Solving Linear Programs: The Dual Simplex Algorithm

Outline

- LP basics
- Primal and dual simplex algorithms
- Implementing the dual simplex algorithm

Some Basic Theory

Linear Program – Definition

A **linear program** (**LP**) in **standard form** is an optimization problem of the form

$$\begin{array}{lll} \textit{Minimize} & c^T x & \\ \textit{Subject to} & Ax = b & \text{(P)} \\ & x \geq 0 & \end{array}$$

Where $c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathbf{R}^{m \times n}$, and x is a vector of n **variables**. $c^T x$ is known as the **objective function**, $Ax = b$ as the **constraints**, and $x \geq 0$ as the **nonnegativity** conditions. b is called the **right-hand side**.

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Dual Linear Program – Definition

The **dual** (or **adjoint) linear program** corresponding to (P) is the optimization problem

$$\begin{array}{lll} \textit{Maximize} & & b^T\pi \\ \textit{Subject to} & A^T\pi \leq c & \\ & \pi \ \textit{free} & \end{array} \qquad \text{(D)}$$

In this context, (P) is referred to as the **primal linear program.**

$$\left(\begin{array}{c} \textbf{Primal} \\ \begin{array}{lll} \textit{Minimize} & & c^T x \\ \textit{Subject to} & Ax = b & \\ & x \geq 0 & \end{array} \end{array} \right)$$

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Weak Duality Theorem

(von Neumann 1947)

Let x be feasible for (P) and π feasible for (D). Then

Maximize $b^T\pi \leq c^Tx$ Minimize

If $b^T\pi = c^Tx$, then x is optimal for (P) and π is optimal for (D); moreover, if either (P) or (D) is **unbounded**, then the other problem is **infeasible**.

Proof:

$$\pi^Tb = \pi^TAx \leq c^Tx \quad \blacksquare$$

$Ax = b$

$\pi^TA \leq c^T \;\&\; x \geq 0$

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Solving Linear Programs

- Three types of algorithms are available
  - Primal simplex algorithms (Dantzig 1947)
  - Dual simplex algorithms (Lemke 1954)
    - Developed in context of game theory
  - Primal-dual log barrier algorithms
    - Interior-point algorithms (Karmarkar 1989)
    - Reference: Primal-Dual Interior Point Methods, S. Wright, 1997, SIAM

Primary focus: Dual simplex algorithms

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Basic Solutions – Definition

Let B be an ordered set of m distinct indices $(B_1, \ldots, B_m)$ taken from $\{1,\ldots,n\}$. B is called a **basis** for (P) if A_B is nonsingular. The variables x_B are known as the **basic variables** and the variables x_N as the **non-basic** variables, where $N = \{1,\ldots,n\} \setminus B$. The corresponding **basic solution** $X \in R^n$ is given by $X_N = 0$ and $X_B = A_B^{-1} b$. B is called (**primal**) **feasible** if $X_B \geq 0$.

Note: $AX = b \Rightarrow A_B X_B + A_N X_N = b \Rightarrow A_B X_B = b \Rightarrow X_B = A_B^{-1} b$

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Primal Simplex Algorithm
(Dantzig, 1947)

Input: A feasible basis B and vectors

$$X_B = A_B^{-1}b \quad \text{and} \quad D_N = c_N - A_N^T A_B^{-T} c_B.$$

- Step 1: (Pricing) If $D_N \geq 0$, stop, B is optimal; else let
$$j = \text{argmin}\{D_k : k \in N\}.$$
- Step 2: (FTRAN) Solve $A_B y = A_j$.
- Step 3: (Ratio test) If $y \leq 0$, stop, (P) is unbounded; else, let
$$i = \text{argmin}\{X_{Bk}/y_k : y_k > 0\}.$$
- Step 4: (BTRAN) Solve $A_B^T z = e_i$.
- Step 5: (Update) Compute $\alpha_N = -A_N^T z$. Let $B_i = j$. Update X_B (using y) and D_N (using α_N)

Note: x_j is called the **entering** variable and x_{Bi} the **leaving** variable. The D_N values are known as **reduced costs** – like partial derivatives of the objective function relative to the nonbasic variables.

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Primal Simplex Example

The Primal Simplex Algorithm

Consider the following simple LP:

$$
\begin{aligned}
\text{Maximize} \quad & 3x_1 + 2x_2 + 2x_3 \\
\text{Subject to} \quad & x_1 + x_2 + x_3 \le 8 \\
& x_1 + x_2 \le 7 \\
& x_1 + 2x_2 \le 12 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

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The Primal Simplex Algorithm

