Robust optimization with the Xpress Optimizer

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About the Xpress suite

- Started by Dash Optimization (Bob **D**aniel, Robert **Ash**ford) in 1988
- Acquired by FICO (Fair Isaac) in 2008

Comes with

- A modeling language, Mosel
- A solver for most optimization problems
- API to create and solve your own optimization problems
- A modeler running on-site and on the FICO Analytic Cloud

Free academic licenses with the Academic Partnership Program[1]

[1] `http://subscribe.fico.com/Academic-Partner-Program`

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About the Xpress suite (cont'd)

Latest version: 7.9, released earlier this year.

Solves:

- LP, QP, QCQP
- MILP, MIQP, MIQCQP, MISOCP
- NLP, MINLP (Nonconvex (MI)NLPs solved to local optimality)
- CP

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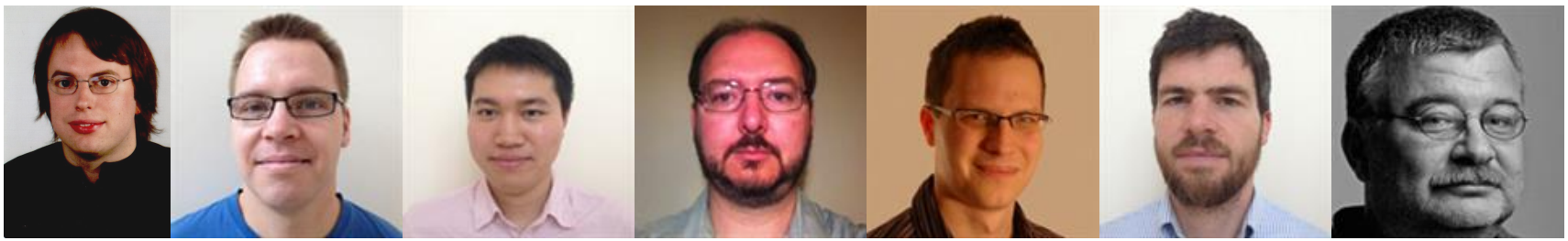
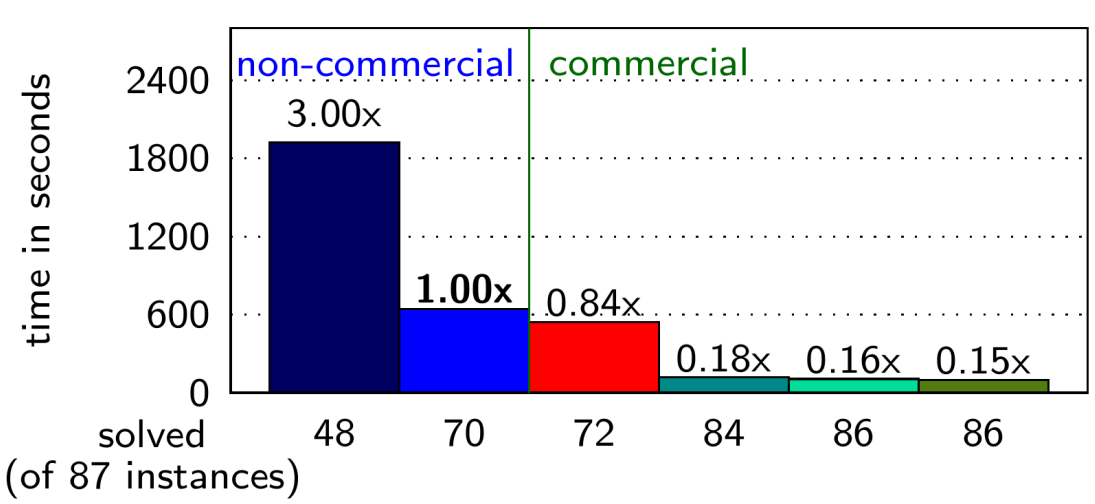
- CP

Development group based in Berlin, Birmingham, Budapest



MILP comparison (courtesy of the SCIP web page)



- CBC 2.9.4
- **SCIP 3.2.0 – SoPlex 2.2.0**
- SCIP 3.2.0 – CPLEX 12.6.2
- Gurobi 6.0.0
- CPLEX 12.6.2
- Xpress 7.9.0

data: Hans Mittelmann
graphics: ZIB

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Today: Robust Optimization

We'll put the solver to work for solving an interesting meta-class of problems: optimization under uncertainty

Examples, exercises, and slides can be downloaded at

`zib.de/berthold/rob-opt.zip`

Outline

- Robust Optimization at a glance
- The opponent's viewpoint
- A ridiculously fast intro to Mosel
- Exercise session I

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- Robust Optimization at a glance
- The opponent's viewpoint
- A ridiculously fast intro to Mosel
- Exercise session I
- Theory and pitfalls of RO
- Types of uncertainty sets
- Exercise session II

The name of the game

- Robust optimization is a paradigm for modeling optimization problems under uncertainty

i.e. One or more of the problem's **parameters** are unknown

- We only have limited information on their value
- We assume these parameters vary in a **well-defined** uncertainty set

Robust optimization

- Real-life problems have uncertainty in their data
- Some might be missing or affected by inaccuracy, measurement errors, etc.
- However, sometimes there's a limit on the uncertainty

$\Rightarrow$ We have an uncertainty set

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$\Rightarrow$ We have an uncertainty set

- There are a few ties with stochastic programming, but RO is an entirely different approach.

Example

$$
\begin{array}{ll}
\min & 2x_1 + 3x_2 \\
\text{s.t.} & u x_1 + 2x_2 \geq 5 \\
& 2x_1 + x_2 \geq v \\
& x_1, x_2 \geq 0
\end{array}
$$

Uncertain parameters: u and v. The only information we have on them is $u \in [1, 2]$, $v \in [5, 6]$.

