An Introduction on SemiDefinite Program

from the viewpoint of computation

Hayato Waki

Institute of Mathematics for Industry, Kyushu University

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Contents and Purpose of this lecture

Subject SemiDefinite Program

Contents

- Part I Formulations & Strong duality on SDP
- Part II Algorithm on SDP – Primal-Dual Interior-Point Methods
- Part III Comments of Computation on SDP

Survey M. Todd, "Semidefinite optimization", Acta Numerica 10 (2001), pp. 515–560.

Purpose

- Better understanding for the next lecture (MOSEK on SDP) by Dr. Dahl
- Know the difficulty in solving SDP in Part III

Message : SDP is convex, but also nonlinear

Properties and applications of SDP

Properties : SDP is an extension of LP

- Duality Theorem
- Solvable by primal-dual interior-point methods with up to a given tolerance

Applications

- Combinatorial problems, e.g., Max-Cut by Goemans and Williams
- Control theory, e.g., $\mathbf{H}_\infty$ control problem
- Lift-and-projection approach for nonconvex quadratic problem
- Lasserre's hierarchy for polynomial optimization problems and complexity theory
- Embedding problems, e.g., sensor networks and molecular conformation
- Statistics and machine learning, etc...

From LP To SDP

$\boxed{\text{LP}}$ Primal and Dual

$$\begin{array}{ll|ll} \min_{\mathbf{x}} & \mathbf{c}^T\mathbf{x} & \max_{(\mathbf{y},\mathbf{s})} & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{a}_j^T\mathbf{x} = \mathbf{b}_j \ (\forall \mathbf{j}) & \text{s.t.} & \mathbf{s} = \mathbf{c} - \sum_{j=1}^{m} \mathbf{y}_j\mathbf{a}_j \\ & \mathbf{x} \in \mathbb{R}^n_+ & & \mathbf{s} \in \mathbb{R}^n_+ \end{array}$$

- Minimize/Maximize linear function over the intersection the affine set and $\mathbb{R}^n_+$
- $\mathbb{R}^n_+$ is closed convex cone in $\mathbb{R}^n$

$\boxed{\text{Extension to SDP}}$

- Extension to the space of symmetric matrices $\mathbb{S}^n$

$$\mathbf{c} \in \mathbb{R}^n \to \mathbf{C} \in \mathbb{S}^n, \mathbf{a}_j \in \mathbb{R}^n \to \mathbf{A}_j \in \mathbb{S}^n$$

- Minimize/Maximize linear function over the intersection the affine set and the set of positive semidefinite matrices

$\boxed{\text{LP}}$ Primal and Dual

$$
\begin{array}{lll|ll}
\min_x & c^T x & & \max_{(y,s)} & b^T y \\
\text{s.t.} & a_j^T x = b_j \ (\forall j) & & \text{s.t.} & s = c - \sum_{j=1}^m y_j a_j \\
 & x \in \mathbb{R}^n_+ & & & s \in \mathbb{R}^n_+
\end{array}
$$

$\boxed{\text{SDP}}$ Primal and Dual

$$
\begin{array}{lll|ll}
\min_X & C \bullet X & & \max_{(y,S)} & b^T y \\
\text{s.t.} & A_j \bullet X = b_j \ (\forall j) & & \text{s.t.} & S = C - \sum_{j=1}^m y_j A_j \\
 & X \in \mathbb{S}^n_+ & & & S \in \mathbb{S}^n_+
\end{array}
$$

- $\mathbb{S}^n$ is the set of $n \times n$ symmetry matrices,
- $\mathbb{S}^n_+$ is the set of $n \times n$ symmetry positive semidefinite matrices, and
- $A \bullet X := \sum_{k=1}^{n} \sum_{\ell=1}^{n} A_{k\ell} X_{k\ell}$.

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1. Definition of positive semidefinite matrices

$\mathbf{X} \in \mathbb{S}^n$ is *positive semidefinite* if for all $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}^T \mathbf{X} \mathbf{z} \geq \mathbf{0}$.
Equivalently, all eigenvalues are nonnegative.

Remark

- Eigendecomposition (Spectral decomposition); $\exists \mathbf{Q} \in \mathbb{R}^{n \times n}$ (orthogonal) and $\exists \lambda_i \geq \mathbf{0}$ such that

$$\mathbf{X} = \mathbf{Q} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \mathbf{Q}^T$$

- See textbooks of linear algebra for proof
- $\Rightarrow \exists \mathbf{B} \in \mathbb{R}^{n \times n}$ such that $\mathbf{X} = \mathbf{B}\mathbf{B}^T$

2. Zero diagonal for positive semidefinite matrices

For $\mathbf{X} \in \mathbb{S}^n_+$, each $\mathbf{X}_{ii}$ is nonnegative. In addition, if $\mathbf{X}_{ii} = \mathbf{0}$ for some $\mathbf{i}$, then $\mathbf{X}_{ij} = \mathbf{X}_{ji} = \mathbf{0}$ for all $\mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}$.