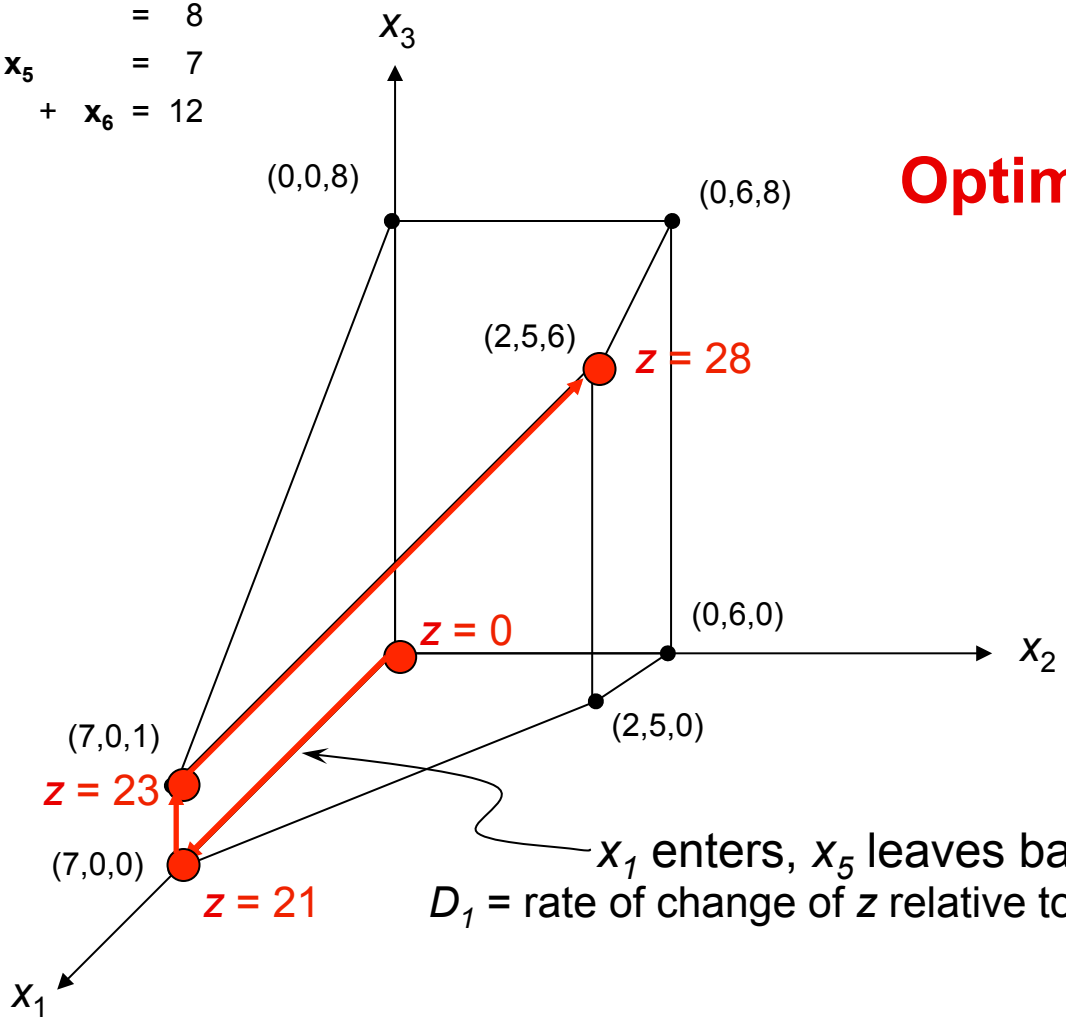
Maximize $z = 3x_1 + 2x_2 + 2x_3$

Add slacks: Initial basis B = (4,5,6)

Maximize $3x_1 + 2x_2 + 2x_3 + 0\mathbf{x_4} + 0\mathbf{x_5} + 0\mathbf{x_6}$

Subject to

$$x_1 + x_3 + \mathbf{x_4} = 8$$
$$x_1 + x_2 + \mathbf{x_5} = 7$$
$$x_1 + 2x_2 + \mathbf{x_6} = 12$$
$$x_1, x_2, x_3, \mathbf{x_4}, \mathbf{x_5}, \mathbf{x_6} \geq 0$$



Optimal

x_1 enters, x_5 leaves basis

D_1 = rate of change of z relative to x_1 = 21/7=3

Dual Simple Algorithm – Setup

Simplex algorithms apply to problems with constraints in equality form. We convert (D) to this form by adding the dual **slacks** *d*:

$$\begin{aligned} \textit{Maximize} \quad & b^T\pi \\ \textit{Subject to} \quad & A^T\pi + d = c \\ & \pi \ \textit{free}, \ d \geq 0 \end{aligned} \qquad \boxed{\Leftrightarrow A^T\pi \leq c}$$

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Dual Simple Algorithm – Setup

$$\begin{aligned} \textit{Maximize} \quad & b^T\pi \\ \textit{Subject to} \quad & A^T\pi + d = c \\ & \pi \ \textit{free}, \ d \geq 0 \end{aligned} \quad \longleftarrow \quad \begin{pmatrix} A_B^T & I_B & 0 \\ A_N^T & 0 & I_N \end{pmatrix} \begin{pmatrix} \pi \\ d_B \\ d_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

Given a basis *B,* the corresponding **dual basic variables** are π and d_N. d_B are the **nonbasic variables**. The corresponding **dual basic solution** Π, D is determined as follows:

$$D_B = 0 \ \Rightarrow \ \Pi = A_B^{-T} c_B \ \Rightarrow \ D_N = c_N - A_N^T \Pi$$

B is **dual feasible** if $D_N \geq 0$.

Dual Simple Algorithm – Setup

$$\begin{aligned} \textit{Maximize} \quad & b^T\pi \\ \textit{Subject to} \quad & A^T\pi + d = c \\ & \pi \ \textit{free}, \ d \geq 0 \end{aligned} \quad \longleftarrow \quad \begin{pmatrix} A_B{}^T & I_B & 0 \\ A_N{}^T & 0 & I_N \end{pmatrix} \begin{pmatrix} \pi \\ d_B \\ d_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}$$

Observation: We may assume that every dual basis has the above form.

