Modelling

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Research Center MATHEON
Mathematics for Key Technologies

Berlin Mathematical School

MODAL
Mathematical Optimization and Data Analysis Laboratories

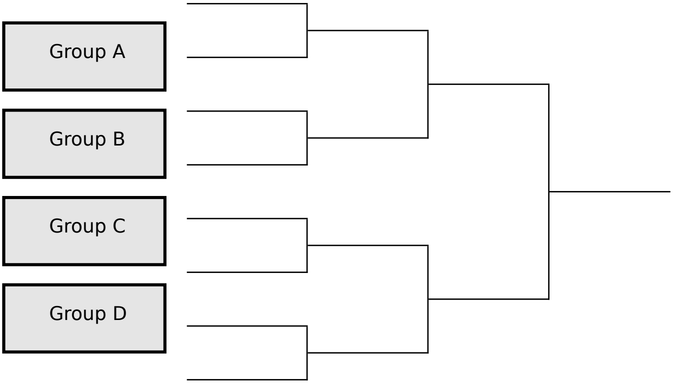
Introduction to Modelling

Stephen J. Maher

The Les Murray Cup

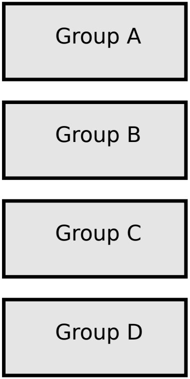
- Sydney Brazilian Social Club invited teams to join a 7-a-side one day football tournament.
- The Glebe Gorillas (my football team) was one participant.
- Varying quality of teams—one ex-Socceroos player participated.

The tournament system



- Knockout tournament with a group stage
- Three teams per group—play each team once.
- Only top two teams from each group will enter knockout.

The Equitable Coach Problem



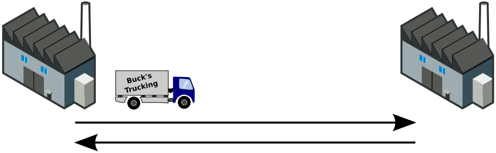
- Only guarantee of two games played.
- Everyone wants to play.
- Everyone would like a fair amount of time on the field.

What is modelling?

Describing a particular situation using a collection of logical and mathematical relationships.

- An objective function is used to evaluate alternative solutions.
- Constraints define the alternative solutions that are feasible for the situation under consideration.

Why build a model?



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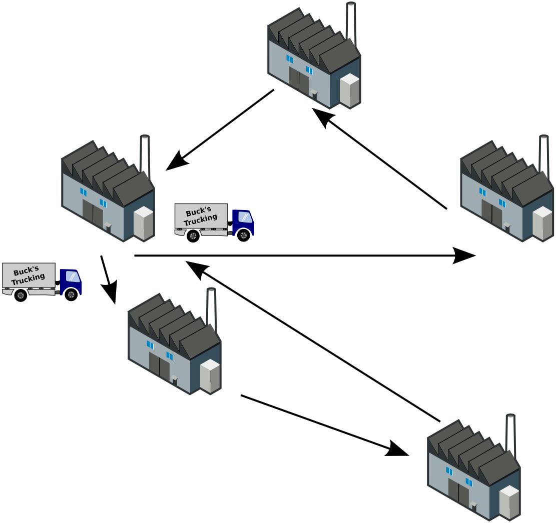
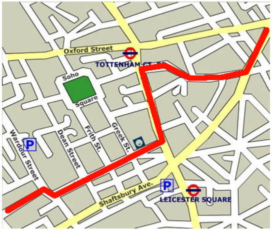


Why build a model?

- Not possible to evaluate all possibilities through experimentation.
- Difficult to examine what-if scenarios.
- Experimentation is not always possible.

Building a model

- Think about what is important to the situation and the problem considered.
- Can be an abstraction of the complete problem.
- Simplification of the problem can yield tractable problems and valuable solutions.



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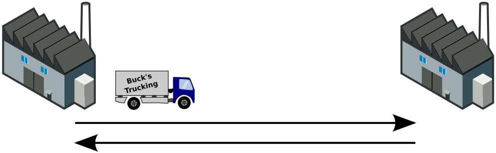
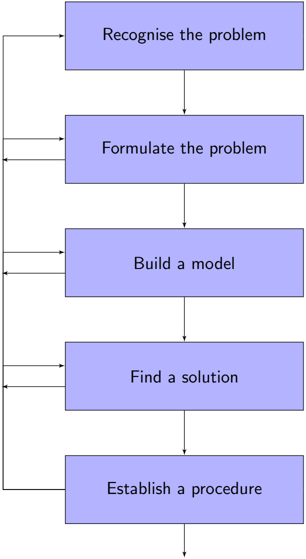


