Combinatorial Optimization @ Work 2009

Linear Programming Exercises

1. Consider the dual problem

$$\begin{aligned} &\text{Maximize } \; b^T\pi \\ &\text{Subject to } \; A^T\pi + d = c \\ &\qquad\qquad\quad d \geq 0 \end{aligned}$$

and let B be a basis with corresponding primal and dual basic solutions X and Π, D, respectively. Thus

$$X_B = A_B^{-1}b$$
$$\Pi = A_B^{-T}c_B$$
$$D_N = c_N - A_N^T\,\Pi \text{ (and } D_B = 0)$$

a. Let $\underline{d}_B = \theta e_i$ where $\theta > 0$. Show that $A^T\underline{\pi} + \underline{d} = c \Rightarrow$

$$\underline{d}_N = D_N - \theta\,\alpha_N$$
$$\underline{\pi} = \Pi - \theta z$$

where

$$A_B^T z = e_i \quad \text{and} \quad \alpha_N = -A_N^T z.$$

b. In a. you derived update formulas for D_N and α_N. These formulas can be used in Step 5 of the DSA. What value of θ should be chosen?

c. If $\underline{B}$ denotes the new basis after B_i has been replaced by j (as in the algorithm), then it can be shown that

$$A_{\underline{B}}^{-1} = (I - (1/y_i)(y - e_i)e_i^T)\,A_B^{-1}$$

where $A_By = A_j$. Verify this formula and then use it fact to give an update for X_B.

2. Consider the LP

$$\begin{aligned} &\text{Minimize} \quad c^Tx \\ &\text{Subject to} \quad Ax = b \quad (\mathrm{P_{BD}}) \\ &\qquad\qquad\quad\; l \leq x \leq u \end{aligned}$$

And assume, for convenience, that all coordinates of l and u are finite. The dual of $(\mathrm{P_{BD}})$ is

$$\begin{aligned} &\textit{Maximize} \;\; b^T\pi + l^T s - u^T r \\ &\textit{Subject to} \;\; A^T\pi + s - r = c \qquad (D_{BD}) \\ &\qquad\qquad\qquad s, r \geq 0 \end{aligned}$$

a. Show that if x is feasible for (P_{BD}) and π,s,r is feasible for (D_{BD}), then

$$b^T\pi + l^T s - u^T r \leq c^T x.$$

b. Now suppose that (B,L,U) is primal feasible and satisfies $D_L \geq 0$ and $D_U \leq 0$. Define

$$s_j = \;\; D_j \text{ if } j \in L, \text{ 0 otherwise}$$
$$r_j = - D_j \text{ if } j \in U, \text{ 0 otherwise}$$

Let $\pi = \Pi$ and show that (π,s,r) is dual feasible.

c. Use a. and b. to show that if a basis for (P_{BD}) is both primal and dual feasible, it is optimal for (P_{BD}) and (D_{BD}).

3. Derive an update formula for $X_{Bi,}$ where d_{Bi} has been selected as the "entering variable" in the dual, when a variable x_j, $j \in N$, is flipped from its lower to its upper bound.