Proof: Assuming that the primal has a basis is equivalent to assuming that *rank(A)=m* (# of rows), and this implies that all π variables can be assumed to be basic.

This observation establishes a 1-1 correspondence between primal and dual bases. ∎

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An Important Fact

If X and Π, D are corresponding primal and dual basic solutions determined by a basis B, then

$$\Pi^T b = c^T X.$$

Hence, by weak duality, if B is both primal and dual feasible, then X is optimal for (P) and Π is optimal for (D).

Proof: $c^T X = c_B^T X_B$ (since $X_N = 0$)
$= \Pi^T A_B X_B$ (since $\Pi = A_B^{-T} c_B$)
$= \Pi^T b$ (since $A_B X_B = b$) ∎

Dual Simplex Algorithm
(Lemke, 1954)

Input: A dual feasible basis B and vectors

$$X_B = A_B^{-1} b \quad \text{and} \quad D_N = c_N - A_N^T B^{-T} c_B.$$

- Step 1: (Pricing) If $X_B \geq 0$, stop, B is optimal; else let
$$i = \text{argmin}\{X_{Bk} : k \in \{1,\ldots,m\}\}.$$
- Step 2: (BTRAN) Solve $A_B^T z = e_i$. Compute $\alpha_N = -A_N^T z$.
- Step 3: (Ratio test) If $\alpha_N \leq 0$, stop, (D) is unbounded; else, let
$$j = \text{argmin}\{D_k/\alpha_k : \alpha_k > 0\}.$$
- Step 4: (FTRAN) Solve $A_B y = A_j$.
- Step 5: (Update) Set $B_i = j$. Update X_B (using y) and D_N (using α_N)

Note: d_{Bi} is the **entering** variable and d_j is the **leaving** variable. (Expressed in terms of the primal: x_{Bi} is the leaving variable and x_j is the entering variable)

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Simplex Algorithms

Input: A primal feasible basis B and vectors

$X_B=A_B^{-1}b$ & $D_N=c_N-A_N^TA_B^{-T}c_B$.

- Step 1: (Pricing) If $D_N \geq 0$, stop, B is optimal; else, let
  $j = argmin\{D_k : k \in N\}$.
- Step 2: (FTRAN) Solve $A_B y=A_j$.
- Step 3: (Ratio test) If $y \leq 0$, stop, (P) is unbounded; else, let
  $i = argmin\{X_{Bk}/y_k : y_k > 0\}$.
- Step 4: (BTRAN) Solve $A_B^T z = e_i$.
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Input: A dual feasible basis B and vectors

$X_B=A_B^{-1}b$ & $D_N=c_N-A_N^TA_B^{-T}c_B$.

- Step 1: (Pricing) If $X_B \geq 0$, stop, B is optimal; else, let
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Summary:
What we have done and what we have to do

- Done
  - Defined primal and dual linear programs
  - Proved the weak duality theorem
  - Introduced the concept of a basis
  - Stated primal and dual simplex algorithms
- To do (for dual simplex algorithm)
  - Show correctness
  - Describe key implementation ideas

Correctness: Dual Simplex Algorithm

- Termination criteria
  - Optimality (DONE – by "An Important Fact")
  - Unboundedness
- Other issues
  - Finding starting dual feasible basis, or showing that no feasible solution exists
  - Input conditions are preserved (i.e., that B is still a feasible basis)
  - Finiteness

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Dual Unboundedness
(⇒ primal infeasible)

- We carry out a key calculation
- As noted earlier, in an iteration of the dual

d_{Bi} enters basis
d_j leaves basis

in

$$\begin{aligned} &\text{Maximize} && b^T\pi \\ &\text{Subject to} && A^T\pi + d = c \\ & && \pi \text{ free}, d \geq 0 \end{aligned}$$

- The idea: Currently $d_{Bi} = 0$, and $X_{Bi} < 0$ has motivated us to increase d_{Bi} to $\theta > 0$, leaving the other components of d_B at 0 (the object being to increase the objective). Letting $\underline{d}, \underline{\pi}$ be the corresponding dual solution as a function of θ, we obtain

$$\underline{d}_B = \theta e_i \qquad \underline{\pi} = \Pi - \theta z \qquad \underline{d}_N = D_N - \theta \alpha_N$$

where α_N and z are as computed in the algorithm.

(Dual Unboundedness – cont.)

- Letting $\underline{d},\underline{\pi}$ be the corresponding dual solution as a function of θ. Using α_N and z from dual algorithm,

$$\underline{d}_B = \theta\, e_i \qquad \underline{d}_N = D_N - \theta\, \alpha_N \qquad \underline{\pi} = \pi - \theta\, z.$$

- Using $\theta > 0$ and $X_{Bi} < 0$ yields

$$\begin{aligned} new\_objective = \underline{\pi}^T b &= (\Pi - \theta\, z)^T b \\ &= \Pi^T b - \theta\, e_i^T A_B^{-1} b = \Pi^T b - \theta\, e_i^T X_B \\ &= old\_objective - \theta\, X_{Bi} > old\_objective \end{aligned}$$

- **Conclusion 1**: If $\alpha_N \leq 0$, then $\underline{d}_N \geq 0\ \forall\ \theta > 0 \Rightarrow$ (D) is unbounded.