The modelling process



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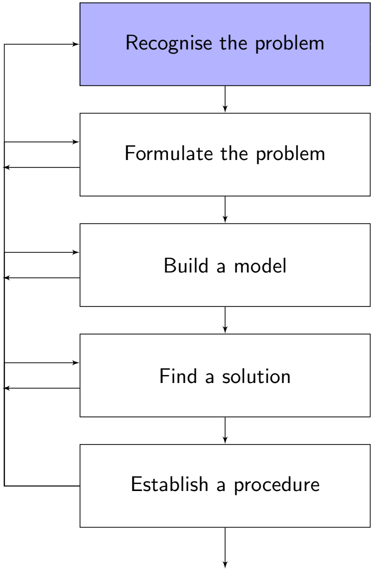
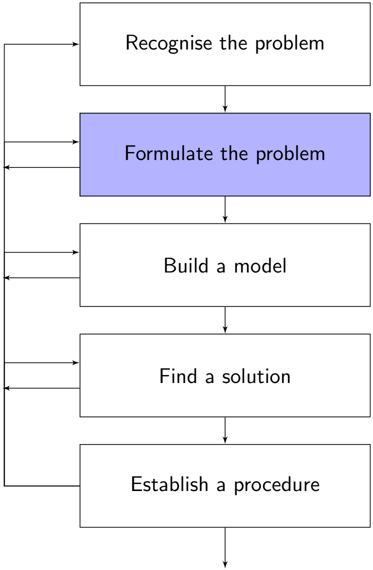


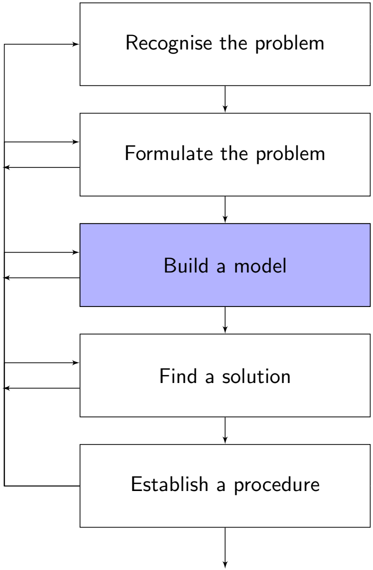
- Identify the problem.
- Could be abstract or real world.

The modelling process



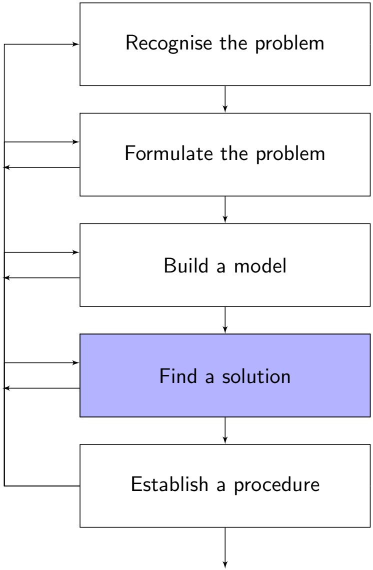
- Identify the important features of the problem.
- Define this problem in mathematical and logical notation.

The modelling process



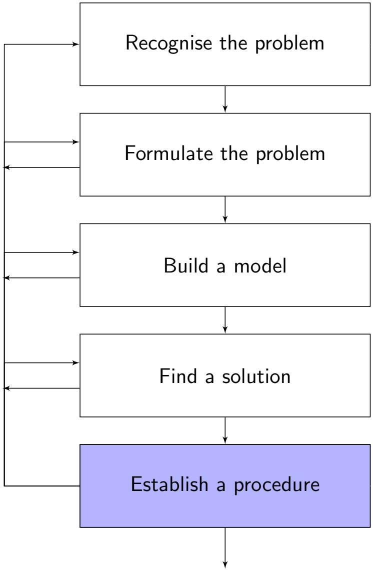
- Transfer the mathematical problem formulation to a model.
- Make use of available modelling tools—direct coding or languages (will be discussed later in this lecture).

The modelling process



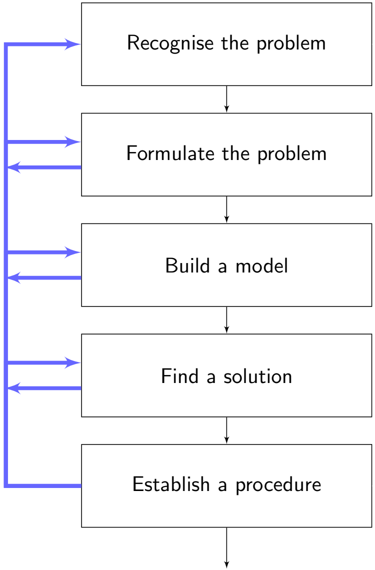
- Employ tools to solve the mathematical model.
- In our case this typically involves MINLP solvers.

The modelling process



- Not normally required to be solved once.
- Design a procedure to implement the solution for the identified problem.