Example of SDP

$$\mathbf{C} = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 10 & 4 \\ 4 & \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} & \\ & -8 \end{pmatrix},$$

$$\mathbf{A}_3 = \begin{pmatrix} & -9 \\ -9 & 2 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 42 & -8 & 20 \end{pmatrix}^T$$

Primal SDP is formulated as follows:

$$\inf_{\mathbf{X}} \left\{ 2x_{11} + x_{22} : \begin{array}{ll} 10x_{11} + 8x_{12} = 42, & -8x_{22} = -8, \\ -18x_{12} + 2x_{22} = 20, & \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in \mathbb{S}^2_+ \end{array} \right\}$$

(Fortunately) the primal solution is uniquely fixed:

$$\mathbf{X} = \begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix} \text{ is positive definite and obj. val.} = \mathbf{11}.$$

Primal SDP is formulated as follows:

$$\inf_{\mathbf{X}} \left\{ 2x_{11} + x_{22} : \begin{array}{ll} 10x_{11} + 8x_{12} = 42, & -8x_{22} = -8, \\ -18x_{12} + 2x_{22} = 20, & \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in \mathbb{S}^2_+ \end{array} \right\}$$

Dual SDP is formulated as follows:

$$\sup_{(\mathbf{y},\mathbf{S})} \left\{ 42y_1 - 8y_2 + 20y_3 : \begin{pmatrix} 2 - 10y_1 & -4y_1 + 9y_3 \\ -4y_1 + 9y_3 & 1 + 8y_2 - 2y_3 \end{pmatrix} \in \mathbb{S}^2_+ \right\}$$

A dual solution is $\mathbf{(1/5, -37/360, 4/45)}$ with the obj. val. $= 11$.

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Application : Computation of lower bounds of nonconvex QP

$\boxed{\text{QP}}$

$$\theta^* := \inf_{\mathbf{x}} \left\{ \mathbf{x}^T Q \mathbf{x} + 2\mathbf{c}^T \mathbf{x} : \mathbf{x}^T Q_j \mathbf{x} + 2\mathbf{c}_j^T \mathbf{x} + r_j \leq 0 \ (j = 1, \ldots, m) \right\}$$

$\boxed{\text{SDP relaxation}}$: Add the following constraint and replace $x_i x_j \to X_{ij}$:

$$\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} (1, \mathbf{x}) \in \mathbb{S}_+^{n+1} \to X \in \mathbb{S}_+^{n+1}$$

$$\therefore \eta^* := \inf_{\mathbf{x}} \left\{ \begin{pmatrix} 0 & \mathbf{c}^T \\ \mathbf{c} & Q \end{pmatrix} \bullet X : \begin{pmatrix} r_j & \mathbf{c}_j^T \\ \mathbf{c}_j & Q_j \end{pmatrix} \bullet X \leq 0, X_{00} = 1, X \in \mathbb{S}_+^{n+1} \right\}$$

$\boxed{\text{Remark}}$

- Handle as SDP
- $\eta^* \leq \theta^*$
- binary $x \in \{0, 1\} \to x^2 - x = 0 \Rightarrow$ MIQP with binary variables $=$ QP

Application : Lasserre's SDP relaxation for Polynomial Optimization Problems

$\boxed{\text{POP}}$: $\mathbf{f}, \mathbf{g_j}$ are polynomials on $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$

$$\theta^* := \inf_{\mathbf{x}} \{\mathbf{f(x)} : \mathbf{g_j(x)} \geq \mathbf{0} \ (\mathbf{j} = \mathbf{1}, \ldots, \mathbf{m})\}$$

$\boxed{\text{Lasserre's SDP relaxation}}$

- Generates a sequence of SDP problems : $\{\mathbb{P}_{\mathbf{r}}\}_{\mathbf{r} \geq 1}^{\infty}$
- Optimal value : $\theta_{\mathbf{r}} \leq \theta_{\mathbf{r}+1} \leq \theta^* \ (\forall \mathbf{r})$
- Under assumptions, $\theta_{\mathbf{r}} \to \theta^* \ (\mathbf{r} \to \infty)$
- $\mathbf{r} = \mathbf{2}, \mathbf{3}$, $\theta_{\mathbf{r}} \approx \theta^*$ in practice
- Strongly connected to sum of square polynomials

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Compared with LP

Similar points

- Weak and Strong duality holds
- PDIPM also works in SDP

Different points

- SDP may have an irrational optimal solution

$$\text{E.g.,}\quad \sup_{\mathbf{y}} \left\{ \mathbf{y} : \begin{pmatrix} 2 & \mathbf{y} \\ \mathbf{y} & 1 \end{pmatrix} \in \mathbb{S}^2_+ \right\}$$

  Optimal solution $\mathbf{y} = \sqrt{2}$, not rational

- Finite optimal value, but $\nexists$ solutions

$$\text{E.g.,}\quad \inf_{\mathbf{y}} \left\{ \mathbf{y}_1 : \begin{pmatrix} \mathbf{y}_1 & 1 \\ 1 & \mathbf{y}_2 \end{pmatrix} \in \mathbb{S}^2_+ \right\}$$

Different points (cont'd)

$\exists$ 2 types of infeasibility

$$
\begin{array}{lll}
\text{(LP) } \exists \mathbf{y}; -\mathbf{A}^T\mathbf{y} \in \mathbb{R}^n_+, \mathbf{b}^T\mathbf{y} > 0 & \Longleftrightarrow & \text{Primal LP is infeasible} \\
\text{(SDP) } \exists \mathbf{y}; -\mathbf{A}^T\mathbf{y} \in \mathbb{S}^n_+, \mathbf{b}^T\mathbf{y} > 0 & \Rightarrow & \text{Primal SDP is infeasible}
\end{array}
$$

