polymake: software for polytope constructions in linear and integer optimization

Michael Joswig

TU Berlin

CO@Work, ZIB, 29 September 2015

joint w/ polymake team

1 `polymake` Basics
Solving an integer linear program

2 One Special Feature
Highly symmetric integer programs
The core point method
Computational results

3 Convex Hull Experiments
Some rules of thumb

4 Epilogue

polymake Basics

polymake Overview

most recent version 2.14 of March 2015

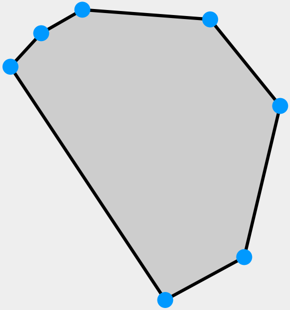
- software for research in mathematics
  - geometric combinatorics: convex polytopes, matroids, ...
  - linear/combinatorial optimization
  - toric/tropical geometry
  - combinatorial topology
- open source, GNU Public License
  - supported platforms: Linux, FreeBSD, MacOS X
  - about 150,000 uloc (C++, Perl, C, Java)
  - interfaces to many other software systems
- co-authored (since 1996) w/ Ewgenij Gawrilow
  - contributions by Benjamin Assarf, Simon Hampe, Katrin Herr, Silke Horn, Lars Kastner, Georg Loho, Benjamin Lorenz, Andreas Paffenholz, Julian Pfeifle, Thomas Rehn, Thilo Rörig, Benjamin Schröter, André Wagner and others

`www.polymake.org`

The Basic Definition

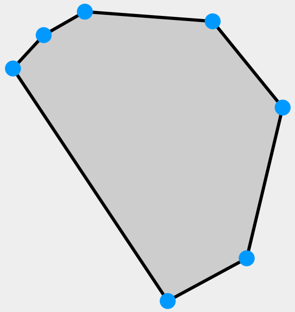
A (convex) polytope is the convex hull of finitely many points (in $\mathbb{R}^d$).

- $=$ intersection of finitely many closed halfspaces (if bounded)
- $=$ set of feasible points of a linear program (if bounded for all choices of linear objective functions)



The Basic Definition

A (convex) polytope is the convex hull of finitely many points (in $\mathbb{R}^d$).



- = intersection of finitely many closed halfspaces (if bounded)
- = set of feasible points of a linear program (if bounded for all choices of linear objective functions)

- conversion from points to inqualities (or vice versa) conceptually simple but still has its challenges

Example: Knapsack Problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{d} u_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{d} w_i x_i \leq b \\ & x_i \in \mathbb{N} \qquad \text{for all } i \in [d] \end{aligned}$$

- $d = \#$ items
- $u_i =$ utility of item i
- $w_i =$ weight of item i
- $b =$ total weight bound

DEMO

Algorithm Overview (Selection)

- convex polytopes, polyhedra and fans
  - convex hulls: `cdd`, `lrs`, `normaliz`, `ppl`, beneath-and-beyond
  - Voronoi diagrams, Delone decompositions
  - Hasse diagrams of face lattices
  - $\rightsquigarrow$ lattice polytopes/toric varieties
- optimization
  - Hilbert bases: `normaliz`, `4ti2`
  - Gomory–Chvátal closures
  - enumerating integer points: `LattE`, bounding box/by projection
- simplicial complexes
- tropical geometry
- graphs, matroids, permutation groups, ...

One Special Feature

The Setup

We consider linear programs $\mathrm{LP}(A, b, c)$ of the form

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \le b\,,\ \ x \in \mathbb{R}^d \end{array}$$

where $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^d$.

Assumptions:

- $P(A, b) := \left\{ x \in \mathbb{R}^d \mid Ax \le b \right\}$ not empty
- optimal solution exists, that is, $\mathrm{LP}(A, b, c)$ bounded
- $c \neq 0$

The Setup

We consider linear programs $\text{LP}(A, b, c)$ of the form

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq b\,,\ \ x \in \mathbb{R}^d \end{array}$$

where $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^d$.

Assumptions:

- $P(A, b) := \left\{x \in \mathbb{R}^d \,\middle|\, Ax \leq b\right\}$ not empty
- optimal solution exists, that is, $\text{LP}(A, b, c)$ bounded
- $c \neq 0$

Notation: $\text{ILP}(A, b, c)$ if additionally $x \in \mathbb{Z}^d$ required

Symmetric Integer Linear Programs

Definition

symmetry of $\mathrm{ILP}(A, b, c)$ = linear automorphism of $\mathrm{LP}(A, b, c)$

- which acts on signed standard basis $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ of $\mathbb{R}^d$ as signed permutation

Symmetric Integer Linear Programs

Definition

symmetry of $\mathrm{ILP}(A, b, c)$ = linear automorphism of $\mathrm{LP}(A, b, c)$

- which acts on signed standard basis $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ of $\mathbb{R}^d$ as signed permutation

