Basics of polyhedral theory, flows and networks

CO@W Berlin

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Graph Theory: Super Quick

- Graph G=(V,E), nodes, edges e=ij
- Digraph D=(V,A), nodes arcs a=(u,v)

Concepts

- Chain, walk, path, cycle, circuit
- clique, stable set, matching
- coloring, clique cover, clique partitioning, edge coloring
- …
- Optimization problems associated with these
- Polynomial time solvability, NP-hardness

I assume that this is known

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Special „simple“ combinatorial optimization problems

Finding a

- minimum spanning tree in a graph
- shortest path in a directed graph
- maximum matching in a graph
- minimum capacity cut separating two given nodes of a graph or digraph
- cost-minimal flow through a network with capacities and costs on all edges
- …

These problems are solvable in polynomial time.

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Special „hard“ combinatorial optimization problems

- travelling salesman problem (the prototype problem)
- location und routing
- set-packing, partitioning, -covering
- max-cut
- linear ordering
- scheduling (with a few exceptions)
- node and edge colouring
- …

These problems are NP-hard
(in the sense of complexity theory).

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Linear Programming

$$\max c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$subject\ \ to$$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$.$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

linear program
in standard form

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Linear Programming

$$\max c^T x \quad Ax = b \quad x \geq 0$$

linear program in standard form

$$\Rightarrow$$

$$\max c^T x \quad Ax \leq b \quad -Ax \leq -b \quad -x \leq 0$$

$$\max c^T x \quad Ax \leq b$$

linear program in "polyhedral form"

$$\Rightarrow$$

$$\max c^T x^+ - c^T x^- \quad Ax^+ + Ax^- + Is = b \quad x^+, x^-, s \geq 0 \quad (x = x^+ - x^-)$$

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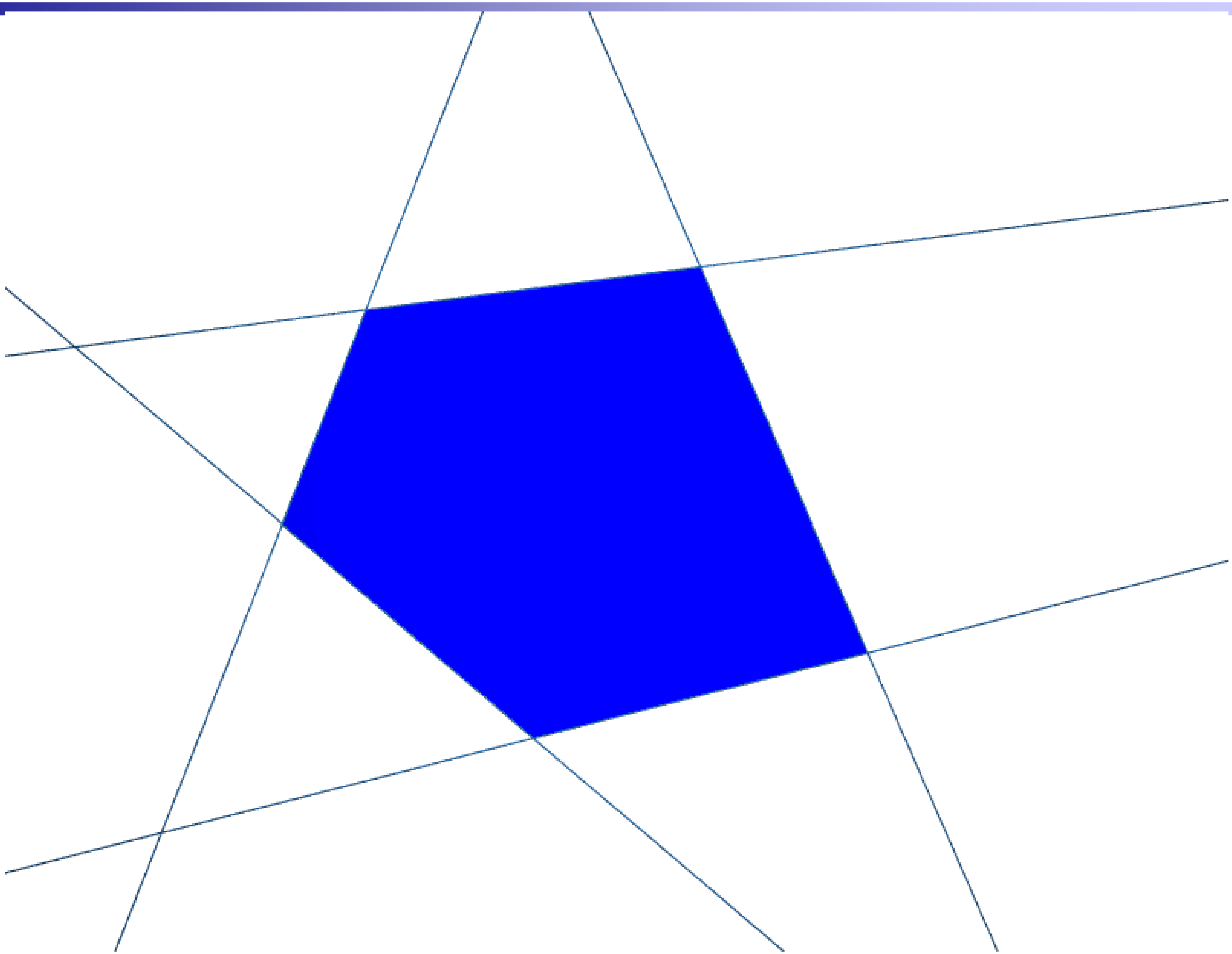
A Polytope in the Plane



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A Polytope in 3-dimensional space

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„beautiful“ polyehedra

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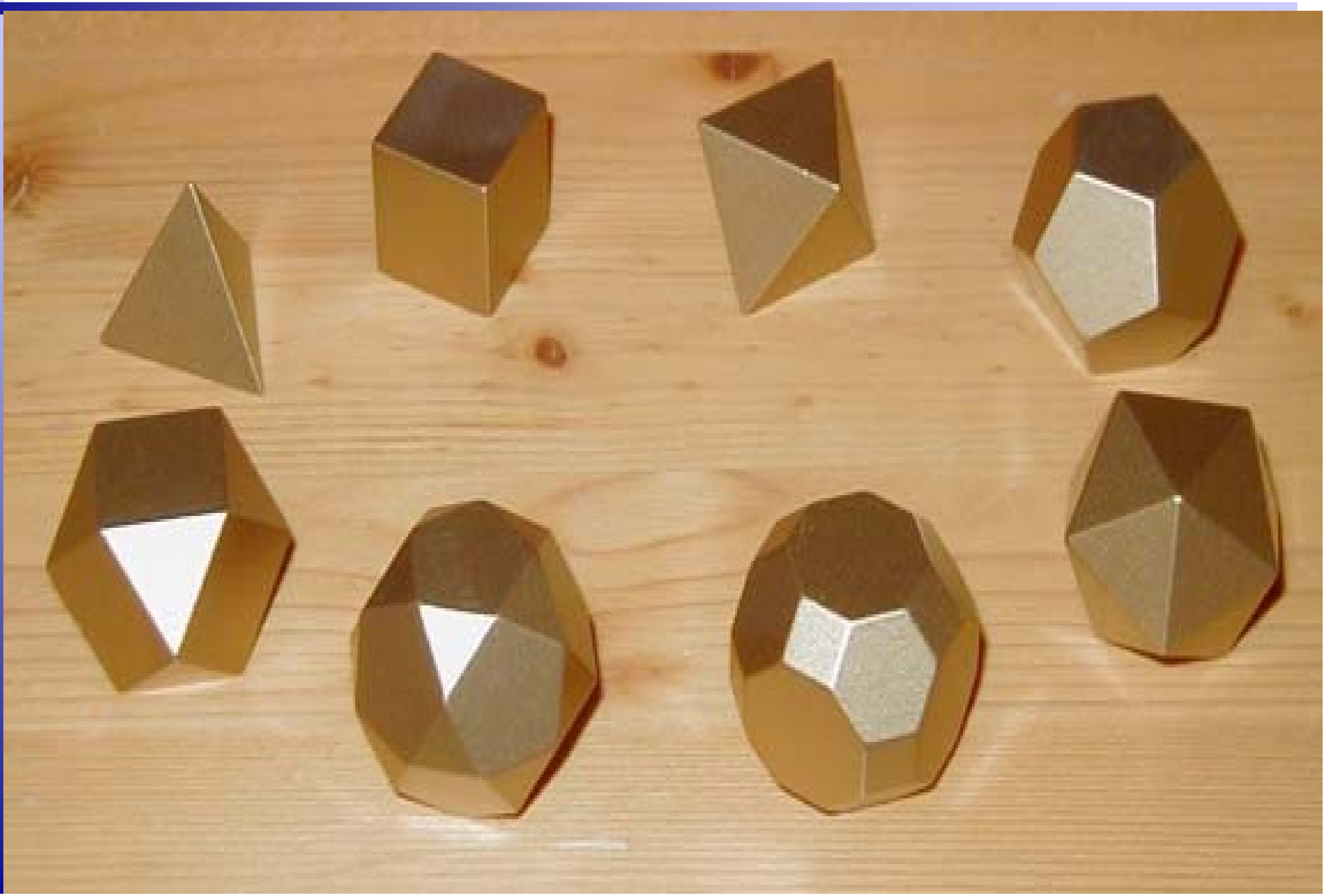