$\boxed{\text{Remark}}$: Need to consider the following cases

- Finite optimal value, but no optimal solutions for Primal and/or Dual
- Difficult to detect the infeasibility completely

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Duality on SDP

$\boxed{\text{Weak duality}}$ for any $\mathbf{X} \in \mathcal{F}_\mathbf{P}$ and $(\mathbf{y}, \mathbf{S}) \in \mathcal{F}_\mathbf{D}$,

$$\mathbf{C} \bullet \mathbf{X} \geq \mathbf{b}^\mathbf{T}\mathbf{y} \quad \therefore \theta^*_\mathbf{P} \geq \theta^*_\mathbf{D}$$

$\boxed{\text{Slater conditon}}$: $\mathbb{S}^\mathbf{n}_{++}$ is the set of positive definite matrices

- Primal satisfies *Slater condition* if $\exists \mathbf{X} \in \mathcal{F}_\mathbf{P}$ such that $\mathbf{X} \in \mathbb{S}^\mathbf{n}_{++}$
- Dual *Slater condition* if $\exists (\mathbf{y}, \mathbf{S}) \in \mathcal{F}_\mathbf{D}$ such that $\mathbf{S} \in \mathbb{S}^\mathbf{n}_{++}$

$\boxed{\text{Strong duality}}$

- Primal satisfies Slater condition and dual is feasible. Then $\theta^*_\mathbf{P} = \theta^*_\mathbf{D}$ and dual has an optimal solution.
- Slater condition are required for both primal and dual for theoretical results on PDIPMs
- See survey on SDP for proof

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3. Inner products on positive semidefinite matrices

For all $\mathbf{X}, \mathbf{S} \in \mathbb{S}^{\mathbf{n}}_{+}$, $\mathbf{X} \bullet \mathbf{S} \geq \mathbf{0}$. Moreover, $\mathbf{X} \bullet \mathbf{S} = \mathbf{0}$ iff $\mathbf{XS} = \mathbf{O_n}$

$\boxed{\text{Proof}}$: $\exists \mathbf{B}$ s. t. $\mathbf{X} = \mathbf{BB^T}$ and $\exists \mathbf{D}$ s.t. $\mathbf{S} = \mathbf{DD^T}$. Then

$$
\begin{aligned}
\mathbf{X} \bullet \mathbf{S} &= \mathrm{Trace}(\mathbf{BB^TDD^T}) = \mathrm{Trace}(\mathbf{D^TBB^TD}) \\
&= \mathrm{Trace}((\mathbf{B^TD})^{\mathbf{T}}(\mathbf{B^TD})) \geq \mathbf{0}
\end{aligned}
$$

Moreover, $\mathbf{X} \bullet \mathbf{S} = \mathbf{0} \Rightarrow \mathbf{B^TD} = \mathbf{O_n} \Rightarrow \mathbf{XS} = \mathbf{O_n}$

$\boxed{\text{Proof of weak duality}}$

In fact, for $\mathbf{X} \in \mathcal{F}_{\mathbf{P}}$ and $(\mathbf{y}, \mathbf{S}) \in \mathcal{F}_{\mathbf{D}}$,

$$
\mathbf{C} \bullet \mathbf{X} - \mathbf{b^Ty} = \left(\mathbf{C} - \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{y_j A_j}\right) \bullet \mathbf{X} = \mathbf{S} \bullet \mathbf{X} \geq \mathbf{0}
$$

because both matrices are positive semidefinite.

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Remark of 3 (cont'd)

- $\mathbf{X} \in \mathcal{F}_{\mathbf{P}}$: optimal in primal and $(\mathbf{y}, \mathbf{S}) \in \mathcal{F}_{\mathbf{D}}$: optimal in dual
- Then, $\theta^*_{\mathbf{P}} - \theta^*_{\mathbf{D}} = \mathbf{X} \bullet \mathbf{S} = \mathbf{0} \iff \mathbf{XS} = \mathbf{O_n}$
- $\mathbf{XS} = \mathbf{O_n}$ is used in PDIPM

SDP with multiple positive semidefinite cones

SDP

$$\begin{array}{ll} \inf_{X_k} & \sum_{k=1}^{N} C^k \bullet X_k \\ \text{s.t.} & \sum_{k=1}^{N} A_j^k \bullet X_k = b_j \ (j = 1, \ldots, m) \\ & X_k \in \mathbb{S}_+^{n_k} \ (k = 1, \ldots, N) \end{array}$$

where $C^k, A_j^k \in \mathbb{S}^{n_k}$

Example

$$A \bullet X \leq d, X \in \mathbb{S}_+^n \Rightarrow A \bullet X + s = d, X \in \mathbb{S}_+^n \text{ and } s \in \mathbb{S}_+^1 (= \mathbb{R}_+)$$

Dual

$$\sup_{y, S_k} \left\{ b^T y : S_k = A_0^k - \sum_{j=1}^{m} y_j A_j^k \in \mathbb{S}_+^{n_k} \ (k = 1, \ldots, N) \right\}$$