Main assumptions

- Once the uncertainty set is defined, the uncertain parameters can take any value in it

$\Rightarrow$ RO does **not** use any probabilistic information on the uncertainty set

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$\Rightarrow$ RO does **not** use any probabilistic information on the uncertainty set

☺ The problem is easier

☹ Throwing away lots of useful info

RO’s challenge

Find a solution that is feasible for **all** possible realizations of the uncertain parameters.

Example

$$\begin{array}{ll} \min & 2x_1 + 3x_2 \\ \text{s.t.} & u_1 x_1 + u_2 x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Uncertainty set: $U = \{(2,1),(1,2)\}$.

- $(x_1, x_2) = (3,0)$ is infeasible because it violates the robust constraint $x_1 + 2x_2 \geq 6$

Example

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- $(x_1, x_2) = (3,0)$ is infeasible because it violates the robust constraint $x_1 + 2x_2 \geq 6$
- $(x_1, x_2) = (0,3)$ is infeasible because it violates the robust constraint $2x_1 + x_2 \geq 6$

Example

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- $(x_1, x_2) = (3,0)$ is infeasible because it violates the robust constraint $x_1 + 2x_2 \geq 6$
- $(x_1, x_2) = (0,3)$ is infeasible because it violates the robust constraint $2x_1 + x_2 \geq 6$
- $(x_1, x_2) = (2,2)$ is feasible and optimal

Example

$$
\begin{array}{ll}
\min & 2x_1 + 3x_2 \\
\text{s.t.} & x_1 + 2x_2 \geq 6 \\
& 2x_1 + x_2 \geq 6 \\
& x_1, x_2 \geq 0
\end{array}
$$

The trick was simply to impose the constraint for all elements of the uncertainty set.

Not-yet-robust optimization

Forget uncertainty for a second. Your optimization problem has

- parameters: known before solving the problem.
- variables: unknown, but we find them by solving a problem

Variables express quantities we make **decisions** on.

- We have control over them, we decide their value
- We want to set them so as to minimize some objective

Robust optimization

In RO, an optimization problem has

- parameters: known before solving the problem.
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- uncertains: unknown, decided by somebody else

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In RO, an optimization problem has

- parameters: known before solving the problem.
- variables: unknown, decided by us
- uncertains: unknown, decided by somebody else

Unlike variables, we have no control over uncertainties.

- In fact, they may be known **after** we made our decision.
- Remember: we have to make our decisions so that they are feasible for any uncertain realization

The opponent's viewpoint

Uncertains in RO are decisions made by an opponent.

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- Think about your favorite supervillain (Voldemort, the Joker, Skynet, Gargamel, Thanos, . . .)

The opponent's viewpoint

Uncertains in RO are decisions made by an opponent.

- Think about your favorite supervillain (Voldemort, the Joker, Skynet, Gargamel, Thanos, . . .)
- **After** we're done optimizing, it's their turn:
  - They see our solution
  - They know the uncertainty set
  - They pick the uncertains that will do us **maximum** harm

  i.e. Make any of our constraints violated

$\Rightarrow$ They are also solving an optimization problem: the uncertains are their variables

- This is akin to a *leader-follower* game

The opponent is quite real

- Nature
- Competitors
- Market
- Customers
- Suppliers
- All of the above, aka Murphy’s law

Example

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & u_1 x_1 + u_2 x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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Uncertainty set: $U = \{(2,1),(1,2)\}$.

If we picked $(x_1, x_2) = (1.1, 2.5)$ as a solution, the opponent would search $(u_1, u_2) \in U$ such that

$$1.1u_1 + 2.5u_2 < 6$$

So $(u_1, u_2) = (2,1)$ would be the opponent's solution, and ours would be proven infeasible.

Example

$$\begin{array}{ll} \min & 2x_1 + 3x_2 \\ \text{s.t.} & u_1 x_1 + u_2 x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Uncertainty set:

$$U = \{(u_1, u_2) \in \mathbb{R}^2_+ : u_1 + u_2 \geq 3, u_1 + 2u_2 \geq 4\}.$$

Suppose we pick again $(x_1, x_2) = (1.1, 2.5)$ as a solution.

Example (cont'd)

The opponent would then solve

$$\begin{array}{ll} \min & 1.1u_1 + 2.5u_2 \\ \text{s.t.} & u_1 + u_2 \geq 3 \\ & u_1 + 2u_2 \geq 4 \end{array}$$

A simple LP that gives $(u_1, u_2) = (4, 0)$, with an objective of $4.4 < 6$.

- The opponent managed to break our constraint,

i.e. Our solution is not robust

Solving RO problems

In order to solve a RO problem, we must

- Assume that the opponent will try to invalidate solution

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$\Rightarrow$ Create a robust counterpart of our optimization problem

The RC is just another optimization problem (if one exists). It just embeds the opponent's optimization problem.

Example

$$
\begin{array}{ll}
\min & 2x_1 + 3x_2 \\
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Example

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Uncertainty set: $u \in [1, 2]$, $v \in [5, 6]$.

The RC is trivial: the constraints will be hardest if $u = 1$ and $v = 6$, so the RC is

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 5 \\ & 2x_1 + x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions?

If you haven't any, here are two:

- What if there are uncertains in the objective?

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If you haven't any, here are two:

- What if there are uncertains in the objective?
- What if two or more constraints are affected by uncertains?

We'll answer these later.

A quick intro to Xpress-Mosel

Mosel is a modeling language for optimization problems.