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“Real” polyhedra

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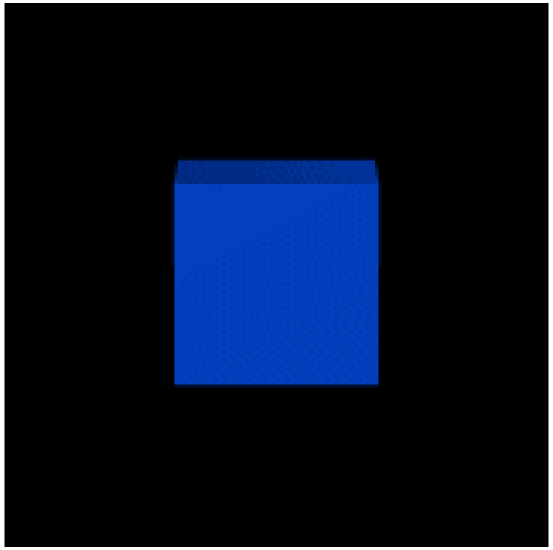
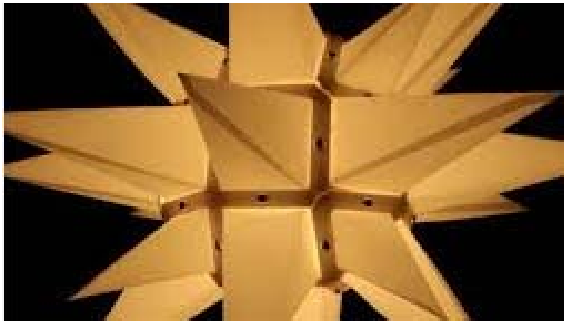
Rhombicuboctahedron

Herrnhuter Stern

Germany's most popular Christmas star





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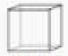
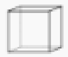
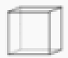
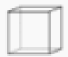
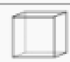
Crystallographic classifications

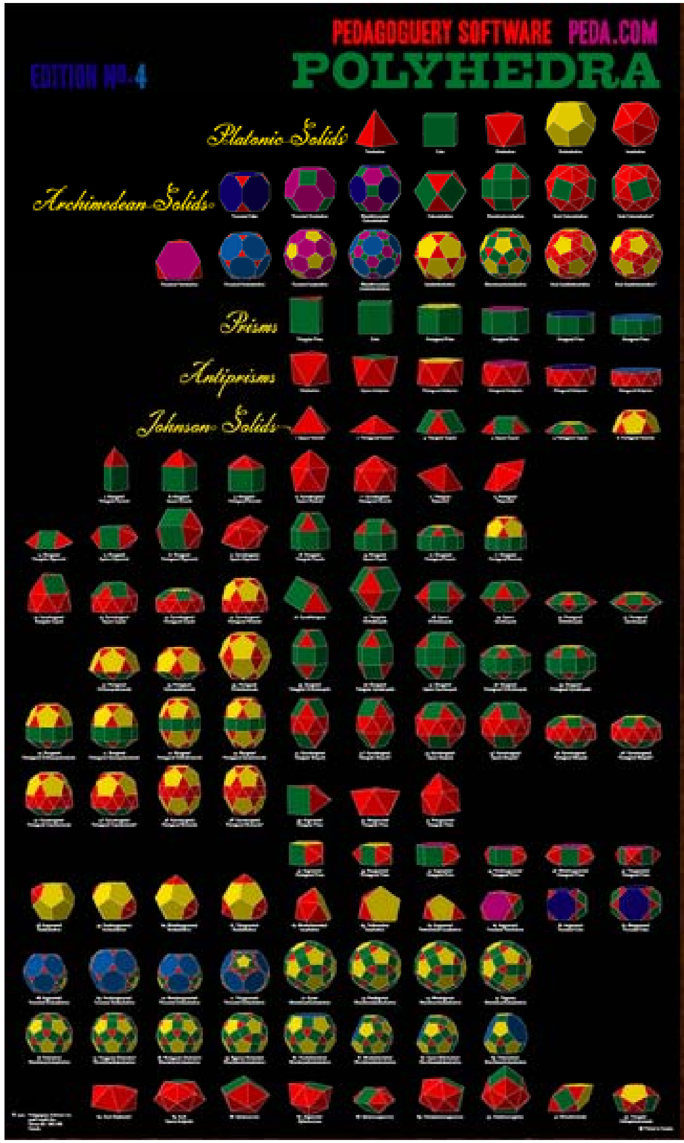
http://de.wikipedia.org/wiki/Kristallmorphologie