Remark

- SDP with $\mathbb{R}^n_+$, Second order cone $\mathbf{L_n}$ and $\mathbb{S}^n_+$ can be handled as SDP and PDIPM works

$$\mathbf{L_n} := \{(\mathbf{x_0}, \mathbf{x}) \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq \mathbf{x_0}\}$$

- Free variable can be accepted

$$\begin{aligned} & \mathbf{A} \bullet \mathbf{X} + \mathbf{a}^T \mathbf{x} = \mathbf{d}, \mathbf{X} \in \mathbb{S}^n_+, \mathbf{x} \in \mathbb{R}^n \\ \Rightarrow \quad & \mathbf{A} \bullet \mathbf{X} + \mathbf{a}^T \mathbf{x_1} - \mathbf{a}^T \mathbf{x_2} = \mathbf{d}, \mathbf{X} \in \mathbb{S}^n_+ \text{ and } \mathbf{x_1}, \mathbf{x_2} \in \mathbb{R}^n_+ \end{aligned}$$

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Classification of Algorithms for SDP

Algorithms for SDP

- Ellipsoid method
- Interior-point methods
- Bundle method
- first-order methods, etc

Interior-point methods

- Path-following algorithm (= Logarithmic barrier function)
- Potential reduction algorithm
- Self-dual homogeneous embeddings

Path-following algorithm

- Primal
- Dual
- Primal-dual

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Path-following method

$\boxed{\text{Optimality conditions}}$: a pair of optimal solutions $(\mathbf{X}, \mathbf{y}, \mathbf{S})$ satisfies

$$\begin{cases} \mathbf{A_j} \bullet \mathbf{X} = \mathbf{b_j}, \mathbf{X} \in \mathbb{S}^n_+, \\ \mathbf{S} = \mathbf{C} - \sum_{j=1}^m \mathbf{y_j}\mathbf{A_j}, \mathbf{S} \in \mathbb{S}^n_+, \\ \mathbf{XS} = \mathbf{O_n} (\iff \mathbf{C} \bullet \mathbf{X} - \mathbf{b}^T\mathbf{y} = 0) \end{cases}$$

$\boxed{\text{Perturbed system}}$: for $\mu > 0$,

$$\begin{cases} \mathbf{A_j} \bullet \mathbf{X} = \mathbf{b_j}, \mathbf{X} \in \mathbb{S}^n_{++}, \\ \mathbf{S} = \mathbf{C} - \sum_{j=1}^m \mathbf{y_j}\mathbf{A_j}, \mathbf{S} \in \mathbb{S}^n_{++}, \\ \mathbf{XS} = \mu \mathbf{I_n} \end{cases}$$

$\boxed{\text{Remark}}$

- for any $\mu > 0$, $\exists$ unique solution $(\mathbf{X}(\mu), \mathbf{y}(\mu), \mathbf{S}(\mu))$
- Central path $\{(\mathbf{X}(\mu), \mathbf{y}(\mu), \mathbf{S}(\mu)) : \mu > 0\}$ is smooth curve and go to a pair of optimal solutions of primal and dual
- Follows the central path = Path-following method

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Algorithm 1: General framework of path-following method

Input: $(\mathbf{X^0}, \mathbf{y^0}, \mathbf{S^0}) \in \mathcal{F}_\mathbf{P} \times \mathcal{F}_\mathbf{D}$ such that $\mathbf{X^0}, \mathbf{S^0} \in \mathbb{S}^\mathbf{n}_{++}$, $\epsilon > \mathbf{0}$, $\mathbf{0} < \theta < \mathbf{1}$ and some parameters

$\mathbf{X} \leftarrow \mathbf{X^0}$, $\mathbf{y} \leftarrow \mathbf{y^0}$ and $\mathbf{S} \leftarrow \mathbf{S^0}$;

while $\mathbf{X} \bullet \mathbf{S} > \epsilon$ **do**

- Compute direction $(\mathbf{\Delta X}, \mathbf{\Delta y}, \mathbf{\Delta S})$ from CPE(μ);
- Compute step size $\alpha_\mathbf{P}, \alpha_\mathbf{D} > \mathbf{0}$;
- $\mathbf{X} \leftarrow \mathbf{X} + \alpha_\mathbf{P}\mathbf{\Delta X}$;
- $\mathbf{y} \leftarrow \mathbf{y} + \alpha_\mathbf{D}\mathbf{\Delta y}$; $\mathbf{S} \leftarrow \mathbf{S} + \alpha_\mathbf{D}\mathbf{\Delta S}$;
- Compute $\mu \leftarrow \theta\mu$;

end

return $(\mathbf{X}, \mathbf{y}, \mathbf{S})$;

Remark

- Infeasible initial guess is acceptable
- # of iteration is polynomial in $\mathbf{n}, \mathbf{m}$ and $\log(\epsilon)$
- Computational cost = Computation of direction

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Computation of direction : Find $(\Delta X, \Delta y, \Delta S)$ such that
$X + \Delta X \in \mathcal{F}_P$, $(y + \Delta y, S + \Delta S) \in \mathcal{F}_D$ and

$$\begin{cases} A_j \bullet \Delta X = 0, \\ \Delta S - \sum_{j=1}^m \Delta y_j A_j = O_n, \\ XS + \Delta XS + X\Delta S = \mu I_n \end{cases}$$