- It can do a lot of other things
- But we'll deal with optimization only

Typical

```
model "hello world"
declarations
  y, x: mpvar              ! This is a comment
end-declarations
y + 2*x <= 10
y + x <= 5
maximize(2*x + 3*y)
writeln("x: ", getsol(x), "; Obj: ", getobjval)
end-model
```

Declaration section

This is where all variables, parameters, sets, and uncertains are declared

```
model "my model"
uses "mmxprs", "mmrobust"
declarations
  x, y: mpvar                    ! variables
  a: array (1..4) of real        ! vector of parameters
  b: real                        ! single parameters
  R: range                       ! set i..j, def'd later
  u: uncertain                   ! uncertain parameter
end-declarations
[...]
```

Constraints

Not too far from how they are formulated in other languages...

```
model "my model"
declarations
  P: range
  a: array(P) of real
  b: real
  x: array(P) of mpvar
  [...]
end-declarations
c1 := sum (i in range) a(i) * x(i) <= b
[...]
```

The "`c1 :=`" is not necessary

Objective

```
maximize (sum (i in P) a(i)* x(i))
writeln ("Obj: ", getobjval)
```

Miscellaneous

Loops:

```
forall (i in P) do
  writeln ("x_", i, ": ", getsol (x(i)))
end-do

forall (i in P: i <= 2) do
  writeln ("x_", i, ": ", getsol (x(i)))
end-do
```

Long comments:

```
(!
  Write whatever you want here.
  I'll just ignore it
!)
```

Variable types and bounds

To be done outside of the declaration section

```
forall (i in P) do
  x(i) is_binary
end-do
y is_integer
z >= 4
z <= z_upper ! z_upper is a parameter
```

Reading data from files

```
initializations from 'datafile.txt'
   a n
end-initializations
```

The file `datafile.txt` must have a specific format

Modules

We have to tell Mosel the modules we want to use. We'll use two:

```
model "my model"
uses "mmxprs", "mmrobust"
[...]
end-model
```

Example: knapsack

$$\begin{array}{ll} \max & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}^\top \boldsymbol{x} \leq b \\ & \boldsymbol{x} \in \{0,1\}^n \end{array}$$

Writing the model is easy. We'll have to set up a data file with n, $\boldsymbol{a}$ and b.

Example: knapsack (cont'd)

```
model "knapsack"
uses "mmxprs"
declarations
  P: range
  c: array(P) of real
  a: array(P) of real
  b: real
  x: array(P) of mpvar
end-declarations
initializations from 'knapsack.dat'
  c a b
end-initializations
forall (i in P) x(i) is_binary
sum (i in P) a(i) * x(i) <= b
maximize (sum (i in P) c(i) * x(i))
end-model
```


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The `knapsack.dat` file

```
n: 6
c: [(1) 34 (2) 37 (3) 41 (4) 49 (5) 52 (6) 55]
a: [(1) 56 (2) 38 (3) 32 (4) 24 (5) 22 (6) 20]
b: 100
```

Example: **robust** knapsack

$$\begin{array}{ll} \max & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & (\boldsymbol{a}+\boldsymbol{u})^\top \boldsymbol{x} \leq b \\ & \boldsymbol{x} \in \{0,1\}^n \end{array}$$

Uncertainty set:

$$U = \{\boldsymbol{u} \in \mathbb{R}^n_+ : \boldsymbol{d}^\top \boldsymbol{u} \leq f\},$$

where $\boldsymbol{d} \in \mathbb{Q}^n_+$ and $f > 0$ is a scalar.

Example: **robust** knapsack (cont'd)

```
model "knapsack"
declarations
  [...]
  x: array(P) of mpvar
  d: array(P) of real          ! data for unc. set
  f: real
  u: array(P) of uncertain     ! unc parameters
end-declarations
initializations from 'knapsack_robust.dat'
  c a b d f
end-initializations
forall (i in P) x(i) is_binary
sum (i in P) (a(i) + u(i)) * x(i) <= b
sum (i in P) d(i) * u(i) <= f
forall (i in P) u(i) >= 0
[...]
```

The knapsack_robust.dat file

```
n: 6
c: [(1) 34 (2) 37 (3) 41 (4) 49 (5) 52 (6) 55]
a: [(1) 56 (2) 38 (3) 32 (4) 24 (5) 22 (6) 20]
d: [(1) 11 (2) 12 (3)  8 (4) 13 (5) 11 (6) 13]
f: 1
b: 100
```

Unanswered questions in RO

- What if there are uncertains in the objective?
- What if two or more constraints are affected by uncertains?

Remember: we are looking for a solution that is feasible **regardless** of the uncertain parameters.

What if there are uncertains in the **objective**?

Suppose the uncertain parameters are only in the objective.

- Feasibility of any solution is independent of the uncertains
- In principle, optimality would depend on the uncertains

$\Rightarrow$ We would have to deal with robust optimality, an entirely different issue

- We'll use a trick to reduce this to a feasibility problem.

Example: knapsack

$$\begin{array}{ll} \max & (\boldsymbol{c}+\boldsymbol{u})^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}^\top \boldsymbol{x} \leq b \\ & \boldsymbol{x} \in \{0,1\}^n \end{array}$$

Uncertainty set: $U = \{\boldsymbol{u} \in \mathbb{R}^n_+ : \boldsymbol{d}^\top \boldsymbol{u} \leq f\}$.

