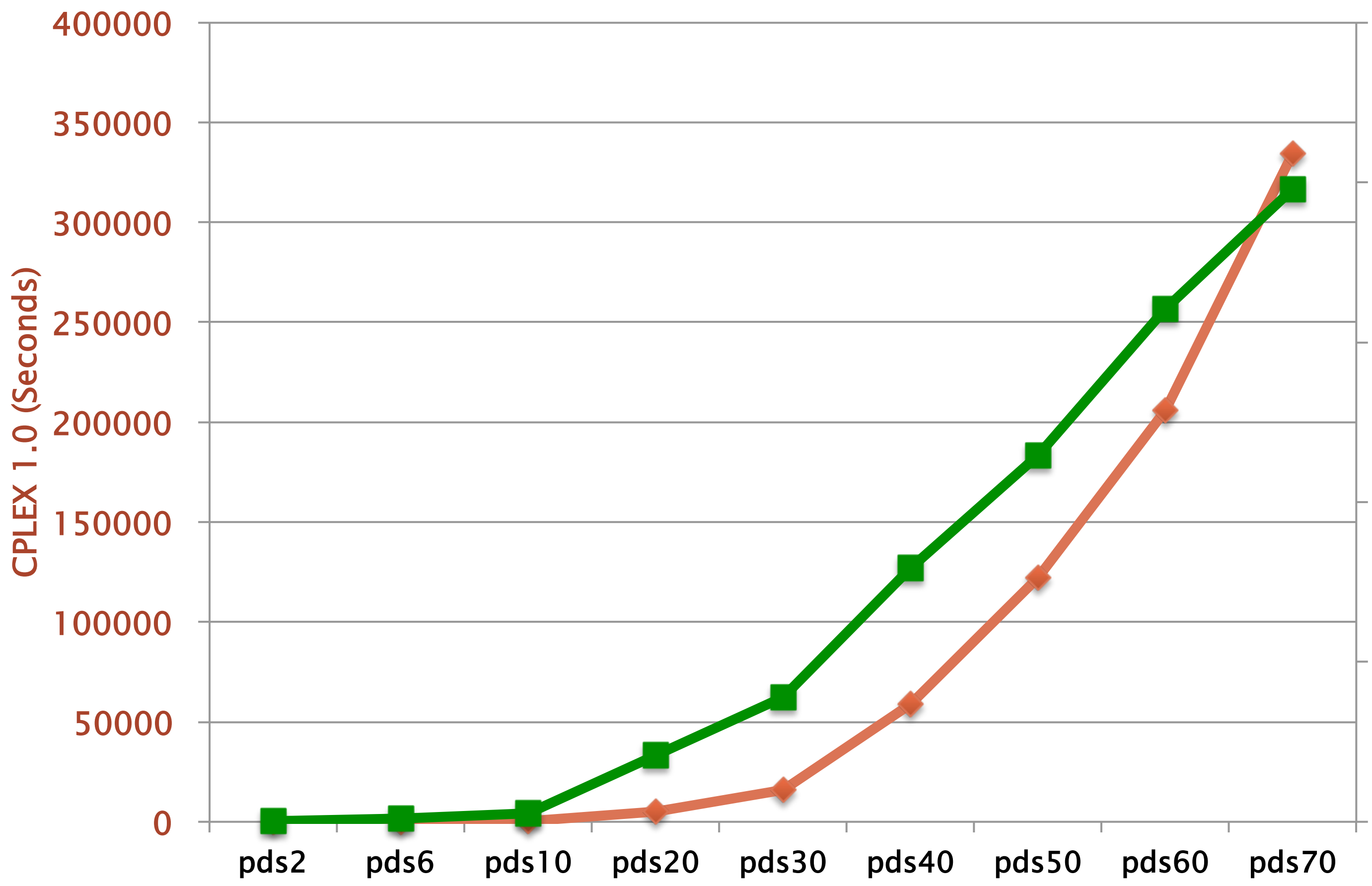
# PDS Models (2002)

"Patient Distribution System": Carolan, Hill, Kennington, Niemi, Wichmann, *An empirical evaluation of the KORBX algorithms for military airlift applications*, Operations Research 38 (1990), pp. 240-248

| MODEL | ROWS | CPLEX1.0 1988 | CPLEX5.0 1997 | CPLEX8.0 2002 | SPEEDUP 1.0→8.0 |
|---|---|---|---|---|---|
| ***pds02*** | 2953 | **0.4** | **0.1** | **0.1** | 4.0 |
| ***pds06*** | 9881 | **26.4** | **2.4** | **0.9** | 29.3 |
| ***pds10*** | 16558 | **208.9** | **13.0** | **2.6** | 80.3 |
| ***pds20*** | 33874 | **5268.8** | **232.6** | **20.9** | 247.3 |
| pds30 | 49944 | **15891.9** | **1154.9** | **39.1** | 406.4 |
| pds40 | 66844 | **58920.3** | **2816.8** | **79.3** | 743.0 |
| pds50 | 83060 | **122195.9** | **8510.9** | **114.6** | 1066.3 |
| pds60 | 99431 | **205798.3** | **7442.6** | **160.5** | 1282.2 |
| pds70 | 114944 | **335292.1** | **21120.4** | **197.8** | 1695.1 |
| | | Primal Simplex | Dual Simplex | Dual Simplex | |

