# polymake: software for polytope constructions in linear and integer optimization

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# polymake Basics

# polymake Overview

most recent version 2.14 of March 2015

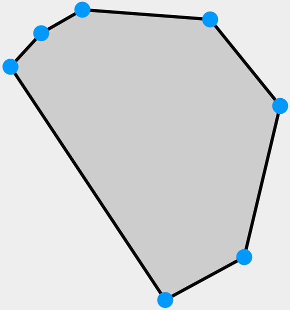
- software for research in mathematics
  - geometric combinatorics: convex polytopes, matroids, ...
  - linear/combinatorial optimization
  - toric/tropical geometry
  - combinatorial topology
- open source, GNU Public License
  - supported platforms: Linux, FreeBSD, MacOS X
  - about 150,000 uloc (C++, Perl, C, Java)
  - interfaces to many other software systems
- co-authored (since 1996) w/ Ewgenij Gawrilow
  - contributions by Benjamin Assarf, Simon Hampe, Katrin Herr, Silke Horn, Lars Kastner, Georg Loho, Benjamin Lorenz, Andreas Paffenholz, Julian Pfeifle, Thomas Rehn, Thilo Rörig, Benjamin Schröter, André Wagner and others

`www.polymake.org`

# The Basic Definition

A (convex) polytope is the convex hull of finitely many points (in $\mathbb{R}^d$).

- $=$ intersection of finitely many closed halfspaces (if bounded)
- $=$ set of feasible points of a linear program (if bounded for all choices of linear objective functions)



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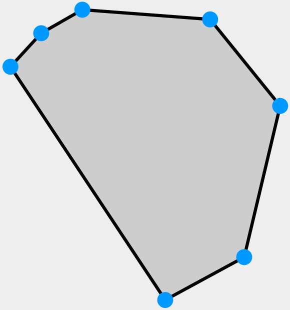

- conversion from points to inqualities (or vice versa) conceptually simple but still has its challenges

# Example: Knapsack Problem

$$\max \quad \sum_{i=1}^{d} u_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{d} w_i x_i \leq b$$

$$x_i \in \mathbb{N} \qquad \text{for all } i \in [d]$$

- $d = \#$ items
- $u_i =$ utility of item $i$
- $w_i =$ weight of item $i$
- $b =$ total weight bound

DEMO

# Algorithm Overview (Selection)

- convex polytopes, polyhedra and fans
  - convex hulls: `cdd`, `lrs`, `normaliz`, `ppl`, `beneath-and-beyond`
  - Voronoi diagrams, Delone decompositions
  - Hasse diagrams of face lattices
  - $\rightsquigarrow$ lattice polytopes/toric varieties
- optimization
  - Hilbert bases: `normaliz`, `4ti2`
  - Gomory–Chvátal closures
  - enumerating integer points: `LattE`, bounding box/by projection
- simplicial complexes
- tropical geometry
- graphs, matroids, permutation groups, ...

# One Special Feature

# The Setup

We consider linear programs $\mathrm{LP}(A, b, c)$ of the form

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq b\,,\; x \in \mathbb{R}^d \end{array}$$

where $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^d$.

Assumptions:

- $P(A, b) := \left\{ x \in \mathbb{R}^d \mid Ax \leq b \right\}$ not empty
- optimal solution exists, that is, $\mathrm{LP}(A, b, c)$ bounded
- $c \neq 0$

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Notation: $\mathrm{ILP}(A, b, c)$ if additionally $x \in \mathbb{Z}^d$ required

# Symmetric Integer Linear Programs

### Definition

symmetry of $\mathrm{ILP}(A, b, c)$ = linear automorphism of $\mathrm{LP}(A, b, c)$

- which acts on signed standard basis $\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ of $\mathbb{R}^d$ as signed permutation

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Facts:

- signed permutations: $\mathrm{O}_n\mathbb{Z} \;\cong\; \mathbb{Z}_2 \wr \mathrm{Sym}(d) \;=\; (\mathbb{Z}_2)^d \rtimes \mathrm{Sym}(d)$
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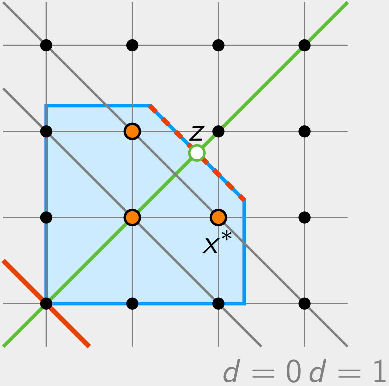
Margot 2002; Friedman 2007; Kaibel & Pfetsch 2008;
Ostrowski & al. 2011; ...