Reformulate

$$
\begin{array}{ll}
\max & z \\
\text{s.t.} & z - (\boldsymbol{c} + \boldsymbol{u})^\top \boldsymbol{x} \leq 0 \\
& \boldsymbol{a}^\top \boldsymbol{x} \leq b \\
& \boldsymbol{x} \in \{0,1\}^n
\end{array}
$$

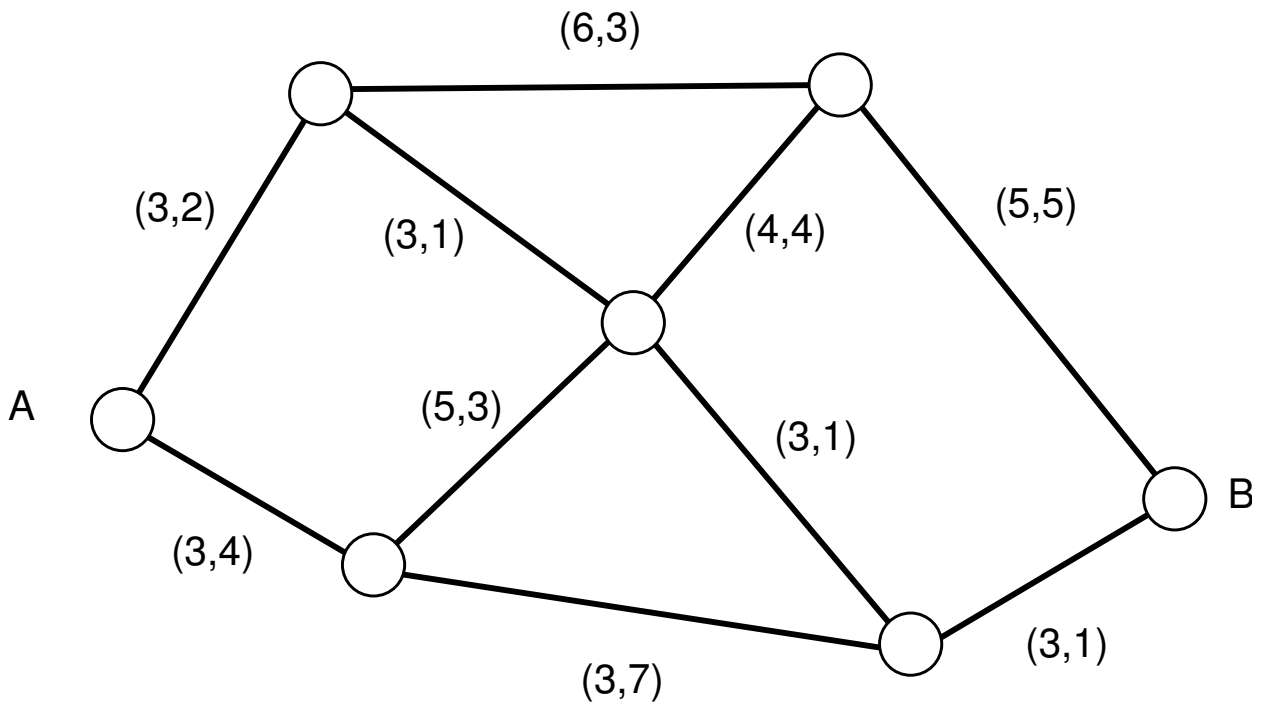
Now we have a robust problem in the classical sense.

Example: robust shortest path

Find the shortest route from A to B on the city's road network.

- It takes c_e minutes to drive on road e
- Unless there's construction work, and then it's $c_e + d_e$
- We don't know where the construction work is
- **But** we know it is on at most k roads

Each link e has (c_e, d_e). $k = 2$ construction zones.



How long to get from A to B?

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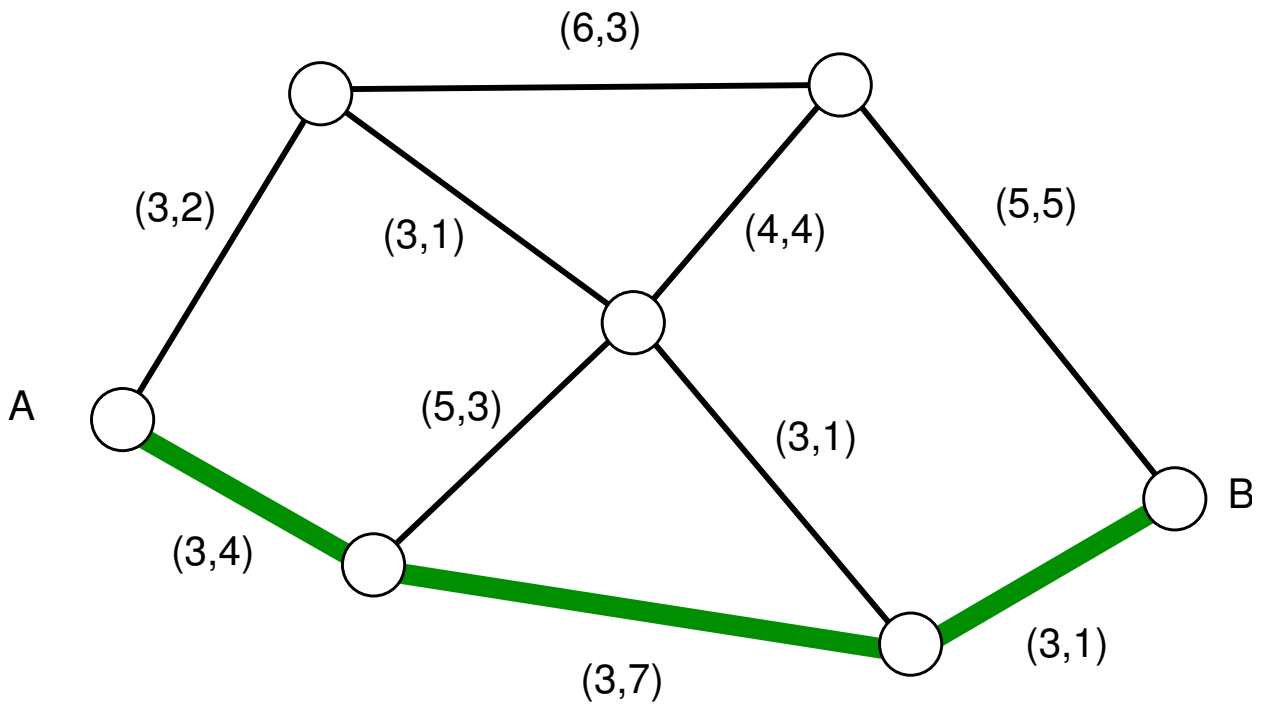
Each link e has (c_e, d_e). $k = 2$ construction zones.