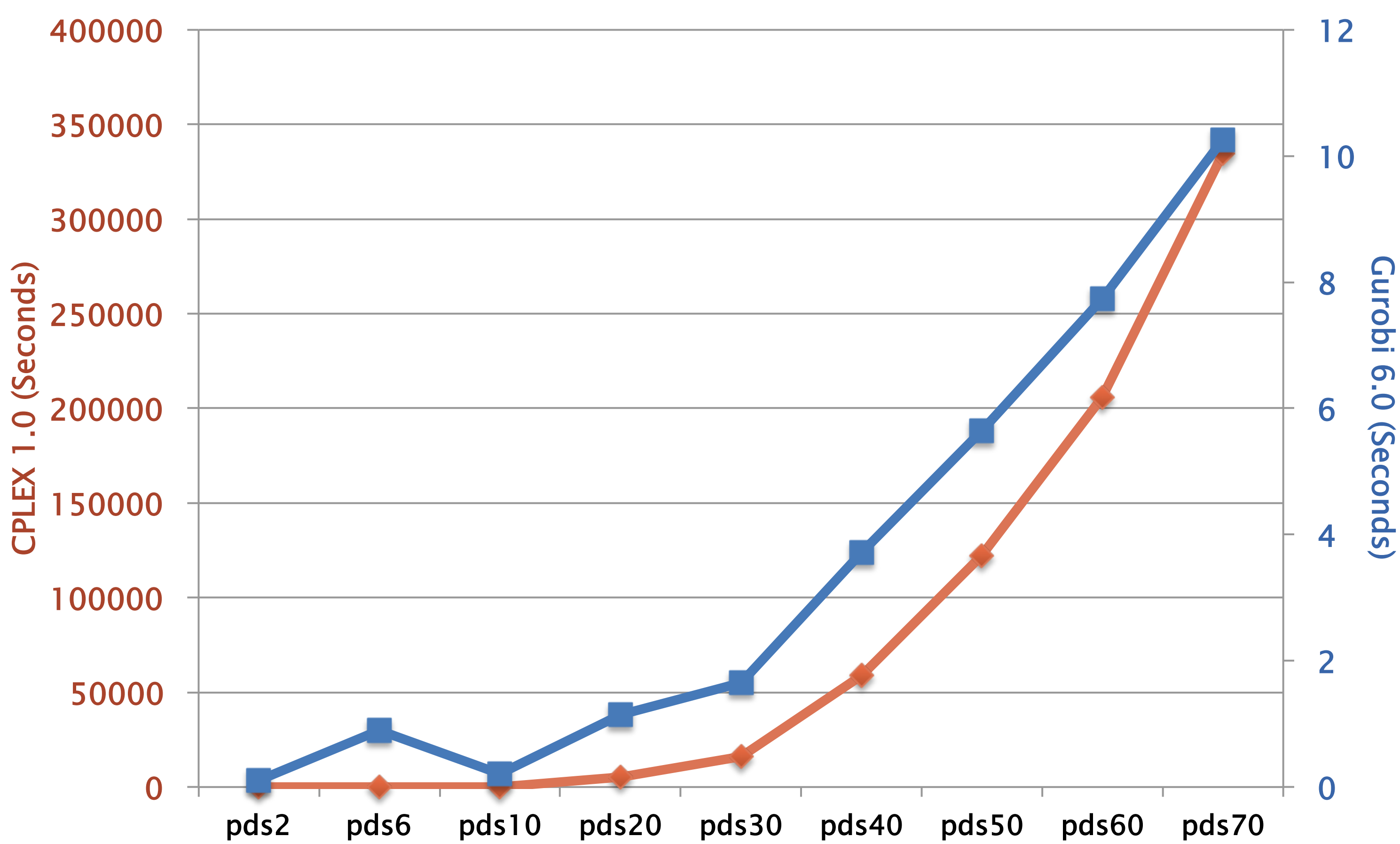
Gurobi Optimization

# Not just faster -- Growth with size: Quadratic *then* & Linear *now*!




Gurobi Optimization

# Gurobi Headline goes here…



# Issue 4
# Ratio Test and Finiteness

The “standard form” dual problem is

$$\begin{array}{rl} \textit{Maximize} & b^T\pi \\ \textit{Subject to} & A^T\pi + d = c \\ & d \geq 0 \end{array}$$

Feasibility means

$$\boldsymbol{d \geq 0}$$

However, in practice this condition is replaced by

$$\boldsymbol{d \geq -\varepsilon\, e}$$

where $e^T=(1,\ldots,1)$ and $\varepsilon = 10^{-6}$, the **feasibility tolerance.**

Reason: Degeneracy. In 1972 Paula Harris suggested exploiting this fact to improve numerical stability.