Kubisches Kristallsystem

Kristallklasse		{hkl}	{hhl} (h>l)	{hll} (h>l)	{hk0}	{111}	{110}	{100}
23	tetraedrisch-pentagondodekaedrisch	 tetraedrisches Pentagondodekaeder (Pentagon-Tritetraeder, Tetartoeder) (12)	 Deltoiddodekaeder (Tetragon-Tritetraeder) (12)	 Triakistetraeder (Tristetraeder, Trigon-Tritetraeder) (12)	 Pentagondodekaeder (Pyritoeder) (12)	 Tetraeder (4)	 Rhombendodekaeder (Granatoeder) (12)	 Würfel (Hexaeder) (6)
$\frac{2}{m}\bar{3}$	disdodekaedrisch	 Disdodekaeder (Dyakisdodekaeder, Diploeder, Diploid) (24)	 Triakisoktaeder (Trisoktaeder, Trigon-Trioktaeder) (24)	 Deltoidikositetraeder (Ikositetraeder, Tetragon-Trioktaeder, Trapezoeder, Leucitoeder) (24)	 Pentagondodekaeder (Pyritoeder) (12)	 Oktaeder (8)	 Rhombendodekaeder (Granatoeder) (12)	 Würfel (Hexaeder) (6)
432	pentagonikositetraedrisch	 Pentagonikositetraeder (Gyroeder, Gyroid) (24)	 Triakisoktaeder (...) (24)	 Deltoidikositetraeder (...) (24)	 Tetrakishexaeder (Tetrahexaeder) (24)	 Oktaeder (8)	 Rhombendodekaeder (Granatoeder) (12)	 Würfel (Hexaeder) (6)
$\bar{4}3m$	hexakistetraedrisch	 Hexakistetraeder (Hexatetraeder) (24)	 Deltoiddodekaeder (Tetragon-Tritetraeder) (12)	 Triakistetraeder (...) (12)	 Tetrakishexaeder (Tetrahexaeder) (24)	 Tetraeder (4)	 Rhombendodekaeder (Granatoeder) (12)	 Würfel (Hexaeder) (6)
$\frac{4}{m}\bar{3}\frac{2}{m}$	hexakisoktaedrisch	 Hexakisoktaeder (Hexaoktaeder) (48)	 Triakisoktaeder (...) (24)	 Deltoidikositetraeder (...) (24)	 Tetrakishexaeder (Tetrahexaeder) (24)	 Oktaeder (8)	 Rhombendodekaeder (Granatoeder) (12)	 Würfel (Hexaeder) (6)

Polyhedra-Poster

http://www.peda.com/posters/Welcome.html



We currently offer one poster for secure online purchasing: our *Polyhedra* poster, which displays all convex polyhedra with regular polygonal faces (a finite sampling of prisms and anti-prisms are included).

It measures 22" x 37" and is printed on glosssy paper. A protective coating was applied during printing.

The poster is shown on the left; to see a close-up of a portion of the poster, move your mouse over the image.

This is the fourth edition of the poster. Other versions of the poster are shown in our Posters Archive.

$14 FOR 1 POSTER
$28 FOR 4 POSTERS
FREE SHIPPING

Poster which displays all convex polyhedra with regular polygonal faces

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http://www.eg-models.de/

EG-Models

EG-Models - a new archive of electronic geometry models
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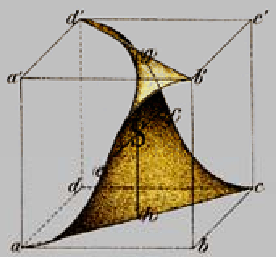


H.A. Schwarz Ges.Math.Abh
Springer Berlin 1890

Note: Some browser versions do not display Java applets. Please, press the 'No Applet' button in the navigation bar to avoid using Java.

Anschauliche Geometrie - A tribute to Hilbert, Cohn-Vossen, Klein and all other geometers.

Electronic Geometry Models

This archive is open for any geometer to publish new geometric models, or to browse this site for material to be used in education and research. These geometry models cover a broad range of mathematical topics from geometry, topology, and to some extent from numerics.

Click "Models" to see the full list of published models. See here for details on the submission and review process.

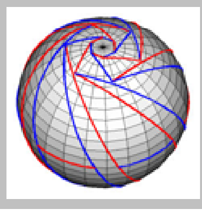
Selection of recently published models

Model 2013.10.001 Bruno Benedetti and Frank H. Lutz *The dunce hat in a minimal non-extendably collapsible 3-ball.*
Section: *Polytopal Complexes*

We obtain a geometric realization of a minimal 8-vertex triangulation of the dunce hat in Euclidean 3-space. We show there is a simplicial 3-ball with 8 vertices that is collapsible, but also collapses onto the dunce hat, which is not collapsible.

Model 2010.11.001 Udo Hertrich-Jeromin and Wayne Rossman *Discrete minimal catenoid in hyperbolic 3-space.*
Section: *Surfaces*

We show a discrete constant mean curvature (in fact, minimal) net of revolution in hyperbolic 3-space (in its Poincare half-space incarnation).

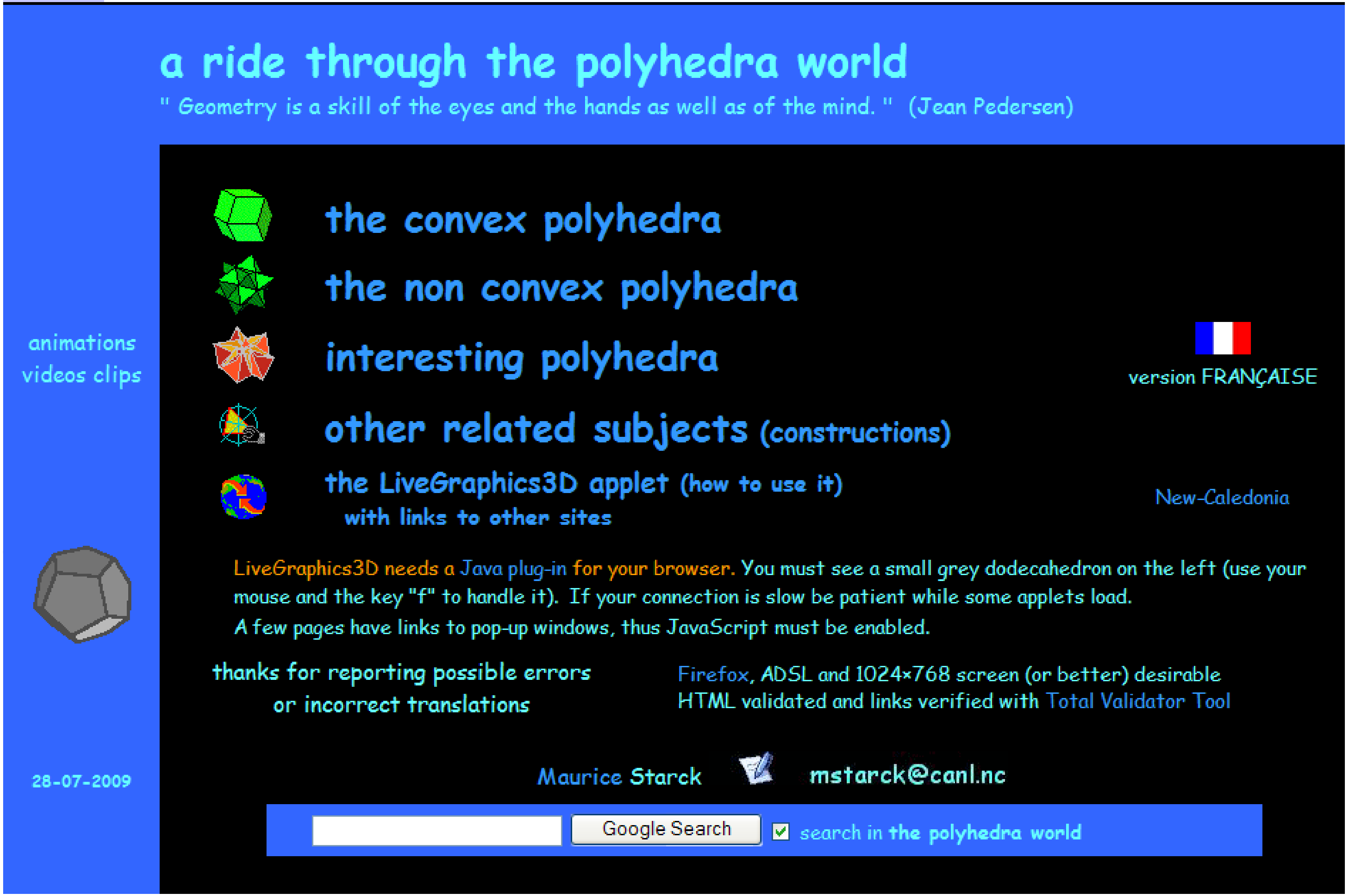


Model 2010.02.002 Marina Knyazeva and Gaiane Panina *Counterexample to a conjecture of Alexandrov.*
Section: *Surfaces*

A pointed graph on the sphere which leads to a counterexample to A.D. Alexandrov's conjecture.