How long to get from A to B? In theory, 9 minutes

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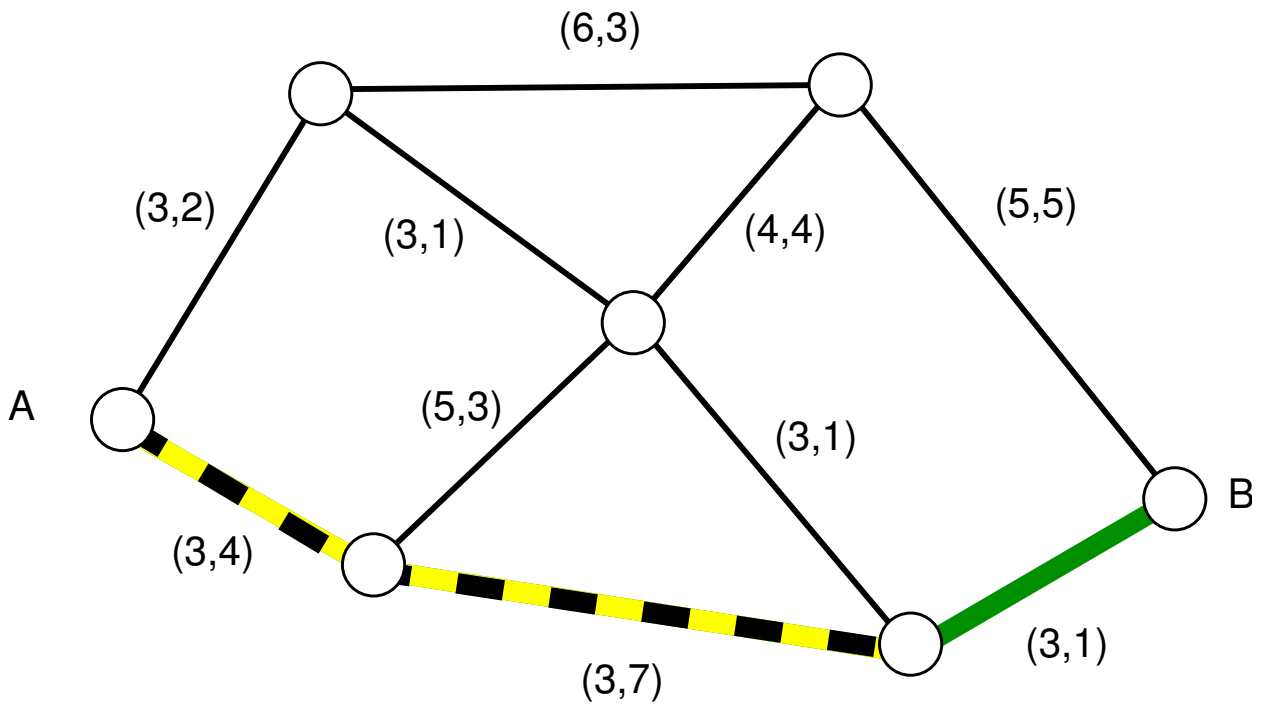
Each link e has (c_e, d_e). $k = 2$ construction zones.



How long to get from A to B? In practice, 20 min.

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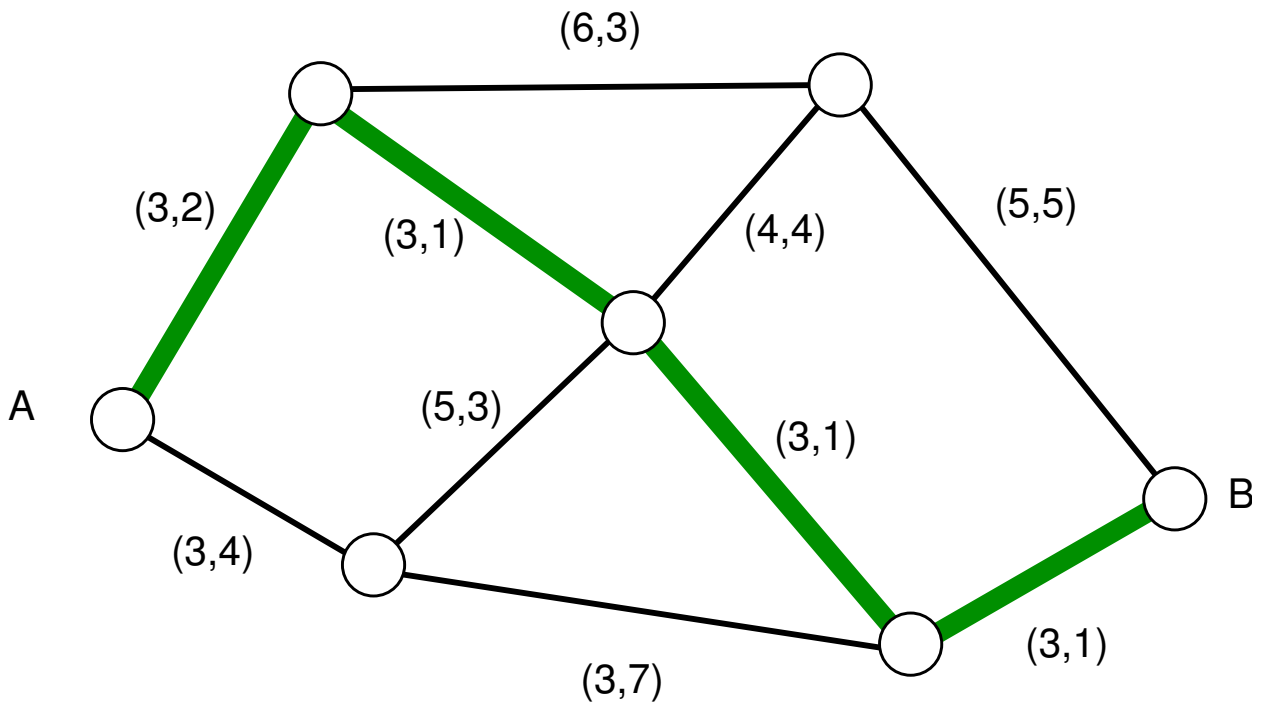
Each link e has (c_e, d_e). $k = 2$ construction zones.