Facts:

- signed permutations: $\mathrm{O}_n\mathbb{Z} \;\cong\; \mathbb{Z}_2 \wr \mathrm{Sym}(d) \;=\; (\mathbb{Z}_2)^d \rtimes \mathrm{Sym}(d)$
- group of combinatorial automorphisms of standard cube/cross polytope

Symmetric Integer Linear Programs

Definition

symmetry of $\mathrm{ILP}(A, b, c)$ = linear automorphism of $\mathrm{LP}(A, b, c)$

- which acts on signed standard basis $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ of $\mathbb{R}^d$ as signed permutation

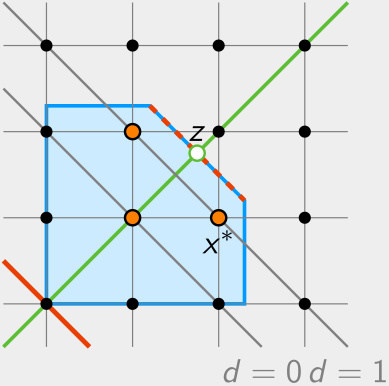
Facts:

- signed permutations: $\mathrm{O}_n\mathbb{Z} \;\cong\; \mathbb{Z}_2 \wr \mathrm{Sym}(d) \;=\; (\mathbb{Z}_2)^d \rtimes \mathrm{Sym}(d)$
- group of combinatorial automorphisms of standard cube/cross polytope

Margot 2002; Friedman 2007; Kaibel & Pfetsch 2008;
Ostrowski & al. 2011; ...

The Core Point Method

Consider $\mathrm{ILP}(A, b, c)$ as above.
Assume that the entire group $\mathrm{Sym}(d)$ acts as symmetries.



Theorem (Bödi, J. & Herr 2013)

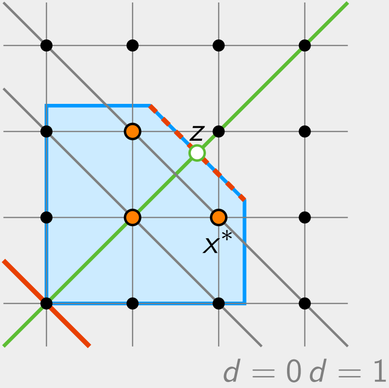
Then the ILP can be solved to optimality in $O(md^2)$ time.

The Core Point Method

Consider $\mathrm{ILP}(A, b, c)$ as above.
Assume that the entire group $\mathrm{Sym}(d)$ acts as symmetries.

Theorem (Bödi, J. & Herr 2013)

Then the ILP can be solved to optimality in $O(md^2)$ time.



- Herr, Rehn & Schürmann 2013: extension of core point algorithm solves MIPLIB 2010 problem `toll-like` w/ `polymake` and `Gurobi`

Computational Results

Wild Input

	CPLEX 12.1.0		polymake 2.13	
d	time LP (s)	time IP (s)	time LP (s)	time IP (s)
3	0.00	0.01	0.00	0.00
4	0.00	0.06	0.01	0.00
5	0.00	0.17	0.01	0.02
6	0.05	0.74	0.04	0.04
7	0.13	2.71	0.09	0.13
8	0.62	10.15	0.24	0.38
9	2.08	42.06	0.69	1.03
10	8.02	135.51	1.86	2.89

Convex Hull Experiments

Example: Max-Cut

- combinatorial optimization problem on $\Gamma = (V, E)$ finite graph

$$\max \sum_{s \in S,\, t \in T,\, \{s,t\} \in E} w(s,t)$$

- maximum over all partitions $S \sqcup T = V$
- w = weight function on E
- each cut $S \sqcup T$ gives rise to subset of E, which can be encoded by its characteristic vector
  - $\rightsquigarrow$ 0/1-polytope

Example: Max-Cut

- combinatorial optimization problem on $\Gamma = (V, E)$ finite graph

$$\max \sum_{s\in S,\, t\in T,\, \{s,t\}\in E} w(s,t)$$

  - maximum over all partitions $S \sqcup T = V$
  - w = weight function on E
  - each cut $S \sqcup T$ gives rise to subset of E, which can be encoded by its characteristic vector
    - $\rightsquigarrow$ 0/1-polytope

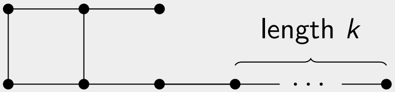
- goal: determine facets of the cut polytopes

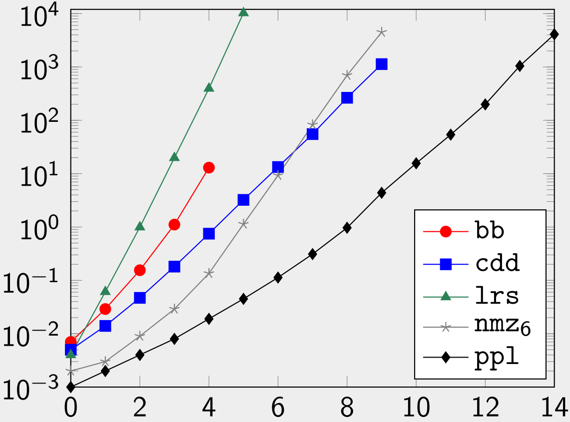
Barahona & al. 1988; Avis, Imai & Ito 2008; Bonato & al. 2014; ...