- **Conclusion 2**: If α_N not ≤ 0, then

$$\begin{aligned} \underline{d}_N \geq 0 &\Rightarrow \theta \leq D_j / \alpha_j \ \ \forall \ \ \alpha_j > 0 \\ &\Rightarrow \theta_{max} = min\{D_j / \alpha_j : \alpha_j > 0\} \end{aligned}$$

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(Dual Unboundedness – cont.)

- **Feasibility preserved: follows from the ratio test.**
- **Nonsingularity preserved:** follows from (also yields update)
  - new $A_B = A_B (I + (y - e_i) e_i^T)$
  - new $A_B^{-1} = (I - (1/y_i)(y - e_i) e_i^T) A_B^{-1}$
- **Finiteness:** If $D_B > 0$ for all dual feasible bases B, then the dual simplex algorithm is finite: The dual objective strictly increases at each iteration ⇒ no basis repeats, and there are a finite number of bases.
- There are various approaches to guaranteeing finiteness in general:
  - **Bland's Rules:** Purely combinatorial, bad in practice.
  - **Gurobi:** A perturbation is added to "guarantee" $D_B > 0$.

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Implementing the Dual Simplex Algorithm

Some Motivation

- Dual simplex vs. primal: Dual > 2x faster
- Dual is the best algorithm for MIP
- There isn't much in books about implementing the dual.

Dual Simplex Algorithm
(Lemke, 1954: Commercial codes ~1990)

Input: A dual feasible basis B and vectors

$$X_B = A_B^{-1} b \quad \text{and} \quad D_N = c_N - A_N^T B^{-T} c_B.$$

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Implementation Issues for Dual Simplex

1. Finding an initial feasible basis or concluding that there is none: Phase I of the simplex algorithm.
2. Pricing: dual steepest edge
3. Solving the linear systems
   - LU factorization and factorization update
   - BTRAN and FTRAN – exploiting sparsity
4. Numerically stable ratio test: Bound shifting and perturbation
5. Bound flipping: Exploiting "boxed" variables to combine many iterations into one.

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Issue 0
Preparation: Bounds on Variables

In practice, simplex algorithms need to accept LPs in the following form:

$$\begin{aligned} &\textit{Minimize} && c^T x \\ &\textit{Subject to} && Ax = b \qquad (P_{BD}) \\ & && l \le x \le u \end{aligned}$$

where l is an n-vector of **lower bounds** and u an n-vector of **upper bounds**. l is allowed to have $-\infty$ entries and u is allowed to have $+\infty$ entries. (Note that (P_{BD}) is in standard form if $l_j = 0$, $u_j = +\infty \; \forall \; j$.)

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(Issue 0 – Bounds on variables)

Basic Solution

A **basis** for (P_{BD}) is a triple (B,L,U) where B is an ordered m–element subset of $\{1,\ldots,n\}$ (just as before), (B,L,U) is a partition of $\{1,\ldots,n\}$, $l_j > -\infty \ \forall\ j \in L$, and $u_j < +\infty \ \forall\ j \in U$. $N = L \cup U$ is the set of **nonbasic** variables. The associated (**primal**) **basic solution** X is given by $X_L = l_L$, $X_U = u_U$ and

$$X_B = A_B^{-1}(b - A_L l_L - A_U u_U).$$

This solution is **feasible** if

$$l_B \leq X_B \leq u_B.$$

The associated **dual basic solution** is defined exactly as before: $D_B = 0$, $\Pi^T A_B = c_B^T$, $D_N = c_N - A_N^T \Pi$. It is **dual feasible** if

$$D_L \geq 0 \ \text{ and } \ D_U \leq 0.$$

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(Issue 0 – Bounds on variables)
The Full Story

- Modify simplex algorithm
  - Only the “Pricing” and “Ratio Test” steps must be changed substantially.
  - The complicated part is the ratio test
- Reference: See Chvátal for the primal

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Issue 1
The Initial Feasible Basis – Phase I

- Two parts to the solution
  1. Finding some initial basis (probably not feasible)
  2. Modified simplex algorithm to find a feasible basis

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(Issue 1 – Initial feasible basis)

Initial Basis

- Primal and dual bases are the same. We begin in the context of the primal. Consider

$$\begin{array}{lll} \textit{Minimize} & c^T x & \\ \textit{Subject to} & Ax = b & (P_{BD}) \\ & l \le x \le u & \end{array}$$

- Assumption: Every variable has some finite bound.
- Trick: Add artificial variables $x_{n+1},\ldots,x_{n+m}$:

$$Ax + I \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = b$$

where $l_j = u_j = 0$ for $j = n+1,\ldots,n+m$.

- Initial basis: $B = (n+1,\ldots,n+m)$ and for each $j \notin B$, pick some finite bound and place j in L or U, as appropriate.
- Free-Variable Refinement: Make free variables non-basic at value 0. This leads to a notion of a *superbasis*, where non-basic variables can be between their bounds.