How long to get from A to B? Nominal 12 min. (worse than 9)

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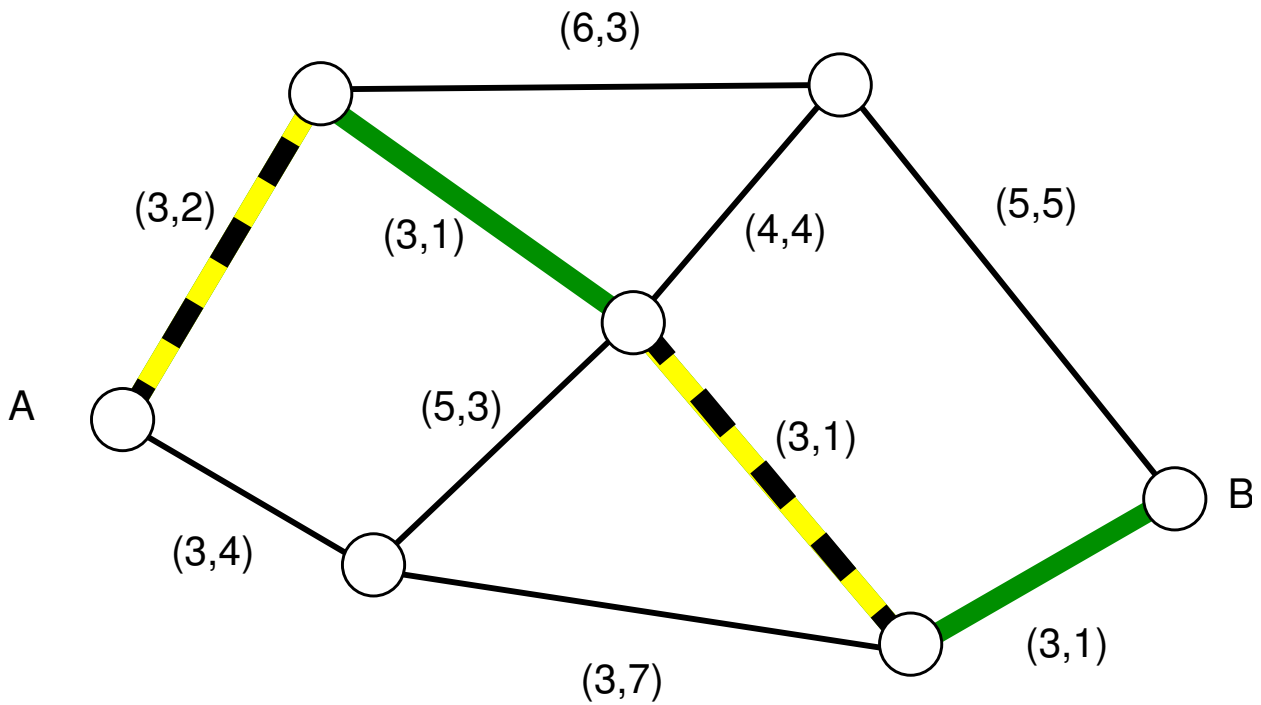
Each link e has (c_e, d_e). $k = 2$ construction zones.



How long to get from A to B? Robust 15 min. (better than 20)

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What if 2+ constraints are affected by uncertains?

Remember: the opponent simply wants to give us a hard time.

His/her method:

1. Observe our solution
2. Find point of uncertainty set so that our solution violates at least one of our constraint
3. So the opponent can focus on one constraint at a time

$\Rightarrow$ Quite a conservative approach: equivalent to one opponent per constraint!

Finally, some theory

So far, we've seen:

- What RO does to a problem
- How to solve some ROs
- How to implement them using Mosel

But how do we solve a general RO?

Robust optimization: a general-enough case

$$
\begin{array}{ll}
\min & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\
& \sum_{i=1}^{n} \alpha_i u_i x_i \leq \beta \\
& \boldsymbol{x} \geq \boldsymbol{0}.
\end{array}
$$

Uncertainty set: $U = \{\boldsymbol{u} \in \mathbb{R}^n_+ : P\boldsymbol{u} \leq \boldsymbol{q}\}$.
U is a polyhedron defined by a system of linear inequalities.

What would an opponent do?

The opponent would look at the single uncertain constraint:

$$\sum_{i=1}^{n} \alpha_i u_i x_i^\star \leq \beta,$$

and think: "$x_i^\star$'s are known. How do I make this violated?"

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$$\sum_{i=1}^{n} \alpha_i u_i x_i^\star \leq \beta,$$

and think: "$x_i^\star$'s are known. How do I make this violated?"

The opponent has a PhD in Optimization, so the answer is:

"I'll maximize $\sum_{i=1}^{n} \alpha_i x_i^\star u_i$! Mwahahahahahahaha!"

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The opponent has a PhD in Optimization, so the answer is:

"I'll maximize $\sum_{i=1}^{n} \alpha_i x_i^\star u_i$! Mwahahahahahahaha!"

The reason: maximizing the left-hand side gives the highest chance to make the constraint violated.

If the solution found is such that $\sum_{i=1}^{n} \alpha_i x_i^\star u_i > \beta$, the opponent wins.