Gurobi Optimization

## (Issue 4 – Ratio test & finiteness)

STANDARD RATIO TEST $j_{enter} = argmin\{D_j / \alpha_j : \alpha_j > 0\}$

**Motivation:** Feasibility $\Rightarrow$ step length $\theta$ satisfies

$$D_N - \theta\alpha_N \geq 0$$

Since the bigger the step length, the bigger the change in the objective, we choose

$$\theta_{max} = min\{D_j / \alpha_j : \alpha_j > 0\}$$

Using $\varepsilon$, we have

$$\theta^{\varepsilon}_{max} = min\{(D_j + \varepsilon)/\alpha_j : \alpha_j > 0\} > \theta_{max}$$

HARRIS RATIO TEST $j_{enter} = argmax\{\alpha_j : \theta_{max} \leq D_j / \alpha_j \leq \theta^{\varepsilon}_{max}, \alpha_j > 0\}$

Gurobi Optimization

# (Issue 4 – Ratio test & finiteness)

- Advantages
  - Numerical stability – $\alpha_{jenter}$ = "pivot element"
  - Degeneracy – Reduces # of 0-length steps

- Disadvantage
  - $D_{jenter} < 0 \Rightarrow$ objective goes in wrong direction

- Solution: BOUND SHIFTING
  - If $D_{jenter} < 0$, we replace the lower bound on $d_{jenter}$ by something less than its current value.
  - Note that this shift changes the problem and must be removed: 5% of cases, this produces dual infeasibility $\Rightarrow$ process is iterated.

Gurobi Optimization

# Example: Bound-Shifting Removal

```
Read MPS format model from file qap12.mps.bz2
Optimize a model with 3192 rows, 8856 columns and 38304 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    0.0000000e+00   1.230000e+02   0.000000e+00      0s
     101    0.0000000e+00   7.229833e+02   0.000000e+00      0s
     173    3.3669520e+00   9.125960e+02   0.000000e+00      0s
....
   49843    5.2387894e+02   5.585623e+01   0.000000e+00     32s
   50213    5.2388556e+02   7.361090e+00   0.000000e+00     32s
   50584    5.2388824e+02   1.648797e+01   0.000000e+00     32s
   50744    5.2388840e+02   0.000000e+00   0.000000e+00     33s
Switch to primal
   50934    5.2289692e+02   0.000000e+00   3.404469e+01     33s
   51123    5.2289527e+02   0.000000e+00   1.021229e+00     33s
   51312    5.2289450e+02   0.000000e+00   2.841123e-01     33s
   51499    5.2289434e+02   0.000000e+00   1.686059e-01     33s
Perturbation ends
   51516    5.2289435e+02   0.000000e+00   0.000000e+00     33s

Solved in 51516 iterations and 33.15 seconds
Optimal objective  5.228943506e+02
```

Shift removed

Shift removed

Gurobi Optimization

## (Issue 4 – Ratio test & finiteness)

**Finiteness:** Bound shifting is closely related to the "perturbation" method employed in Gurobi if no progress is being made in the objective.

**If "insufficient" progress is being made, replace**

$$d_j \geq -\varepsilon \qquad j = 1,\ldots,n$$

by

$$d_j \geq -\varepsilon - \varepsilon_j \qquad j = 1,\ldots,n,$$

where $\varepsilon_j$ is pseudo-random uniform on $[0,\varepsilon]$. This makes the probability of a 0-length step very small, and in practice has been sufficient to guarantee finiteness.

Gurobi Optimization

# Issue 5
# Bound Flipping (Long-Step Dual)

- A basis is given by a triple (B,L,U)
  - L = non-basics at lower bound: Feasibility $D_L \geq 0$
  - U = non-basics at upper bound: Feasibility $D_U \leq 0$

- **Ratio test:** Suppose $X_{Bi}$ is the leaving variable, and the step length is blocked by some variable $d_j$, $j \in L$, where $d_j$ is about to become negative and $u_j < +\infty$:
  - **Flipping means:** Move *j* from *L* to *U*.
  - **Check:** Do an update to see if $X_{Bi}$ is still favorable (just as we did in Phase I!)

- Can combine many iterations into a single iteration.

Gurobi Optimization

# Example: Bound Flipping

```
Read MPS format model from file fit2d.mps.bz2
Optimize a model with 25 rows, 10500 columns and 129018 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0   -9.1662550e+04   9.553095e+03   0.000000e+00      0s
    6023   -6.8464293e+04   0.000000e+00   0.000000e+00      1s

Solved in 6023 iterations and 0.82 seconds
Optimal objective -6.846429329e+04
```

w/o flipping

```
Read MPS format model from file fit2d.mps.bz2
Optimize a model with 25 rows, 10500 columns and 129018 nonzero

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0   -9.1662550e+04   9.553095e+03   0.000000e+00      0s
     255   -6.8464293e+04   0.000000e+00   0.000000e+00      0s

Solved in 255 iterations and 0.07 seconds
Optimal objective -6.846429329e+04
```

w/ flipping

Gurobi Optimization