(Issue 1 – Initial feasible basis)

Solving the Phase I

- If the initial basis is not dual feasible, we consider the problem:

$$\begin{aligned} &\textit{Maximize} && \Sigma\,(d_j : d_j < 0) \\ &\textit{Subject to} && A^T\pi + d = c \end{aligned}$$

- This problem is "locally linear": Define $\kappa \in R^n$ by $\kappa_j = 1$ if $D_j < 0$, and 0 otherwise. Let

$$K = \{j: D_j < 0\} \quad \textit{and} \quad \underline{K} = \{j: D_j \geq 0\}$$

Then our "local" problem becomes

$$\begin{aligned} &\textit{Maximize} && \kappa^T d \\ &\textit{Subject to} && A^T\pi + d = c \\ & && d_K \leq 0,\ d_{\underline{K}} \geq 0 \end{aligned}$$

- Apply dual simplex, and whenever d_j for $j \in K$ becomes 0, move it to $\underline{K}$.

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Solving Phase I:
An Interesting Computation

- Suppose d_{Bi} is the entering variable. Then $X_{Bi} < 0$ where X_B is obtained using the following formula:

$$X_B = A_B^{-1} A_N \kappa$$

- Suppose now that d_j is determined to be the leaving variable. Then in terms of the phase I objective, this means κ_j is replace by $\kappa_j + \varepsilon e_j$, where $\varepsilon \in \{0, +1, -1\}$. It can then be shown that

$$\underline{x}_{Bi} = X_{Bi} + \varepsilon \alpha_j$$

- Conclusion: If $x_{Bi} < 0$, then the current iteration can continue without the necessity of changing the basis.

- Advantages
  - Multiple iterations are combined into one.
  - x_{Bi} will tend not to change sign precisely when α_j is small. Thus this procedure tends to avoid unstable pivots.

Issue 2
Pricing

- The textbook rule is TERRIBLE: For a problem in standard form, select the entering variable using the formula

$$j = \text{argmin}\{X_{Bi} : i = 1,\ldots,m\}$$

- Geometry is wrong: Maximizes rate of change relative to axis; better to do relative to edge.

- Goldfard and Forrest 1992 suggested the following steepest-edge alternative

$$j = \text{argmin}\{X_{Bi} / \eta_i : i = 1,\ldots,m\}$$

where $\eta_i = ||e_i^T A_B^{-1}||_2$, and gave an efficient update.

- Note that there are two ingredients in the success of Dual SE:
  - *Significantly reduced iteration counts*
  - *The fact that there is a very efficient update for* η

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Example: Pricing
Model: dfl001

Pricing: Greatest infeasibility

```
Solved in 281829 iterations and 118.68 seconds
Optimal objective  1.126639304e+07
```

Pricing: Goldfarb-Forrest steepest-edge

```
Solved in 18412 iterations and 5.36 seconds
Optimal objective  1.126639304e+07
```

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Issue 3
Solving FTRAN, BTRAN

- Computing LU factorization: See Suhl & Suhl (1990). "Computing sparse LU factorization for large-scale linear programming basis", ORSA Journal on Computing 2, 325-335.

- Updating the Factorization: Forrest-Tomlin update is the method of choice. See Chvátal Chapter 24.
  - **There are multiple, individually relatively minor tweaks that collectively have a significant effect on update efficiency.**

- Further exploiting sparsity: This is the main recent development.ds

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(Issue 3 – Solving FTRAN & BTRAN)

We must solve two linear systems per iteration:

FTRAN	BTRAN
$A_B y = A_j$	$A_B^T z = e_i$

where

A_B = basis matrix (very sparse)

A_j = entering column (very sparse)

e_i = unit vector (very sparse)

$\Rightarrow y$ an z are typically very sparse

Example: Model pla85900 (from TSP)

Constraints	85900
Variables	144185
Average |y|	15.5

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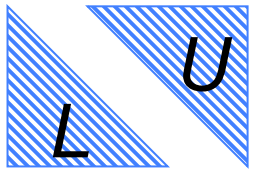
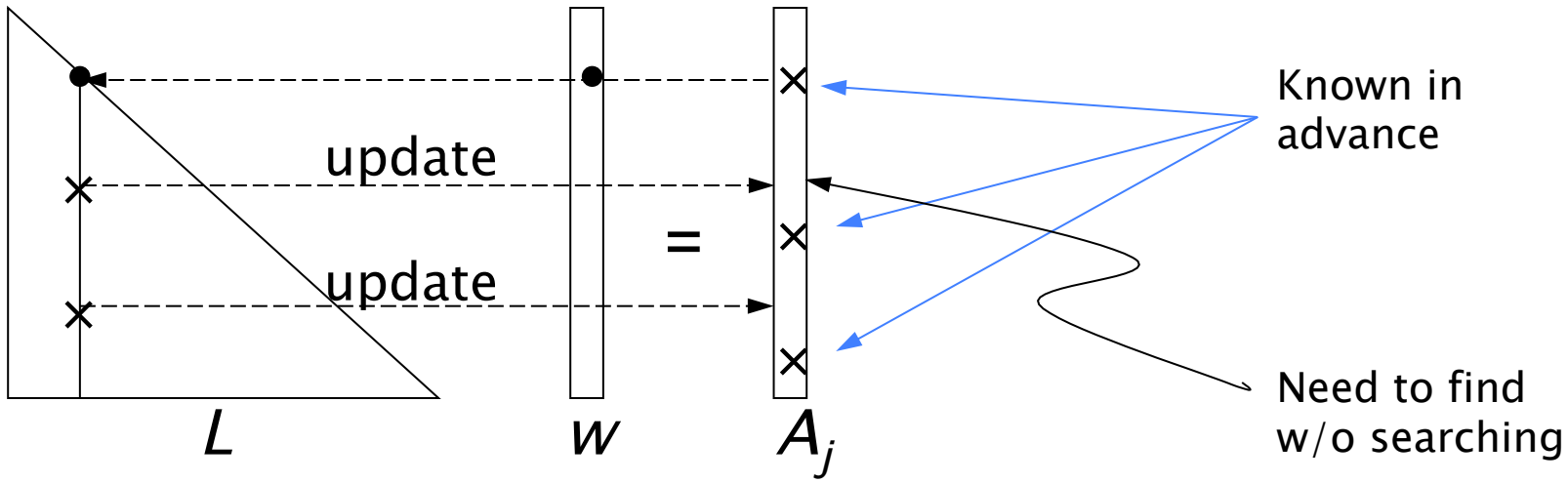
A_B =