This graph is interesting and important not only because of its funny combinatorics, but also because it leads to a counterexample to A.D. Alexandrov's uniqueness conjecture for smooth convex surfaces.

http://www.ac-noumea.nc/maths/amc/polyhedr/index_.htm



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convex polyhedra 1 - Microsoft Internet Explorer

Datei Bearbeiten Ansicht Favoriten Extras ?

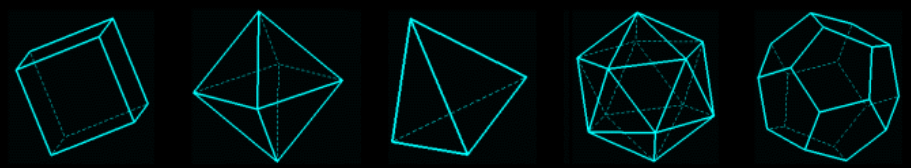
Zurück Suchen Favoriten Medien

Adresse http://www.ac-noumea.nc/maths/amc/polyhedr/convex1_.htm Wechseln zu Links

Plato's five regular polyhedra

The regular polyhedra are, in the space, the analogues of the regular polygons in the plane ; their faces are regular and identical polygons, and their vertices, regular and identical, are regularly distributed on a sphere. Their analogues in dimension four are the regular polytopes.
As we do for the polygons, we recognize a convex polyhedron by the very fact that all its diagonals (segments which join two vertices not joined by an edge) are inside the polyhedron.
Whereas there exist an infinity of regular convex polygons, the regular convex polyhedra are only five.

proof : The angle of a regular polygon with n sides is 180°(n-2)/n : 60° (triangle), 90° (square), 108° (pentagon), 120° (hexagon)...
On a vertex of a regular polyhedron the sum of the face's angles (there are at least three) must be smaller than 360°.
Since 6x60° = 4x90° = 3x120° = 360° < 4x108°, there are only five possibilities: 3, 4, or 5 triangles, 3 squares or 3 pentagons.

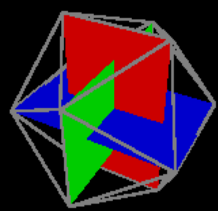


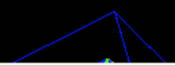
name	cube	octahedron	tetrahedron	icosahedron	dodecahedron
faces	6 squares	8 equil.triangles	4 equil.triangles	20 equil.triangles	12 regul.pentagons
vertices	8	6	4	12	20
edges	12	12	6	30	30
faces angle	90°	109°28'	70°32'	138°11'	116°34'

The LiveGraphics3D applet by Martin Kraus (University of Stuttgart) allows you to move these polyhedra with your mouse.

The regular octahedron's edges are the sides of three squares with the same centre and orthogonal by pairs.

The regular icosahedron's vertices are the vertices of three golden rectangles (sides in golden ratio 1.618...) with the same centre and orthogonal by pairs.





Four vertices of a cube are the vertices of a regular tetrahedron ; so we can make a regular tetrahedron by cutting four "corners" of a cube.

LiveGraphics3D 1.54: Please drag to rotate. Internet

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Kepler’s solar system

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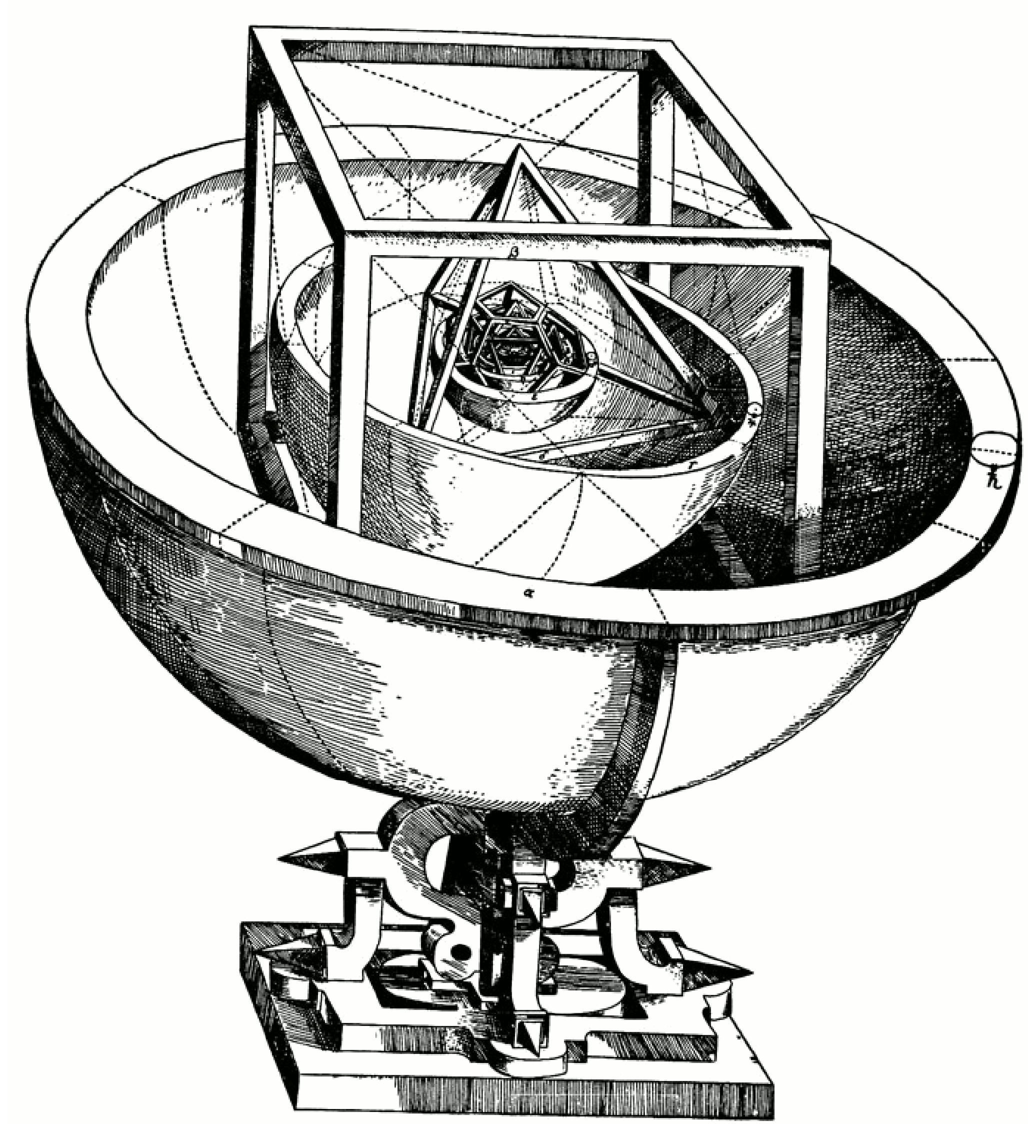