What would an opponent do?

If the constraint were of the opposite sign,

$$\sum_{i=1}^{n} \alpha_i u_i x_i \geq \beta,$$

the plan (equally evil) would be
"I'll m**in**imize $\sum_{i=1}^{n} \alpha_i u_i x_i$! Mwahahahahahahaha!"

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the plan (equally evil) would be
"I'll m**in**imize $\sum_{i=1}^{n} \alpha_i u_i x_i$! Mwahahahahahahaha!"
Equality constraints are simply split into $\leq$ and $\geq$ constraints

The robust counterpart

$$
\begin{array}{ll}
\min & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \max_{\boldsymbol{u} \in U} \sum_{i=1}^{n} \alpha_i u_i x_i \leq \beta \\
& \boldsymbol{x} \geq \boldsymbol{0}.
\end{array}
$$

The opponent's optimization problem

$$\begin{array}{ll} \max & \sum_{i=1}^{n} \alpha_i x_i u_i \\ \text{s.t.} & \boldsymbol{P}\boldsymbol{u} \leq \boldsymbol{q} \\ & \boldsymbol{u} \geq \boldsymbol{0}. \end{array}$$

Here, x_i's are given. The variables (i.e. the decision that are controlled by the opponent) are the u_i's

Duality, our savior

If the opponent has this problem:

$$\begin{array}{ll} \max & \sum_{i=1}^{n} \alpha_i x_i u_i \\ \text{s.t.} & \boldsymbol{P}\boldsymbol{u} \leq \boldsymbol{q} \\ & \boldsymbol{u} \geq \boldsymbol{0}. \end{array}$$

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The dual is (for dual variables $\boldsymbol{y}$)

$$\begin{array}{ll} \min & \boldsymbol{q}^\top \boldsymbol{y} \\ \text{s.t.} & P_{i,\cdot}^\top \boldsymbol{y} \geq \alpha_i x_i \quad \forall i = 1, 2, \ldots, n \\ & \boldsymbol{y} \geq \boldsymbol{0}. \end{array}$$

Duality, our savior

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$$\begin{array}{ll} \max & \sum_{i=1}^{n} \alpha_i x_i u_i \\ \text{s.t.} & \boldsymbol{P}\boldsymbol{u} \leq \boldsymbol{q} \\ & \boldsymbol{u} \geq \boldsymbol{0}. \end{array}$$

The dual is (for dual variables $\boldsymbol{y}$)

$$\begin{array}{lll} \min & \boldsymbol{q}^\top \boldsymbol{y} & \\ \text{s.t.} & P_{i,\cdot}^\top \boldsymbol{y} \geq \alpha_i x_i & \forall i = 1, 2, \ldots, n \\ & \boldsymbol{y} \geq \boldsymbol{0}. & \end{array}$$

However, strong duality dictates:

If an optimal solution $(\boldsymbol{u}, \boldsymbol{y})$ exists, then $\sum_{i=1}^{n} \alpha_i x_i u_i = \boldsymbol{q}^\top \boldsymbol{y}$

The robust counterpart[2]

From a difficult (nonlinear) problem

$$
\begin{array}{ll}
\min & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & \max_{\boldsymbol{u} \in U} \sum_{i=1}^{n} \alpha_i u_i x_i \leq \beta \\
 & \boldsymbol{x} \geq \boldsymbol{0}.
\end{array}
$$

[2] Soyster, Ben Lev, Toof, "Conservative linear programming with mixed multiple objectives." *Omega 5.2* (1977): 193-205.

The robust counterpart[2]

From a difficult (nonlinear) problem

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \max_{\boldsymbol{u}\in U} \sum_{i=1}^{n} \alpha_i u_i x_i \leq \beta \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

we get a much nicer problem (an LP!):

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{q}^\top \boldsymbol{y} \leq \beta \\ & P_{i,\cdot}^\top \boldsymbol{y} \geq \alpha_i x_i \quad \forall i = 1, 2, \ldots, n \\ & \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}. \end{array}$$

[2] Soyster, Ben Lev, Toof, "Conservative linear programming with mixed multiple objectives." *Omega 5.2* (1977): 193-205.

Non-polyhedral uncertainty sets

In the general case, we have seen that a polyhedral uncertainty set can be dealt with easily.

- If the original problem is a LP, the RC is an LP as well
- If the original problem is a MILP, the RC is a MILP

So polyhedral uncertainty doesn't add to the complexity of the problem

Ellipsoidal uncertainty

Suppose the uncertainty set has been estimated as an ellipsoid defined by mean/covariance data:

$$U = \{\boldsymbol{u} \in \mathbb{R}^n : (\boldsymbol{u} - \bar{\boldsymbol{u}})^\top Q(\boldsymbol{u} - \bar{\boldsymbol{u}}) \leq \epsilon\}$$

Hence $\bar{\boldsymbol{u}}$ is the mean value of $\boldsymbol{u}$ and Q is the covariance matrix of $\boldsymbol{u}$.

Ellipsoidal uncertainty $\rightarrow$ a conic RC

It is easy to show (we won't) that the resulting robust counterpart becomes

- a SOCP if the initial problem was a LP
- a MISOCP if the initial problem was a MILP

Ellipsoidal uncertainty in Mosel

```
model "quadknapsack"
declarations
  P: range
  c: array(P) of real
  a: array(P) of real
  b: real
  x: array(P) of mpvar
  u: array(P) of uncertain  ! unc parameters
end-declarations
initializations from 'knapsack_robust_eps.dat'
  c a b eps
end-initializations
forall (i in P) x(i) is_binary
sum (i in P) (a(i) + u(i)) * x(i) <= b
sum (i in P) u(i)^2 <= eps
[...]
```


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Scenario uncertainty

Suppose there is a database of K historical values of the vector $\boldsymbol{u}$ of uncertain parameters: $U = \{\boldsymbol{u}^1, \boldsymbol{u}^2, \cdots, \boldsymbol{u}^K\}$.