The modelling process



FEEDBACK

- Each stage helps refine the previous stages.
- The modelling process aids the understanding of the problem.
- The problem understanding develops and the solution approach becomes clearer.

The Equitable Coach Problem

What is the problem?

- The team could only play two games and everyone wants to have time on the field.
- It is important to be fair to every player, while not being too disruptive to the play, i.e. not too many interchanges.
- It is also important to put a strong team on the field.

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- What are the positional preferences of the players?
- Are you equitable for only a single game or across the two matches?
- Complicated: Does winning the first match impact your decision about being equitable for the second?

Problem definition

OBJECTIVE: For a single game identify the equitable number of playing minutes for each player.

- 7 players on the field, 12 players in a team
- 20 minutes games $\Rightarrow$ 140 player minutes

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Easy Solution: Every player get 11.6667 minutes of play

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Still Easy Solution: Keeper gets 20 min, every other player get 10.909 minutes of play

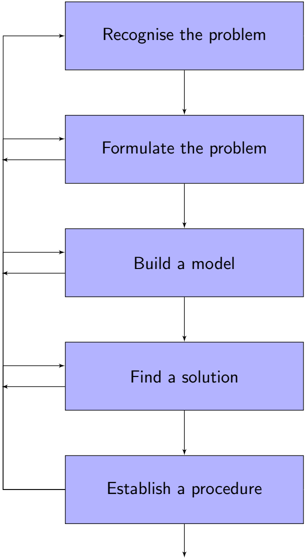
Problem definition

OBJECTIVE: For a single game identify the equitable number of playing minutes for each player given that there is only one goal keeper and players have positional preferences.

- 7 players on the field, 12 players in a team
- 20 minutes games $\Rightarrow$ 140 player minutes
- 4 different position types, Goal Keeper, Defence, Midfield and Striker.

Solution is not as easy. Need a more sophisticated mathematical model.

The modelling process - revision



Modelling tips and tricks

Finding the maximum of the minimum of functions

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- $f(x) = \max_{i \in I}\{\min g_i(x)\}$.
- This can be modelled as a linear program by,

$$
\begin{aligned}
f(x) = \min_{x,\varphi} \quad & \varphi \\
\text{s.t.} \quad & \varphi \geq g_i(x) \quad \forall i \in I \\
& x \geq 0
\end{aligned}
\tag{1}
$$

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Minimising an absolute value

- Consider an optimisation problem with an objective $\min |f(x)|$.
- Introduce two variables x^+ and x^-, where $x^+, x^- \geq 0$.
- An optimisation problem to minimise an absolute value is given by,

$$\begin{aligned} \min_{x,x^+,x^-} \quad & x^+ + x^- \\ \text{s.t.} \quad & f(x) = x^+ - x^- \\ & g_i(x) \geq b \quad \forall i \in I \\ & x \geq 0 \\ & x^+, x^- \geq 0 \end{aligned} \tag{2}$$

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Notation

- R is the set of all players r in the team,
- N is the set of all positions i in the team,
- variable $x_{r,i}$ is the number of playing minutes for player r in position i.

Modelling tips and tricks

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Maximum number of playing minutes

- Assuming a minimisation problem.
- Introduce variable φ with a positive objective coefficient.
- Include the constraints

$$\varphi \geq \sum_{i} x_{r,i} \quad \forall r \in R. \tag{3}$$

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$$\sum_i x_{r,i} - \sum_{r',i} \frac{x_{r',i}}{|R|} = \epsilon_r^+ - \epsilon_r^- \quad \forall r \in R. \tag{4}$$

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- Set the objective function to

$$\sum_r \epsilon_r^+ + \epsilon_r^- \tag{5}$$

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- Introduce variables x^+ and x^-, where $x^+, x^- \geq 0$.
- Introduce the binary variable y, i.e. $y \in \{0, 1\}$.
- The absolute value of x is given by

$$\begin{aligned} x &= x^+ - x^-, \\ 0 &\leq x^+ \leq Uy, \\ 0 &\leq x^- \leq |L|(1 - y). \end{aligned} \tag{6}$$

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Time discretisation

- Continuous time may not be always necessary.
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- Define a set of time periods T.
- Define the binary variables $x_{r,i,t}$ that indicate player r is in position i during time period t.
- The Equitable Coach Problem is now focuses on the number of time periods, not the amount of time.

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Linearisation of binary product

- Consider the binary variables x_1 and x_2.
- The product, $x_1 x_2$ can be replaced with the addition of another binary variable y.

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$$\begin{aligned} y &\leq x_1 \\ y &\leq x_2 \\ y &\geq x_1 + x_2 - 1 \\ y &\in \{0, 1\} \end{aligned} \tag{7}$$

Modelling tips and tricks

Linearisation of binary/continuous product

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$$\begin{aligned} y &\leq U x_1 \\ y &\leq x_2 \\ y &\geq x_2 - U(1 - x_1) \\ y &\geq 0 \end{aligned} \tag{8}$$