Remark

- ΔX *may not* be symmetry. So, change $XS = \mu I_n$ by
$$\frac{1}{2}\left(PXSP^{-1} + P^{-T}SXP^{T}\right) = \mu I_n,$$
where P is nonsingular
- Possible choice of P

$$\begin{aligned} P &= S^{1/2} \text{ (HRVW/KSH/M)} \\ P &= X^{-1/2} \text{ (dual HRVW/KSH/M)} \\ P &= W^{1/2}, W = X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2} \text{ (NT)} \\ P &= \ldots \text{ More than 20 types of directions by Todd} \end{aligned}$$

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Computational cost in PDIPM

$\boxed{\text{1. Construction of linear system on } \mathbf{\Delta y}}$ for HRVW/KSH/M direction,

$$\mathbf{M\Delta y} = (\text{RHS}), \text{where } \mathbf{M} = \left(\text{Trace}(\mathbf{A_i X A_j S^{-1}})\right)_{\mathbf{1 \leq i,j \leq m}}$$

- Use of sparsity in $\mathbf{A_j}$ is necessary for computation of $\mathbf{M}$
- Almost the same for other search directions

$\boxed{\text{2. Solving the linear system}}$

- $\mathbf{M}$ is dense $\Rightarrow$ takes $\mathbf{O(m^3)}$ computation by Cholesky decomposition
- $\mathbf{M}$ is often sparse in SDP relax for POP $\Rightarrow$ sparse Cholesky decomposition works well

After them , $\mathbf{\Delta S} = \sum_{\mathbf{j=1}}^{\mathbf{m}} \mathbf{\Delta y_j A_j}$ and obtain $\mathbf{\Delta X}$.

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Sparsity in SDP

Example **Q** is nonsingular and dense. Then $\mathbb{P}_1$ is equivalent to $\mathbb{P}_2$:

$$\mathbb{P}_1 \quad : \quad \inf_{\mathbf{X}} \left\{ \mathbf{C} \bullet \mathbf{X} : \mathbf{E_i} \bullet \mathbf{X} = 1 \ (\mathbf{i} = 1, \ldots, \mathbf{n}), \mathbf{X} \in \mathbb{S}^{\mathbf{n}}_{+} \right\},$$

$$\mathbb{P}_2 \quad : \quad \inf_{\mathbf{X}} \left\{ (\mathbf{Q}^T \mathbf{C} \mathbf{Q}) \bullet \mathbf{X} : (\mathbf{Q}^T \mathbf{E_i} \mathbf{Q}) \bullet \mathbf{X} = 1 \ (\mathbf{i} = 1, \ldots, \mathbf{n}), \mathbf{X} \in \mathbb{S}^{\mathbf{n}}_{+} \right\}$$

where

$$(\mathbf{E_i})_{\mathbf{pq}} = \begin{cases} \mathbf{1} & \text{if } \mathbf{p} = \mathbf{q} = \mathbf{i} \\ \mathbf{0} & \text{o.w.} \end{cases} \quad (\mathbf{p}, \mathbf{q} = 1, \ldots, \mathbf{n})$$

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$\boxed{\text{CPU time}}$: Solved by SeDuMi 1.3 on the MacBook Air (1.7 GHz Intel Core i7)

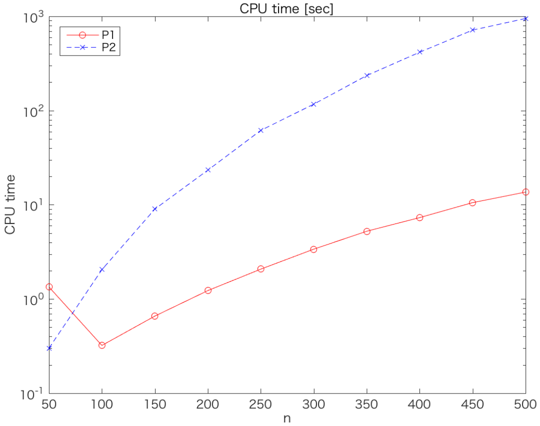

Figure : CPU time on $\mathbb{P}_1$ and $\mathbb{P}_2$

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Software

Information from `http://plato.asu.edu/ftp/sparse_sdp.html`

- SeDuMi, SDPT3 (MATLAB)
- SDPA (C++, MATLAB)
- CSDP (C, MATLAB)
- DSDP (C, MATLAB)
- MOSEK

Remark

- Based on PDIPM for almost all software
- Performance depends on SDP problems

Modelling languages on SDP : they can call the above software

- YALMIP
- CVX

Slater conditions

Strong duality

- Require Slater conditions for Primal or Dual
- PDIPM requires Slater conditions for both Primal and Dual
- Sufficient conditions for optimal solutions
- If either Primal or Dual does not satisfy Slater conditions, ...