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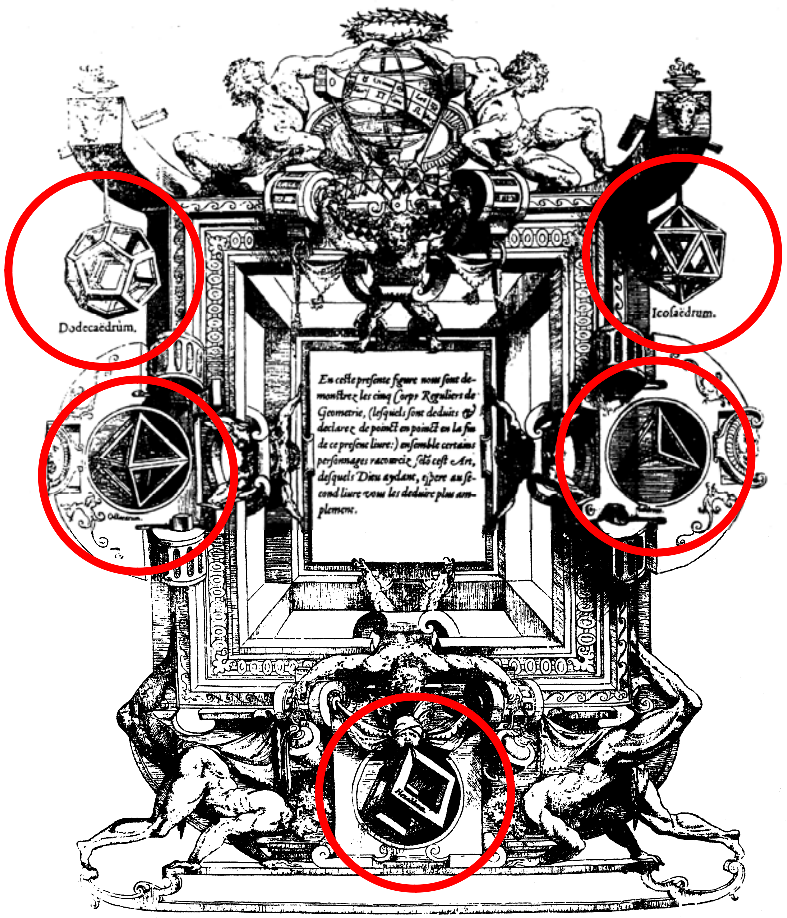
Polyhedra have fascinated people during all periods of our history



From *Livre de Perspective* by Jean Cousin, 1568.

- book illustrations
- magic objects
- pieces of art
- objects of symmetry
- models of the universe

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Definitions

Linear programming lives (for our purposes) in the n-dimensional real (in practice: rational) vector space.

- convex polyhedral cone: conic combination (i. e., nonnegative linear combination or conical hull) of finitely many points
  K = cone(E), E a finite set in □^{n}.
- polytope: convex hull of finitely many points:
  P = conv(V), V a finite set in □^{n}.
- polyhedron: intersection of finitely many halfspaces

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

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Important theorems of polyhedral theory (LP-view)

When is a polyhedron nonempty?

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Important theorems of polyhedral theory (LP-view)

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \le b$$

is empty, if and only if there is a vector y such that

$$y \ge 0,\ y^T A = 0^T,\ y^T b < 0^T$$

Theorem of the alternative

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Important theorems of polyhedral theory (LP-view)

Which forms of representation do polyhedra have?

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Important theorems of polyhedral theory (LP-view)

Which forms of representation do polyhedra have?

Minkowski (1896), Weyl (1935), Steinitz (1916) Motzkin (1936)

Theorem: For a subset P of $\mathbf{R}^n$ the following are equivalent:

(1) P is a polyhedron.

(2) P is the intersection of finitely many halfspaces, i.e., there exist a matrix A und ein vector b with

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}.$$ (exterior representation)

(3) P is the sum of a convex polytope and a finitely generated (polyhedral) cone, i.e., there exist finite sets V and E with

$$P = \mathrm{conv}(\mathrm{V}) + \mathrm{cone}(\mathrm{E}).$$ (interior representation)

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Representations of polyhedra

Carathéodory's Theorem (1911), 1873 Berlin – 1950 München

Let $x \in P = \mathrm{conv(V)+cone(E)}$, there exist

$$v_0,\dots,v_s \in \mathrm{V},\ \lambda_0,\dots,\lambda_s \in \mathbf{R}_+,\ \sum_{i=0}^{s} \lambda_i = 1$$

and $\mathrm{e}_{s+1},\dots,e_t \in \mathrm{E},\ \ \mu_{\mathrm{s+1}},\dots,\mu_{\mathrm{t}} \in \mathbf{R}_+$ with $\mathrm{t} \le \mathrm{n}$ such that

$$x = \sum_{i=1}^{s} \lambda_i v_i + \sum_{\mathrm{i=s+1}}^{\mathrm{t}} \mu_{\mathrm{i}} e_i$$