# The Core Point Method

Consider $\mathrm{ILP}(A, b, c)$ as above.
Assume that the entire group $\mathrm{Sym}(d)$ acts as symmetries.

**Theorem (Bödi, J. & Herr 2013)**

*Then the ILP can be solved to optimality in $O(md^2)$ time.*



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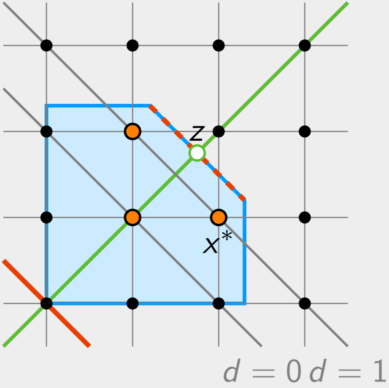


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- Herr, Rehn & Schürmann 2013: extension of core point algorithm solves MIPLIB 2010 problem `toll-like` w/ `polymake` and `Gurobi`

# Computational Results

## Wild Input

| | CPLEX 12.1.0 | | polymake 2.13 | |
|---|---|---|---|---|
| $d$ | time LP (s) | time IP (s) | time LP (s) | time IP (s) |
| 3 | 0.00 | 0.01 | 0.00 | 0.00 |
| 4 | 0.00 | 0.06 | 0.01 | 0.00 |
| 5 | 0.00 | 0.17 | 0.01 | 0.02 |
| 6 | 0.05 | 0.74 | 0.04 | 0.04 |
| 7 | 0.13 | 2.71 | 0.09 | 0.13 |
| 8 | 0.62 | 10.15 | 0.24 | 0.38 |
| 9 | 2.08 | 42.06 | 0.69 | 1.03 |
| 10 | 8.02 | 135.51 | 1.86 | 2.89 |

# Convex Hull Experiments

# Example: Max-Cut

- combinatorial optimization problem on $\Gamma = (V, E)$ finite graph

$$\max \sum_{s \in S,\, t \in T,\, \{s,t\} \in E} w(s, t)$$

  - maximum over all partitions $S \sqcup T = V$
  - $w$ = weight function on $E$
  - each cut $S \sqcup T$ gives rise to subset of $E$, which can be encoded by its characteristic vector
    - $\rightsquigarrow$ 0/1-polytope

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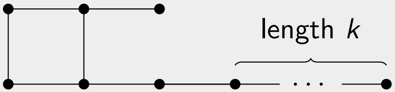
- goal: determine facets of the cut polytopes

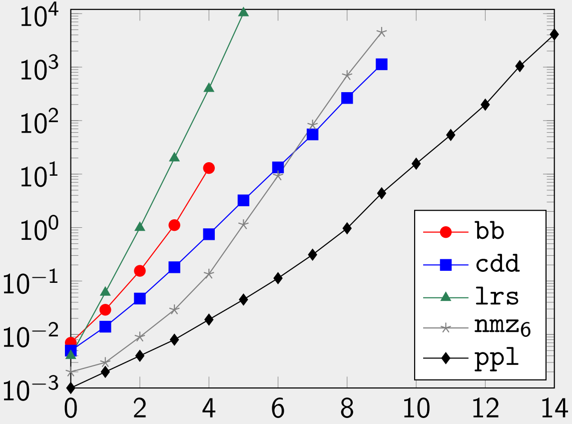
Barahona & al. 1988; Avis, Imai & Ito 2008; Bonato & al. 2014; ...