E.g., Lasserre's SDP relaxation

$$\mathbb{P} : \inf_{x} \left\{ x : x^2 - 1 \geq 0, x \geq 0 \right\}$$

- Gererate SDP relaxation problems $\mathbb{P}_1, \mathbb{P}_2, \ldots,$
- Slater condition fails in all SDP relaxation & all optimal values are $\mathbf{0}$
- SeDuMi and SDPA returns wrong value $\mathbf{1}$
- All SDP relaxation problems are sensitive to numerical errors in the computation of floating points

E.g., Graph Equipartition

- $\mathbf{G(V,E)}$: a weighted undirected graph $\Rightarrow$ Partition the vertex set $\mathbf{V}$ into $\mathbf{L}$ and $\mathbf{R}$
- the minimum total weight of the cut subject to $|\mathbf{L}| = |\mathbf{R}|$
- QOP formulation

$$\inf_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \sum w_{ij}(1 - x_i x_j) : \sum_{i=1}^{n} x_i = 0, x_i{}^2 = 1 \ (i = 1, \ldots, n) \right\}$$

E.g., Graph Equipartition (cont'd)

- SDP relaxation problem: constant matrices $\mathbf{W}$, $\mathbf{E}$ and $\mathbf{E_i}$

$$\inf_{\mathbf{X}\in\mathbb{S}^n_+} \{\mathbf{W} \bullet \mathbf{X} \mid \mathbf{E} \bullet \mathbf{X} = 0, \mathbf{E_i} \bullet \mathbf{X} = 1\}$$

- Since $\mathbf{E} \in \mathbb{S}^n_+$, $\nexists \mathbf{X} \in \mathbb{S}^n_{++}$ s.t. $\mathbf{E} \bullet \mathbf{X} = \mathbf{0} \Rightarrow$ Slater cond. fails
- Inaccurate value and/or many iterations

Table : SeDuMi 1.3 with ϵ=1.0e-8

SDPLIB	iter	cpusec	duality gap
gpp124-1	30	2.40	-4.63e-05
gpp250-1	29	10.19	-1.60e-04
gpp500-1	34	61.58	-1.90e-04
gpp124-4	40	3.02	-2.14e-08
gpp500-2	40	76.88	-8.26e-06

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E.g., Graph Equipartition (cont'd)

$$\inf_{\mathbf{X} \in \mathbb{S}^n_+} \{\mathbf{W} \bullet \mathbf{X} \mid \mathbf{E} \bullet \mathbf{X} = 0, \mathbf{E_i} \bullet \mathbf{X} = 1\}$$

- Transformation of SDP by $\mathbf{V}$:

$$\mathbf{V} = \begin{pmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}$$

- $\mathbf{X} \to \mathbf{V}^{-\mathbf{T}}\mathbf{X}\mathbf{V}^{-1} =: \mathbf{Z}$ and $\mathbf{E} \to \mathbf{V}\mathbf{E}\mathbf{V}^{\mathbf{T}}$
- Then, $\mathbf{X} \in \mathbb{S}^n_+ \iff \mathbf{Z} \in \mathbb{S}^n_+$ and $\mathbf{E} \bullet \mathbf{X} = 0 \iff \mathbf{Z_{nn}} = 0$
- Eliminate $\mathbf{n}$th row and column from transformed SDP $\Rightarrow$ Slater cond. holds

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E.g., Graph Equipartition (cont'd)

Table : Numerical Results by SeDuMi 1.3 with ϵ=1.0e-8.

	Slater fails			Slater holds		
Problems	iter	cpusec	d.gap	d.gap	cpusec	iter
gpp100	30	1.78	-2.46e-07	-4.97e-09	0.73	16
gpp124-1	30	2.34	-4.63e-05	-1.75e-08	1.12	19
gpp124-2	26	1.76	-1.41e-06	-1.11e-09	1.03	18
gpp124-3	30	2.56	-4.41e-07	-3.05e-09	1.01	17
gpp124-4	40	2.93	-2.14e-08	-9.52e-11	1.09	17
gpp250-1	29	8.81	-1.60e-04	-1.82e-08	4.71	21
gpp250-2	29	8.61	-1.49e-05	-9.74e-09	4.19	19
gpp250-3	34	9.48	-3.97e-07	-8.12e-10	4.08	18
gpp250-4	35	11.28	-8.80e-07	-7.43e-10	4.37	19
gpp500-1	34	53.45	-1.90e-04	-2.76e-08	31.49	24
gpp500-2	40	68.47	-8.26e-06	-2.20e-09	28.98	22
gpp500-3	28	54.81	-1.00e-05	-2.39e-09	31.35	21
gpp500-4	28	55.06	-1.02e-06	-8.96e-10	32.06	23

Comments : If does not satisfy Slater conditions, ...

- PDIPM computes inaccurate values and/or spends many iter.
- But, reduce the size of SDP

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Comments

- A simple (?) transformation generates an SDP in which Slater cond. holds
- More elementary approach :

$$(\mathrm{QOP}) \quad : \quad \inf_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}\sum w_{ij}(1 - x_i x_j) : \sum_{i=1}^{n} x_i = 0, {x_i}^2 = 1 \right\}$$

$$(\mathrm{QOP'}) \quad : \quad \text{obtained by substituting } x_1 = -\sum_{i=2}^{n} x_i \text{ in (QOP)}$$

$$\begin{array}{ccc} (\mathrm{QOP}) & \xrightarrow{\text{equiv.}} & (\mathrm{QOP'}) \\ \downarrow \text{SDP relax.} & & \text{SDP relax.} \downarrow \\ (\mathrm{SDP}) & \xrightarrow[\text{equiv.}]{\text{V}} & (\mathrm{SDP'}) \end{array}$$