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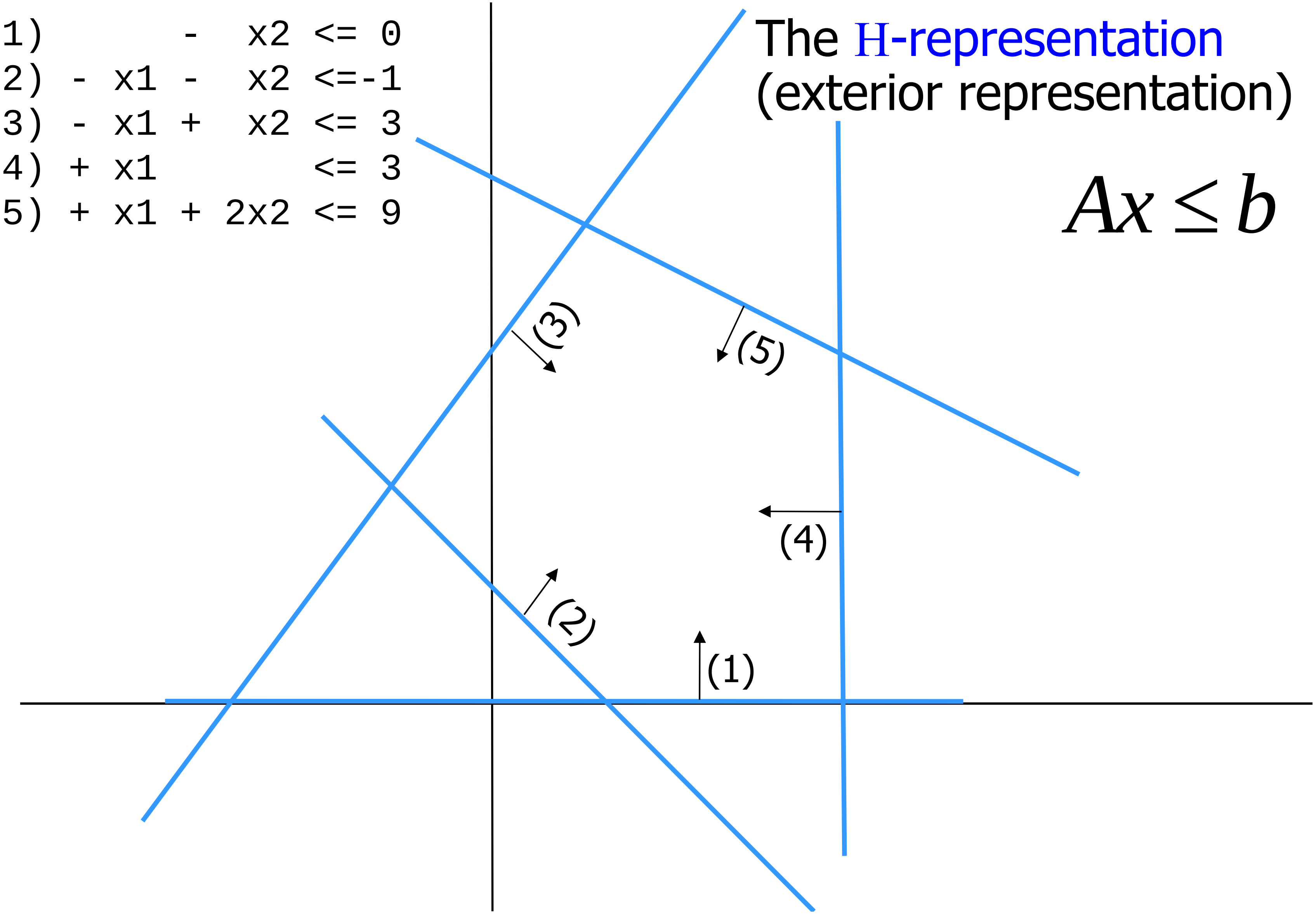
Representations of polyhedra


```
(1)         -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```

The H-representation
(exterior representation)

$$Ax \leq b$$



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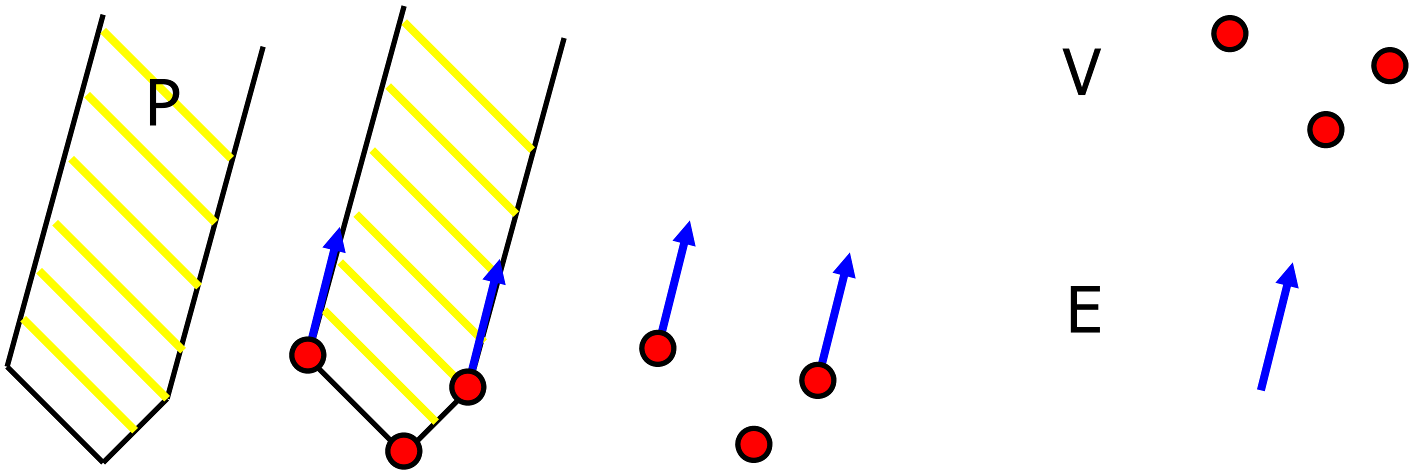
Representations of polyhedra

The ς-representation (interior representation)

$$P = \mathrm{conv}(V) + \mathrm{cone}(E).$$



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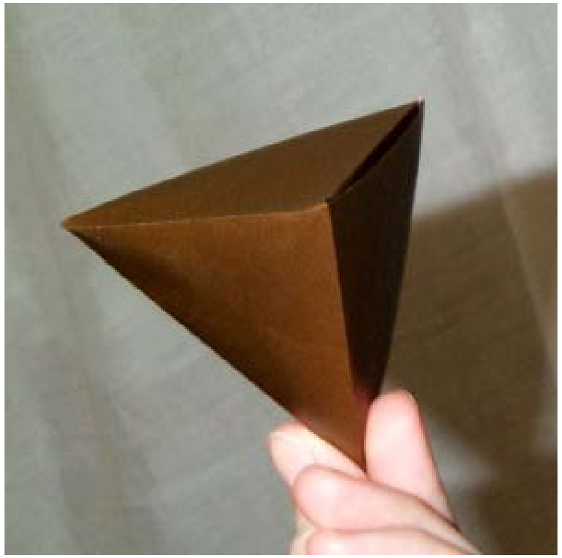
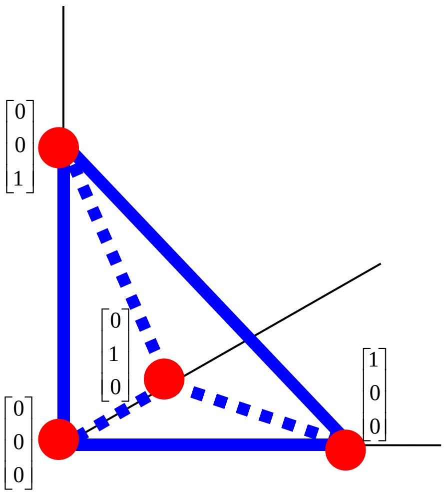
Example: the Tetrahedron



$$y \in conv\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$



$$\begin{aligned} y_1 + y_2 + y_3 &\le 1 \\ y_1 &\ge 0 \\ y_2 &\ge 0 \\ y_3 &\ge 0 \end{aligned}$$

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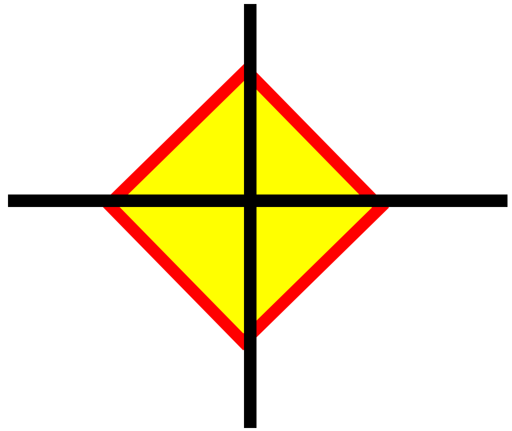
Example: the cross polytope

2n points

$$P = conv\{e_i, -e_i \mid i = 1, \ldots, n\} \subseteq \mathbf{R}^n$$



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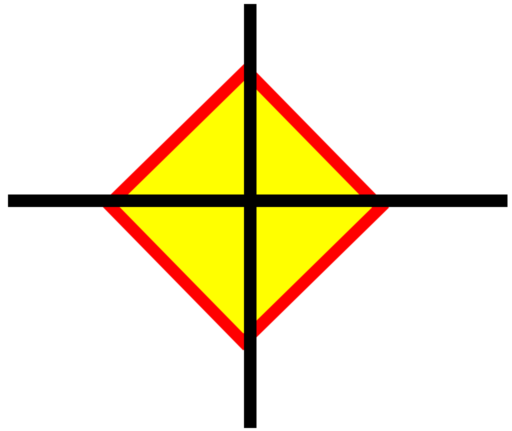
Example: the cross polytope