DEMO

# Facets of Cut Polytopes

## variable dimension



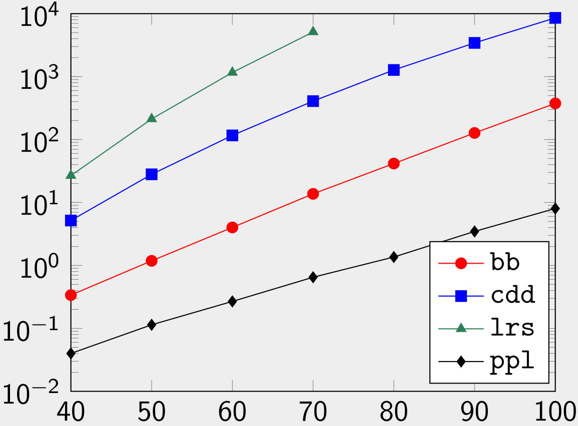

- $d = k + 6$
- $n = 2^{k+5} = \#$ cuts
- $m = 2d + 8 = 2k + 20$

- Barahona 1983: facets known if no $K_5$-minor

# Knapsack Integer Hulls

## fixed dimension, variable right hand side

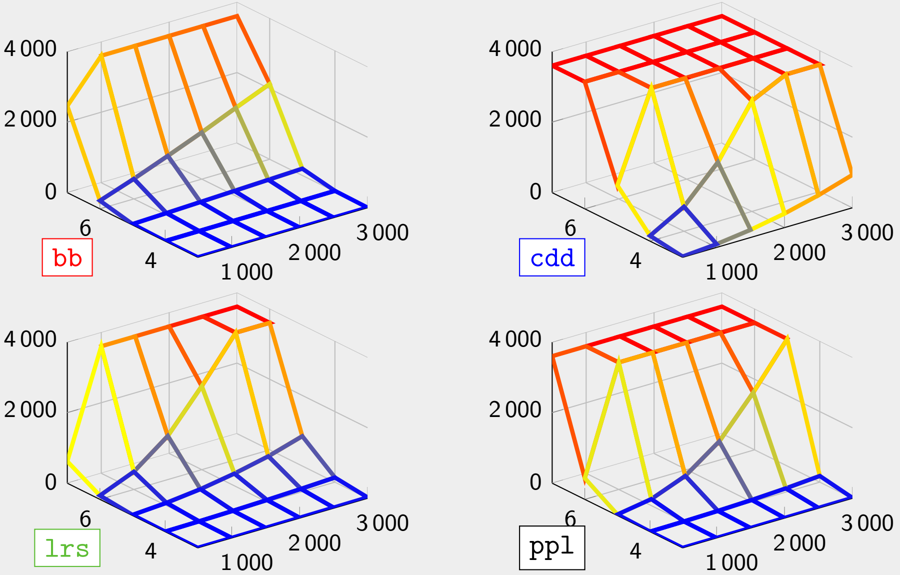
$$a_1 = 2,\ a_2 = 3,\ a_i = a_{i-2} + a_{i-1}$$
$$F_d(b) = \{x \in \mathbb{R}^d_{\geq 0} \mid a^\top x \leq b\}$$




- $d = 5$
- $n = 1366,\ 3173,\ 6509,\ 12182,\ 21245,\ 35025,\ 55157$
- $m = 12,\ 15,\ 12,\ 12,\ 8,\ 13,\ 15$

# Voronoi Diagrams of Random Points in a Box

variable dimension, variable number of points



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# Some Rules of Thumb

1. If you do not know anything about your input, try double description.
   - `cdd`, `ppl`, `nmz`
2. Do use double description for computing the facets of 0/1-polytopes.
   - `cdd`, `ppl`
3. On random input beneath-and-beyond often behaves very well.
   - `bb`
4. Use reverse search for partial information and non-degenerate input.
   - `lrs`

# Epilogue

# References


- Benjamin Assarf, Ewgenij Gawrilow, Katrin Herr, Michael Joswig, Benjamin Lorenz, Andreas Paffenholz, and Thomas Rehn, `polymake` in linear and integer programming, 2014, Preprint `arXiv:1408.4653`.
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- Katrin Herr, Thomas Rehn, and Achill Schürmann, Exploiting symmetry in integer convex optimization using core points, Oper. Res. Lett. **41** (2013), no. 3, 298–304. MR 3048847