Triangular solve: $Lw=A_j$ $(A_B y = L(\underbrace{Uy}_{w}) = A_j)$



Graph structure: Define an acyclic digraph $D = (\{1,\ldots,m\}, E)$ where $(i,j)\in E \Leftrightarrow l_{ij} \neq 0$ and $i \neq j$.

Solving using D: Let $X = \{i\in V: L_{ij} \neq 0\}$. Compute

$$\underline{X} = \{i\in V: \exists \text{ a directed path from } i \text{ to } X\}.$$

$\underline{X}$ can be computed in time linear in $|E(\underline{X})|+|\underline{X}|$.

PDS Models (2002)

"Patient Distribution System": Carolan, Hill, Kennington, Niemi, Wichmann, *An empirical evaluation of the KORBX algorithms for military airlift applications*, Operations Research 38 (1990), pp. 240–248

MODEL	ROWS	CPLEX1.0 1988	CPLEX5.0 1997	CPLEX8.0 2002	SPEEDUP 1.0→8.0
pds02	2953	**0.4**	**0.1**	**0.1**	**4.0**
pds06	9881	**26.4**	**2.4**	**0.9**	**29.3**
pds10	16558	**208.9**	**13.0**	**2.6**	**80.3**
pds20	33874	**5268.8**	**232.6**	**20.9**	**247.3**
pds30	49944	**15891.9**	**1154.9**	**39.1**	**406.4**
pds40	66844	**58920.3**	**2816.8**	**79.3**	**743.0**
pds50	83060	**122195.9**	**8510.9**	**114.6**	**1066.3**
pds60	99431	**205798.3**	**7442.6**	**160.5**	**1282.2**
pds70	114944	**335292.1**	**21120.4**	**197.8**	**1695.1**
		Primal Simplex	Dual Simplex	Dual Simplex	

