`polymake` Exercises for CO@Work

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The official `polymake` version for the workshop is Release 2.14. It is available on:

`http://polymake.org/doku.php/download/start`

For help with the properties, functions, and methods you can use the help function in the shell (either with the `help` command or `F1`). You can also check the online reference documentation, the wiki, and the forum on `www.polymake.org`.

1 Introduction to the polymake shell

1. Construct a 3-dimensional unit simplex from scratch by defining the property `POINTS`.

   Here is one of various possibilities: Define a 3-dimensional unit matrix, append a row with 0's, and prepend a column of 1's in front.

   Compare this with the output of the command `simplex` and visualize the polytope (try to vary the colors of the facets and vertices...).

2. Define a 3-cube. Extract the facet matrix into a separate variable. Print the entry at position (2,3). Change one of the entries, and use this to define a new polytope.

3. Determine all binomial coefficients $\binom{a}{b}$ for $1 \leq a, b, \leq 100$ whose second to last digit is 7.

4. Produce the hypersimplex $H(3,6)$ (i.e., the convex hull of all $0/1$-vectors of length 6 with exactly 3 ones) by using an outer description, and secondly by an inner description. Compare with the client `hypersimplex`.

5. Define a square S and a pyramid P over a 6-gon.
   (a) Determine their join J, their product Π, and their Minkowski sum M.
   (b) What are the dimensions of J, Π, M?
   (c) Turn all faces of J into separate polytopes. Project them to full dimensional polytopes.

6. Let P denote the convex hull in $\mathbb{R}^3$ of the nine points with coordinates (i, i^2, i^3) for $i = -4, -3, -2, -1, 0, 1, 2, 3, 4$. Compute the volume of P.

7. Generate some random polytopes with `rand_sphere(5,100,seed=>123)` (to know what the parameters mean check `help`. You can vary the parameters...) and test the different convex hull algorithms on your polytopes (there are `cdd`, `lrs`, `beneath_beyond`, `ppl`).

8. Select eight points at random on the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Determine (experimentally) the expected number of facets of their convex hull.

2 Optimization/geometry oriented exercises

1. Eliminate the unknown z from the system of linear inequalities

$$0 \leq x + y - 2z \leq 1 \text{ and } 0 \leq x - 2y + z \leq 1 \text{ and } 0 \leq -2x + y + z \leq 1.$$

2. Construct an infeasible linear program. How does `polymake` treat it? Employ the `properties` user function. Find out what attachments are and what they are used for.

3. The *hexadecachoron* is the convex hull of the even vertices of the 4-cube:

$$\begin{aligned} P \quad = \quad \operatorname{conv}\{ \quad & (0,0,0,0),\ (0,0,1,1),\ (0,1,0,1),\ (0,1,1,0), \\ & (1,0,0,1),\ (1,0,1,0),\ (1,1,0,0),\ (1,1,1,1)\ \}. \end{aligned}$$

   Determine the f-vector of P. Is P simple? Is P simplicial? Draw a Schlegel diagram. How about the analogous polytope in 5 dimensions?

4. Construct a 3-polytope and apply the function `fan::normal_fan`. Visualize and explore.

5. How many 4×4-matrices with non-negative integer entries and zeros on the diagonal have all row sums and all column sums equal to m? Determine this number for $0 \leq m \leq 7$ and formulate a conjecture.

6. Consider the polytope P given as the convex hull of

$$\begin{aligned} P \quad = \quad \operatorname{conv}\{ \quad & (1,1,0),\ (1,1,1),\ (0,1,0),\ (0,1,1), \\ & (1,0,0),\ (1,0,1),\ (0,0,7),\ (0,0,8)\ \}. \end{aligned}$$

   Check that the polytope is not normal (i.e. not every lattice point in kP is a sum of k lattice points in P). Verify that the polytope is very ample (i.e. the Hilbert basis of the cone spanned at every vertex of P is contained in P).

   Note: there are actually properties computing these things directly, but you might want to try yourself first.

7. Compute the circuits of a graphical matroid. What is the dimension of its matroid polytope?

8. Compute the Voronoi diagram of the lattice points of the $-1/1$-cube in dimension 3.

9. Define a (3×4)-matrix M with constant row sum and constant column sum. Construct a polytope which is the convex hull of the rows of M. Choose a cost function and solve the corresponding linear program. Try to find a way to solve the associated integer program.

10. Determine a TDI system of inequalities for the cube and the cross polytope. There is a function `make_totally_dual_integral`.

11. Consider all vectors in $\mathbb{R}^6$ obtained from $(0,0,1,1,2,2)$ by permuting coordinates, let P denote their convex hull. Write the set P as the solution set of a system of linear inequalities in six unknowns.

12. Construct 50 combinatorially distinct 4-dimensional polytopes, each of which has precisely 30 vertices and is neither simple nor simplicial.