2n points

$$P = conv\{e_i, -e_i \mid i = 1,\ldots,n\} \subseteq \mathbf{R}^n$$



2^n inequalities

$$P = \left\{x \in \mathbf{R}^n \mid a^T x \leq 1 \ \forall \ a \in \{-1,1\}^n\right\}$$

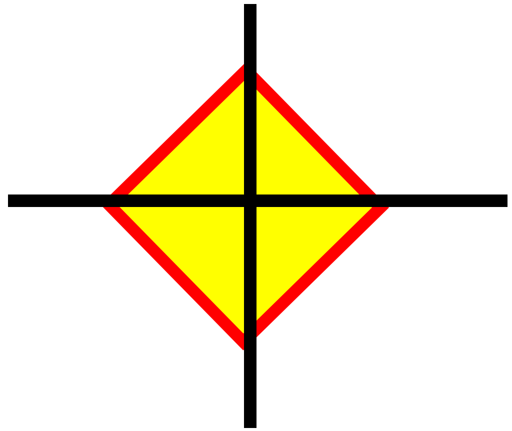
Example: the cross polytope

2n points

$$P = conv\{e_i, -e_i \mid i = 1, \ldots, n\} \subseteq \mathbf{R}^n$$

The "power" of |.|.

$$P = \left\{ x \in \mathbf{R}^n \mid \sum_{i=1}^{n} |x_i| \leq 1 \right\}$$



2^n inequalities

$$P = \left\{ x \in \mathbf{R}^n \mid a^T x \leq 1 \; \forall \; a \in \{-1, 1\}^n \right\}$$

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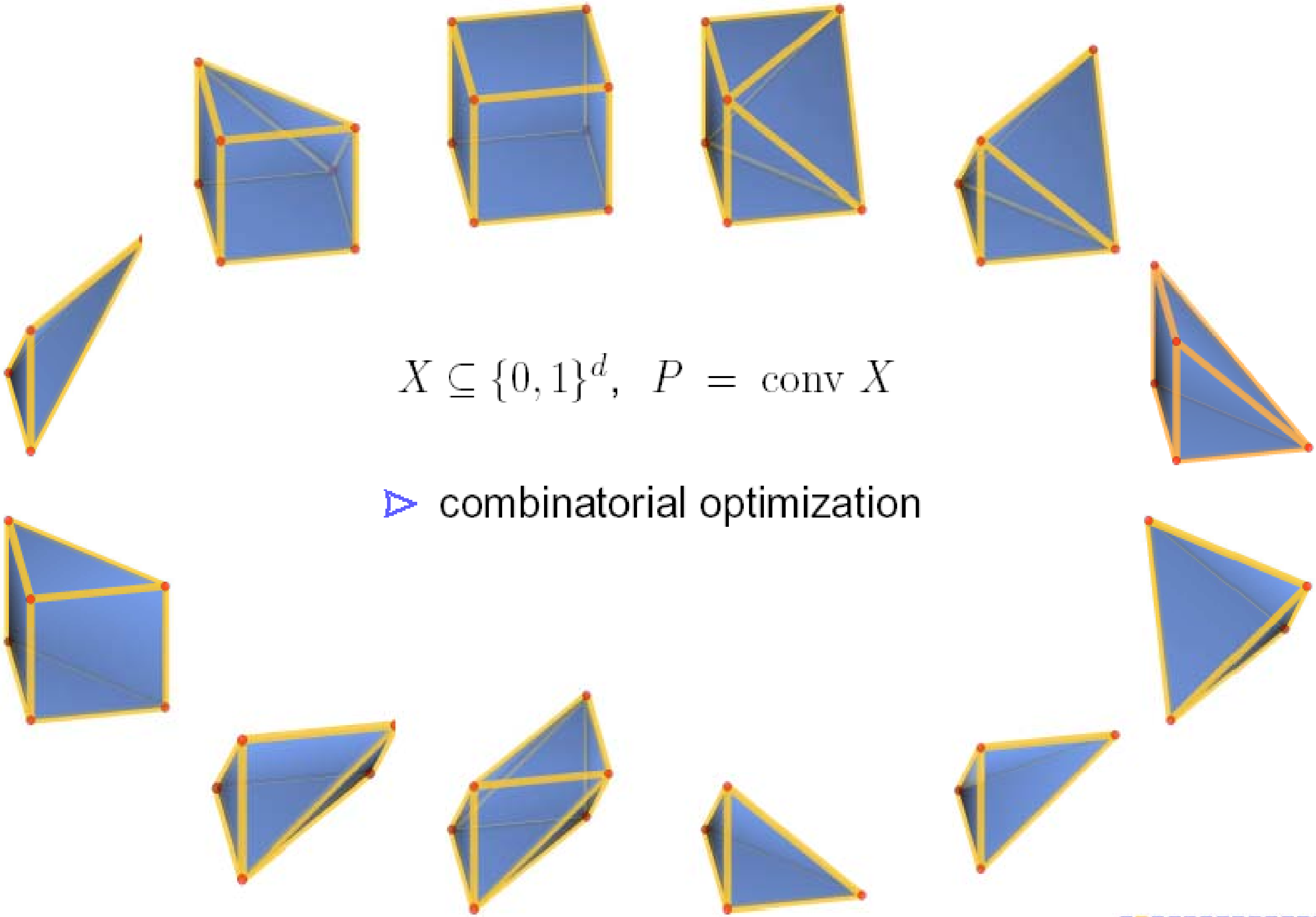
All 3-dimensional 0/1-polytopes



$$X \subseteq \{0,1\}^d, \quad P = \text{conv } X$$

▷ combinatorial optimization

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2. Polyhedra
3. Algorithms for polyhedra
   - Fourier-Motzkin elimination
   - some Web resources
4. Semi-algebraic geometry
5. Faces of polyhedra
6. Flows, networks, min-max results

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Polyedra in linear programming

- The solution sets of linear programs are polyhedra.