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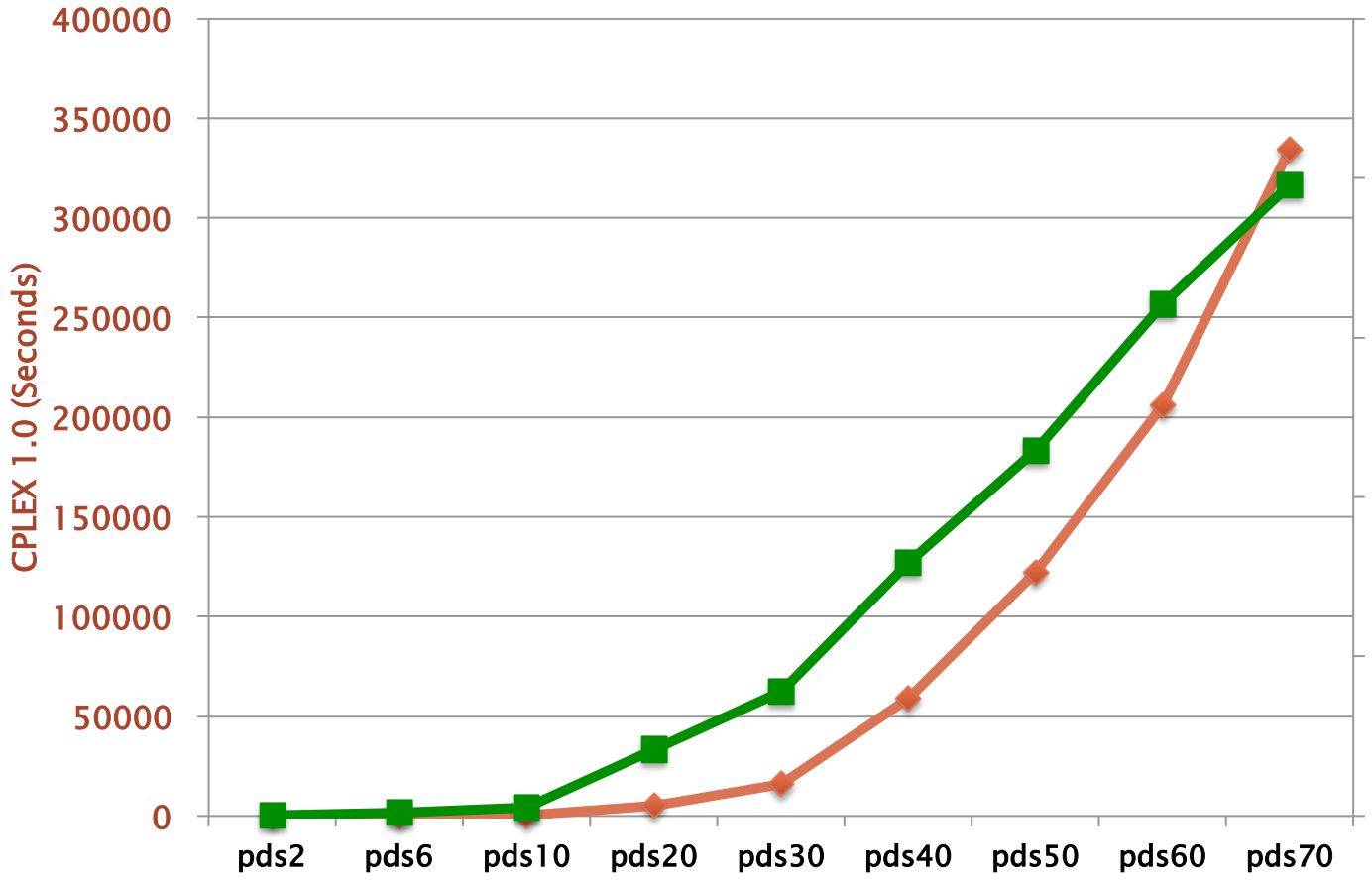
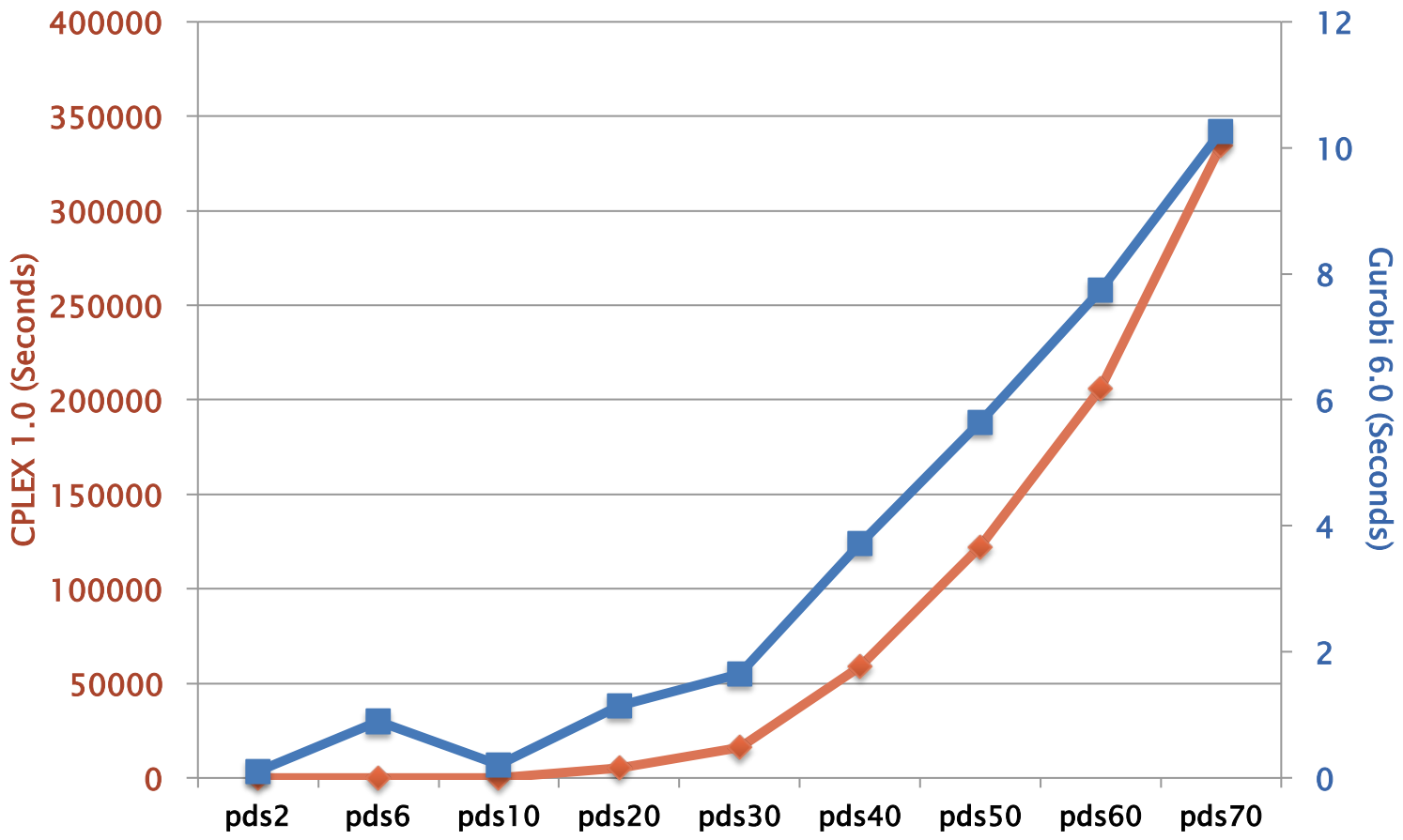
Not just faster -- Growth with size: Quadratic *then* & Linear *now*!