DEMO

Facets of Cut Polytopes

variable dimension





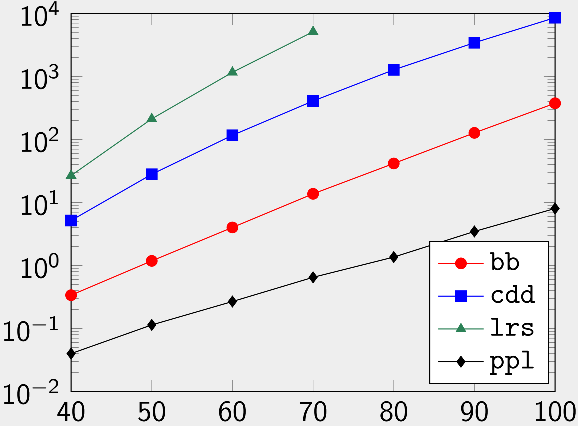
- $d = k + 6$
- $n = 2^{k+5} = \#$ cuts
- $m = 2d + 8 = 2k + 20$

- Barahona 1983: facets known if no K_5-minor

Knapsack Integer Hulls

fixed dimension, variable right hand side

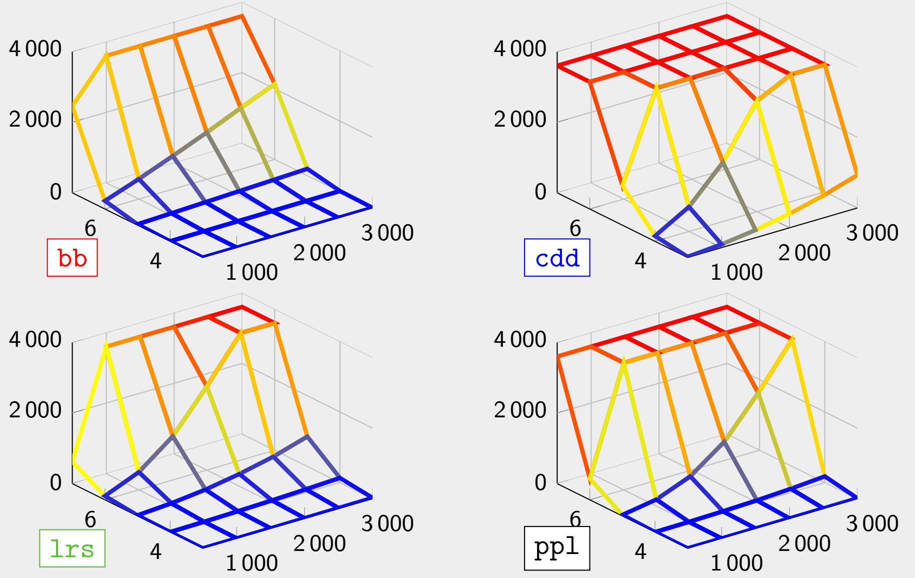
$$a_1 = 2,\ a_2 = 3,\ a_i = a_{i-2} + a_{i-1}$$
$$F_d(b) = \{x \in \mathbb{R}^d_{\geq 0} \mid a^\top x \leq b\}$$



- $d = 5$
- $n = 1366, 3173, 6509, 12182, 21245, 35025, 55157$
- $m = 12, 15, 12, 12, 8, 13, 15$

Voronoi Diagrams of Random Points in a Box

variable dimension, variable number of points



DEMO

Some Rules of Thumb

1. If you do not know anything about your input, try double description.
   - `cdd`, `ppl`, `nmz`
2. Do use double description for computing the facets of 0/1-polytopes.
   - `cdd`, `ppl`
3. On random input beneath-and-beyond often behaves very well.
   - `bb`
4. Use reverse search for partial information and non-degenerate input.
   - `lrs`

Epilogue

References

- Benjamin Assarf, Ewgenij Gawrilow, Katrin Herr, Michael Joswig, Benjamin Lorenz, Andreas Paffenholz, and Thomas Rehn, `polymake` in linear and integer programming, 2014, Preprint `arXiv:1408.4653`.
- Richard Bödi, Katrin Herr, and Michael Joswig, Algorithms for highly symmetric linear and integer programs, Math. Program. **137** (2013), no. 1-2, Ser. A, 65–90. MR 3010420
- Katrin Herr, Thomas Rehn, and Achill Schürmann, Exploiting symmetry in integer convex optimization using core points, Oper. Res. Lett. **41** (2013), no. 3, 298–304. MR 3048847