- General case : separate x into basic and nonbasic variables & substitute basic variables $\Rightarrow$ SDP relax

$$\inf_{x} \left\{ x^T Q x + 2c^T x : a_j^T x = b_j \ (j = 1, \ldots, m), x_k \in \{0, 1\} \right\}$$

Extension

$\boxed{\text{SDP}}$

$$\inf_{\mathbf{X}} \left\{ \mathbf{C} \bullet \mathbf{X} : \mathbf{A_j} \bullet \mathbf{X} = \mathbf{b_j}, \mathbf{X} \in \mathbb{S}^{\mathbf{n}}_{+} \right\}$$

Slater condition fails in Primal $\iff$ $\exists \mathbf{y} \in \mathbb{R}^{\mathbf{m}} \setminus \{\mathbf{0}\}$ such that

$$\mathbf{b}^{\mathbf{T}}\mathbf{y} \geq \mathbf{0}, -\sum_{\mathbf{j}} \mathbf{y_j}\mathbf{A_j} \in \mathbb{S}^{\mathbf{n}}_{+}$$

Moreover, if $\exists \mathbf{y}$ such that $\mathbf{b}^{\mathbf{T}}\mathbf{y} > \mathbf{0}$, then Primal is infeasible

$\boxed{\text{Proof of } (\Leftarrow)}$: Suppose the contrary that Slater condition holds in Primal. $\exists \hat{\mathbf{X}}$ such that $\mathbf{A_j} \bullet \hat{\mathbf{X}} = \mathbf{b_j}$ and $\hat{\mathbf{X}} \in \mathbb{S}^{\mathbf{n}}_{++}$.

$$\mathbf{0} \ \leq \ \mathbf{b}^{\mathbf{T}}\mathbf{y} = \sum_{\mathbf{j}} (\mathbf{A_j} \bullet \hat{\mathbf{X}})\mathbf{y_j} = \left(\sum_{\mathbf{j}} \mathbf{A_j}\mathbf{y_j} \right) \bullet \hat{\mathbf{X}} < \mathbf{0} (\text{contradiction})$$

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Facial Reduction

$\boxed{\text{Idea}}$: Let $\mathbf{W} := -\sum_j \mathbf{A_j y_j} \in \mathbb{S}^n_+$ and $\mathbf{b^T y = 0}$

- For any feasible solutions $\mathbf{X}$ in Primal,

$$\mathbf{W \bullet X} = -\sum_j (\mathbf{A_j \bullet X})\mathbf{y_j} = -\mathbf{b^T y} = \mathbf{0}.$$

- Primal is equivalent to

$$\inf_{\mathbf{X}} \left\{ \mathbf{C \bullet X} : \mathbf{A_j \bullet X} = \mathbf{b_j}, \mathbf{X} \in \mathbb{S}^n_+ \cap \{\mathbf{W}\}^\perp \right\}$$

where $\{\mathbf{W}\}^\perp := \{\mathbf{X} : \mathbf{X \bullet W} = \mathbf{0}\}$

- The set $\mathbb{S}^n_+ \cap \{\mathbf{W}\}^\perp$ has nice structure

$$\mathbb{S}^n_+ \cap \{\mathbf{W}\}^\perp = \left\{ \mathbf{X} \in \mathbb{S}^n : \mathbf{X} = \mathbf{Q} \begin{pmatrix} \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \mathbf{Q^T}, \mathbf{M} \in \mathbb{S}^r_+ \right\}$$

Idea (cont'd)

$$\mathbb{S}^n_+ \cap \{\mathbf{W}\}^\perp = \left\{\mathbf{X} \in \mathbb{S}^n : \mathbf{X} = \mathbf{Q}\begin{pmatrix} \mathbf{M} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}\mathbf{Q}^T, \mathbf{M} \in \mathbb{S}^r_+\right\}$$

- Assume $\mathbf{Q} = \mathbf{I_n}$. Then Primal is equivalent to

$$\inf_{\mathbf{X}} \left\{\tilde{\mathbf{C}} \bullet \mathbf{X} : \tilde{\mathbf{A}}_j \bullet \mathbf{X} = \mathbf{b}_j, \mathbf{X} \in \mathbb{S}^r_+\right\}$$

  where $\tilde{\mathbf{A}}_j$ is $\mathbf{r} \times \mathbf{r}$ principal matrix
- Compare this SDP with Primal $\Rightarrow$ the size $\mathbf{n} \to \mathbf{r}$
- May not satisfy Slater cond.
- $\Rightarrow$ Find $\mathbf{y}$ and $\mathbf{W}$ for the smaller Primal
- This procedure terminates in finitely many iterations
- This procedure is called Facial Reduction Algorithm and acceptable for dual

Histroy of FRA

- Borwein-Wolkowicz in 1980 for general convex optimization
- Ramana, Ramana-Tunçel-Wolkowicz for SDP
- Pataki simplified FRA for the extension
- Apply FRA into SDP relax. for Graph Partition, Quadratic Assignment, Sensor Network by Wolkowicz group
- Apply FRA into SDP relax. for Polynomial Optimization in Waki-Muramatsu
- ...