- Problem: be robust against all previous occurrences of the vector $\boldsymbol{u}$
- Solution: specify $\boldsymbol{u}$ as a (discrete) set of occurrences

So the robust counterpart is

$$\begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \sum_{i=1}^{n} \alpha_i u_i^k x_i \leq \beta \quad \forall k = 1, 2, \ldots, K \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

Example: knapsack with historical values

```
declarations:
  [...]
  hist_data: array (1..4, set of uncertain) of real
  [...]
end-declarations

hist_data (1,u(1)) := 12
hist_data (1,u(2)) := 14
hist_data (1,u(3)) := 20
[...]
hist_data (4,u(3)) := 10

scenario (hist_data)
```

References

- Ben-Tal, A., El Ghaoui, L., Nemirovski, A. (2009). Robust optimization. Princeton University Press.
- Bertsimas, D., Sim, M. (2004). The price of robustness. *Operations Research*, 52(1), 35-53.

Thank You

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Real-world cases: Liquid gas production[3]

Production planning at a liquid oxygen/nitrogen plant:

- Given customer demands at the beginning of the month
- Plan production for each day of the month
- Store extra daily production in inventory (big tanks)
- Maximum production P, tank capacity C

[3] Latifoglu, C., Belotti, P., Snyder, L.V. (2013). Models for production planning under power interruptions. *Naval Research Logistics* 60(5):413-431.

Production planning under energy uncertainty

- Production very energy-intensive $\Rightarrow$ expensive electric bill
- Interruptible Load Contract (ILC): power company can suspend supply in periods of high demand (summer)
- At most k interruptions each month (8 hours each)
- **Cheaper** (per kWh) than with uninterrupted contract

Production planning under energy uncertainty

- Production very energy-intensive $\Rightarrow$ expensive electric bill
- Interruptible Load Contract (ILC): power company can suspend supply in periods of high demand (summer)
- At most k interruptions each month (8 hours each)
- **Cheaper** (per kWh) than with uninterrupted contract

The power supplier won't tell us when the interruptions will be.

- Treat interruptions as uncertains

$\Rightarrow$ Plan production so that even with the worst-case k interruptions we satisfy customer demand

Original model

```
declarations
  produce:    array (PERIODS, GASES) of mpvar
  inventory:  array (PERIODS, GASES) of mpvar

end-declarations

forall(t in PERIODS, g in GASES) do
  inventory(0,g) + sum(tp in PERIODS | tp <= t)
    (produce (tp,g) - DEMAND(tp,g)) >= 0
  inventory (t,g) <= INV_CAP  (g)
  produce   (t,g) <= PROD_CAP (g)
end-do


minimize(sum (t in PERIODS, g in GASES)
 (PROD_COST * produce(t,g) + INV_COST * inventory(t,g)))
```

Robust model

```
declarations
  produce:    array (PERIODS, GASES) of mpvar
  inventory:  array (PERIODS, GASES) of mpvar
  interrupt:  array (PERIODS) of uncertain
end-declarations

forall(t in PERIODS, g in GASES) do
  inventory(0,g) + sum(tp in PERIODS | tp <= t)
    ((1-interrupt(tp))*produce(tp,g)-DEMAND(tp,g))>=0
  inventory (t,g) <= INV_CAP  (g)
  produce   (t,g) <= PROD_CAP (g)
end-do

sum(t in PERIODS) interrupt (t) <= MAX_NINTERR

minimize(sum (t in PERIODS, g in GASES)
 (PROD_COST * produce(t,g) + INV_COST * inventory(t,g)))
```

Modeling advantage

Modeling this is much easier with the `mmrobust` module.

The alternative: add the dual constraints to the model.

- Error prone
- Only suited to a user with good knowledge of optimization, duality, and robust optimization

Original problem

$$
\begin{array}{lll}
\min & \sum_{t \in T} c_g^{\text{prod}} \text{prod}_{tg} + c_g^{\text{inv}} \text{inv}_{tg} & \\
\text{s.t.} & \text{inv}_{0,g} + \sum_{\tau \le t} (\text{prod}_{\tau,g} - \text{dem}_{\tau,g}) \ge 0 & \forall t \in T \\
& 0 \le \text{prod}_{t,g} \le P_g & \forall t \in T, \tau \in T \\
& 0 \le \text{inv}_{t,g} \le C_g & \forall t \in T, \tau \in T
\end{array}
$$

Robust problem

$$
\begin{array}{lll}
\min & \sum_{t\in T} c_g^{\text{prod}}\text{prod}_{tg} + c_g^{\text{inv}}\text{inv}_{tg} & \\
\text{s.t.} & \text{inv}_{0,g} + \sum_{\tau\le t}(\text{prod}_{\tau,g} - \text{dem}_{\tau,g}) & \\
& \quad -\sum_{\tau\in T}\mu_\tau^t + k\sigma^t \ge 0 & \forall t \in T \\
& \mu_\tau^t + \sigma^t \ge \text{prod}_{\tau,g} & \forall t \in T, \tau \le t \\
& \mu_\tau^t + \sigma^t \ge 0 & \forall t \in T, \tau > t \\
& \mu_\tau^t \ge 0, \sigma^t \ge 0 & \forall t \in T, \tau \in T \\
& 0 \le \text{prod}_{t,g} \le P_g & \forall t \in T, \tau \in T \\
& 0 \le \text{inv}_{t,g} \le C_g & \forall t \in T, \tau \in T
\end{array}
$$

Exercise session II (25-30 minutes)

Complete exercise4.mos.

Exercise session I (25-30 minutes)

Complete exercise1.mos, exercise2.mos, and exercise3.mos.