Gurobi Optimization

Gurobi Headline goes here…



Gurobi Optimization

Issue 4
Ratio Test and Finiteness

The "standard form" dual problem is

$$\begin{aligned} \textit{Maximize} \quad & b^T\pi \\ \textit{Subject to} \quad & A^T\pi + d = c \\ & d \geq 0 \end{aligned}$$

Feasibility means

$$d \geq 0$$

However, in practice this condition is replaced by

$$d \geq -\varepsilon e$$

where $e^T=(1,\ldots,1)$ and $\varepsilon = 10^{-6}$, the **feasibility tolerance.**

Reason: Degeneracy. In 1972 Paula Harris suggested exploiting this fact to improve numerical stability.

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(Issue 4 – Ratio test & finiteness)

STANDARD RATIO TEST $\quad j_{enter} = argmin\{D_j / \alpha_j : \alpha_j > 0\}$

Motivation: Feasibility ⇒ step length θ satisfies

$$D_N - \theta\alpha_N \geq 0$$

Since the bigger the step length, the bigger the change in the objective, we choose

$$\theta_{max} = min\{D_j / \alpha_j : \alpha_j > 0\}$$

Using ε, we have

$$\theta^{\varepsilon}_{max} = min\{(D_j+\varepsilon)/\alpha_j : \alpha_j > 0\} > \theta_{max}$$

HARRIS RATIO TEST $\quad j_{enter} = argmax\{\alpha_j : \theta_{max} \leq D_j / \alpha_j \leq \theta^{\varepsilon}_{max}, \alpha_j > 0\}$

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- Advantages
  - Numerical stability – α_{jenter} = "pivot element"
  - Degeneracy – Reduces # of 0-length steps
- Disadvantage
  - $D_{jenter} < 0 \Rightarrow$ objective goes in wrong direction
- Solution: BOUND SHIFTING
  - If $D_{jenter} < 0$, we replace the lower bound on d_{jenter} by something less than its current value.
  - Note that this shift changes the problem and must be removed: 5% of cases, this produces dual infeasibility ⇒ process is iterated.

Example: Bound-Shifting Removal

```
Read MPS format model from file qap12.mps.bz2
Optimize a model with 3192 rows, 8856 columns and 38304 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    0.0000000e+00   1.230000e+02   0.000000e+00      0s
     101    0.0000000e+00   7.229833e+02   0.000000e+00      0s
     173    3.3669520e+00   9.125960e+02   0.000000e+00      0s
....
   49843    5.2387894e+02   5.585623e+01   0.000000e+00     32s
   50213    5.2388556e+02   7.361090e+00   0.000000e+00     32s
   50584    5.2388824e+02   1.648797e+01   0.000000e+00     32s
   50744    5.2388840e+02   0.000000e+00   0.000000e+00     33s
Switch to primal
   50934    5.2289692e+02   0.000000e+00   3.404469e+01     33s
   51123    5.2289527e+02   0.000000e+00   1.021229e+00     33s
   51312    5.2289450e+02   0.000000e+00   2.841123e-01     33s
   51499    5.2289434e+02   0.000000e+00   1.686059e-01     33s
Perturbation ends
   51516    5.2289435e+02   0.000000e+00   0.000000e+00     33s

Solved in 51516 iterations and 33.15 seconds
Optimal objective  5.228943506e+02
```

Shift removed

Shift removed

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Finiteness: Bound shifting is closely related to the "perturbation" method employed in Gurobi if no progress is being made in the objective.

If "insufficient" progress is being made, replace

$$d_j \geq -\varepsilon \qquad j = 1,\ldots,n$$

by

$$d_j \geq -\varepsilon - \varepsilon_j \qquad j = 1,\ldots,n,$$

where ε_j is pseudo-random uniform on $[0,\varepsilon]$. This makes the probability of a 0-length step very small, and in practice has been sufficient to guarantee finiteness.

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Issue 5
Bound Flipping (Long-Step Dual)

- A basis is given by a triple (B,L,U)
  - L = non-basics at lower bound: Feasibility $D_L \geq 0$
  - U = non-basics at upper bound: Feasibility $D_U \leq 0$
- **Ratio test:** Suppose X_{Bi} is the leaving variable, and the step length is blocked by some variable d_j, $j \in L$, where d_j is about to become negative and $u_j < +\infty$:
  - **Flipping means:** Move j from L to U.
  - **Check:** Do an update to see if X_{Bi} is still favorable (just as we did in Phase I!)
- Can combine many iterations into a single iteration.

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Example: Bound Flipping

```
Read MPS format model from file fit2d.mps.bz2
Optimize a model with 25 rows, 10500 columns and 129018 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0   -9.1662550e+04   9.553095e+03   0.000000e+00      0s
    6023   -6.8464293e+04   0.000000e+00   0.000000e+00      1s

Solved in 6023 iterations and 0.82 seconds
Optimal objective -6.846429329e+04
```

w/o flipping

```
Read MPS format model from file fit2d.mps.bz2
Optimize a model with 25 rows, 10500 columns and 129018 nonzero

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0   -9.1662550e+04   9.553095e+03   0.000000e+00      0s
     255   -6.8464293e+04   0.000000e+00   0.000000e+00      0s

Solved in 255 iterations and 0.07 seconds
Optimal objective -6.846429329e+04
```

w/ flipping

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