Summary on Slater condition

- Hope that both Primal and dual satisfy Slater conditions
- Otherwise, may not have any optimal solutions, and wrong value may be obtained
- Obtain inaccurate solutions even if exists optimal solutions, but, one can reduce the size of SDP
- FRA is a general framework to remove the difficulty in Slater cond.

In modeling to SDP...

- Need to be careful in even dual to guarantee the existence of optimal solutions in dual
- A rigorous solution for FRA is necessary

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Status of infeasibility

Feasiblity and infeasiblity

$$\inf_{\mathbf{X}} \left\{ \mathbf{C} \bullet \mathbf{X} : \mathbf{A_j} \bullet \mathbf{X} = \mathbf{b_j}, \mathbf{X} \in \mathbb{S}^{\mathbf{n}}_{+} \right\}$$

- Strongly feasible if SDP satisfies Slater cond.
- Weakly feasible if SDP is feasible but, does not satisfies Slater cond.
- Strongly infeasible if $\exists$ improving ray **d**, *i.e.*,

$$\mathbf{b}^{\mathbf{T}}\mathbf{d} > \mathbf{0}, -\sum_{\mathbf{j}} \mathbf{d_j A_j} \in \mathbb{S}^{\mathbf{n}}_{+}.$$

- Weakly infeasible if SDP is infeasible, but $\nexists$ improving ray

Remark

- Weak infeasibility does not occur in LP
- SOCP and conic optimization also have the four status

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$\boxed{\text{Example}}$: Infeasible SDPs

$$\mathbb{P}_1 \qquad \inf_{\mathbf{X}} \left\{ \mathbf{C} \bullet \mathbf{X} : \begin{pmatrix} \mathbf{1} & \\ & \mathbf{1} \end{pmatrix} \bullet \mathbf{X} = \mathbf{0}, \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \bullet \mathbf{X} = \mathbf{2}, \mathbf{X} \in \mathbb{S}^{\mathbf{2}}_{+} \right\},$$

$$\mathbb{P}_2 \qquad \inf_{\mathbf{X}} \left\{ \mathbf{C} \bullet \mathbf{X} : \begin{pmatrix} & \\ & \mathbf{1} \end{pmatrix} \bullet \mathbf{X} = \mathbf{0}, \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \bullet \mathbf{X} = \mathbf{2}, \mathbf{X} \in \mathbb{S}^{\mathbf{2}}_{+} \right\}$$

$\boxed{\text{Comments}}$

- $\mathbb{P}_1$ is strongly infeasible because $\exists$ certificate $\mathbf{y} = (\mathbf{-1}, \mathbf{1})$
- $\mathbb{P}_2$ is weakly infeasible because $\nexists$ certificate

$\boxed{\text{Characterization of weak infeasibility}}$

- Weakly infeasible SDP; for all $\epsilon >$, $\exists \mathbf{X} \in \mathbb{S}^n_+$

$$|\mathbf{A_j} \bullet \mathbf{X} - \mathbf{b_j}| < \epsilon \ (\mathbf{j} = \mathbf{1}, \ldots, \mathbf{m})$$

- More elementary characterization of Weak infeasibility by recent work by Liu and Pataki

$\boxed{\text{Example } \mathbb{P}_2}$ Perturb $\mathbf{b_1} = \mathbf{0} \to \epsilon > \mathbf{0}$

$$\mathbb{P}_2 : \inf_{\mathbf{X}} \left\{ \mathbf{C} \bullet \mathbf{X} : \begin{pmatrix} & \\ & \mathbf{1} \end{pmatrix} \bullet \mathbf{X} = \epsilon, \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \bullet \mathbf{X} = \mathbf{2}, \mathbf{X} \in \mathbb{S}^2_+ \right\}$$

Then, perturbed $\mathbb{P}_1$ is feasible:

$$\mathbf{X} = \begin{pmatrix} \mathbf{1}/\epsilon & \mathbf{1} \\ \mathbf{1} & \epsilon \end{pmatrix}$$

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$\boxed{\text{Pathological?}}$

$$(\mathrm{POP}) : \inf_{\mathbf{x},\mathbf{y}} \{-\mathbf{x} - \mathbf{y} : \mathbf{x}\mathbf{y} \le \mathbf{1/2}, \mathbf{x} \ge \mathbf{1/2}, \mathbf{y} \ge \mathbf{1/2}\}$$

- Optimal value is $-\mathbf{1.5}$
- Apply Lasserre's SDP hierarchy
- All SDP relaxation is weakly infeasible (in Waki 2012)
- SeDuMi and SDPA returns $-\mathbf{1.5}$ for higher oder SDP relaxation
- Sufficient conditions of (POP) for SDP relaxation to be weakly infeasible (in Waki 2012)

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Summary on infeasibility

- Weak infeasibility may occur in SDP, SOCP and conic optimization, but not in LP
- Difficult to detect this type of infeasibility by software
- But, software returns good values for weak infeasible SDP

Summary

- Introduce a part of theoretical and practical aspects in SDP
- Skip applications of SDP, *e.g.*, SDP relaxation for combinatorial problems
- Can read papers on SDP
- Not so easy to handle SDP because it is convex but nonlinear programming

Further Reading I

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Acta Numerica 10 (2001), pp. 515–560.

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