- If a polyhedron $P = \text{conv(V)+cone(E)}$ is given explicitly via finite sets V und E, linear programming is trivial.

- In linear programming, polyhedra are always given in H-representation. Each solution method has its „standard form".

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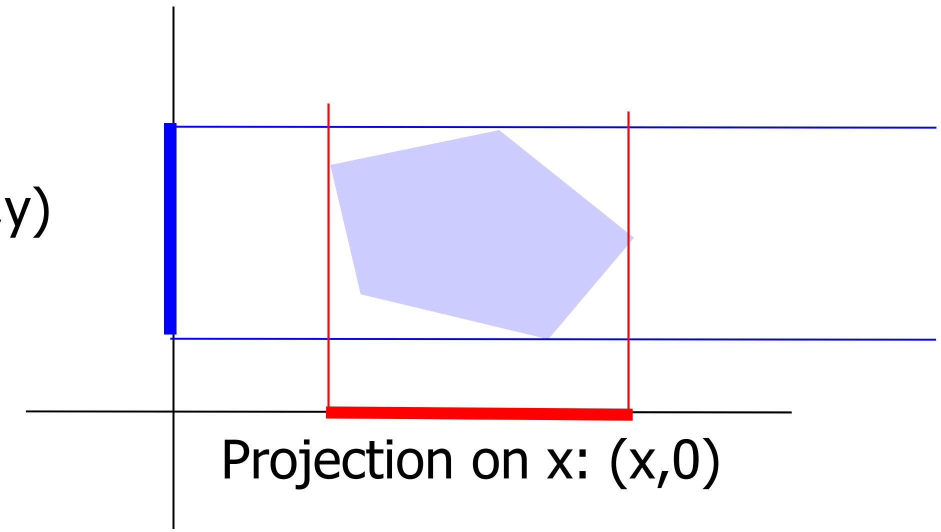
Fourier-Motzkin Elimination

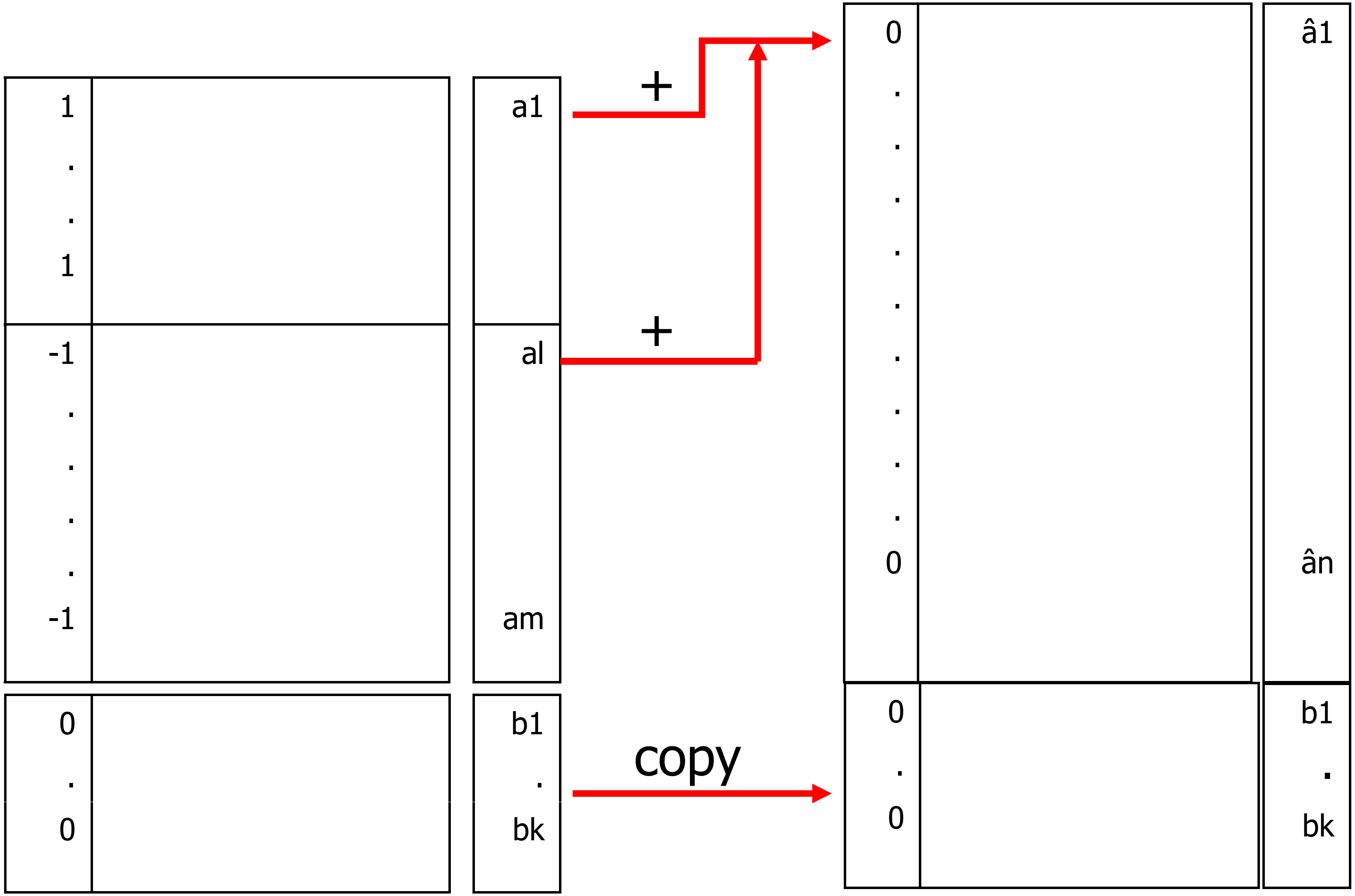
- Fourier, 1847
- Motzkin, 1938
- **Method**: successive projection of a polyhedron in n-dimensional space into a vector space of dimension n-1 by elimination of one variable.

Projection on y: (0,y)



Projection on x: (x,0)

A Fourier-Motzkin step



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Fourier-Motzkin elimination proves the Farkas Lemma

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

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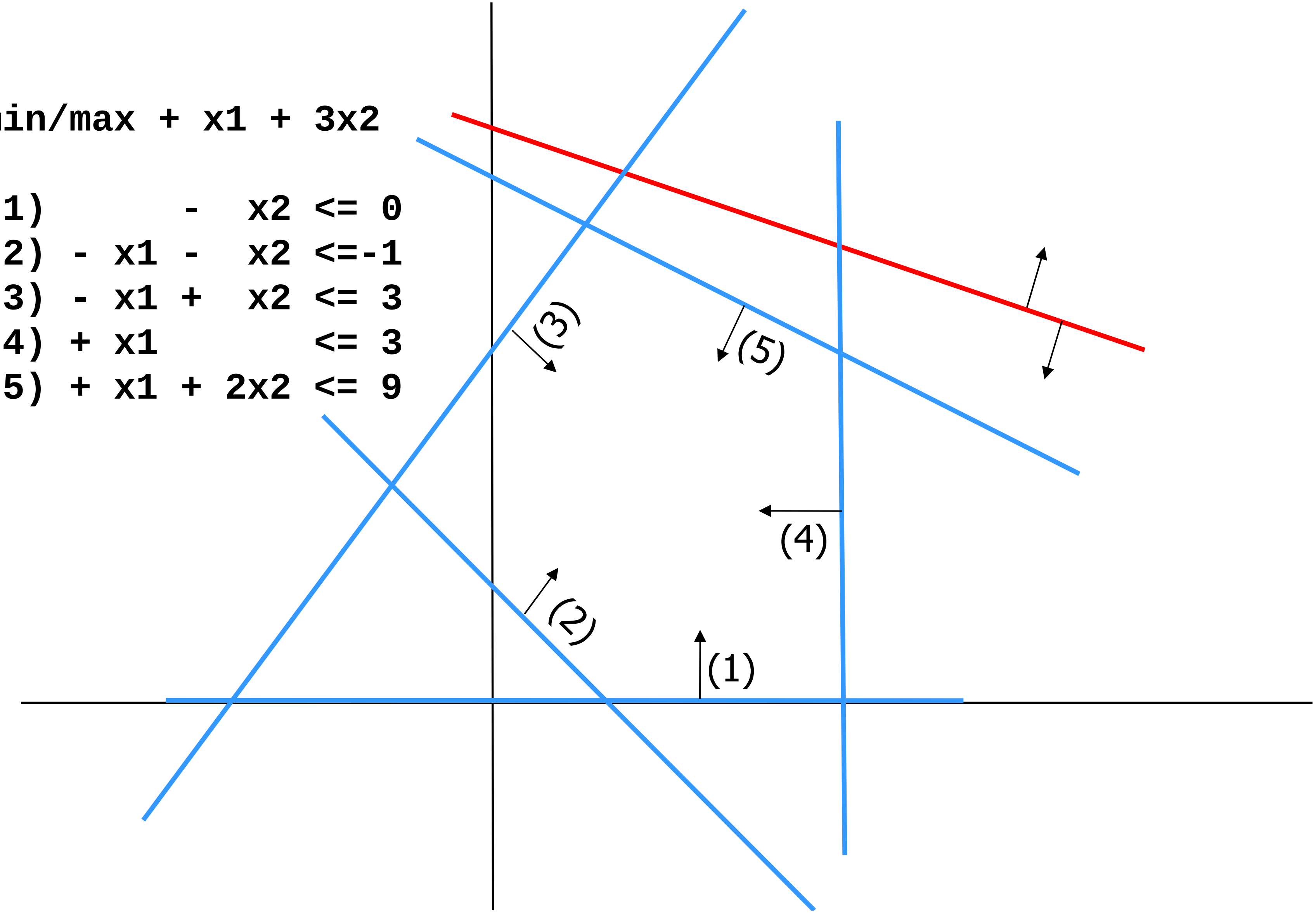
Fourier-Motzkin Elimination: an example


```
min/max + x1 + 3x2

(1)       -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1         <= 3
(5) + x1 + 2x2 <= 9
```



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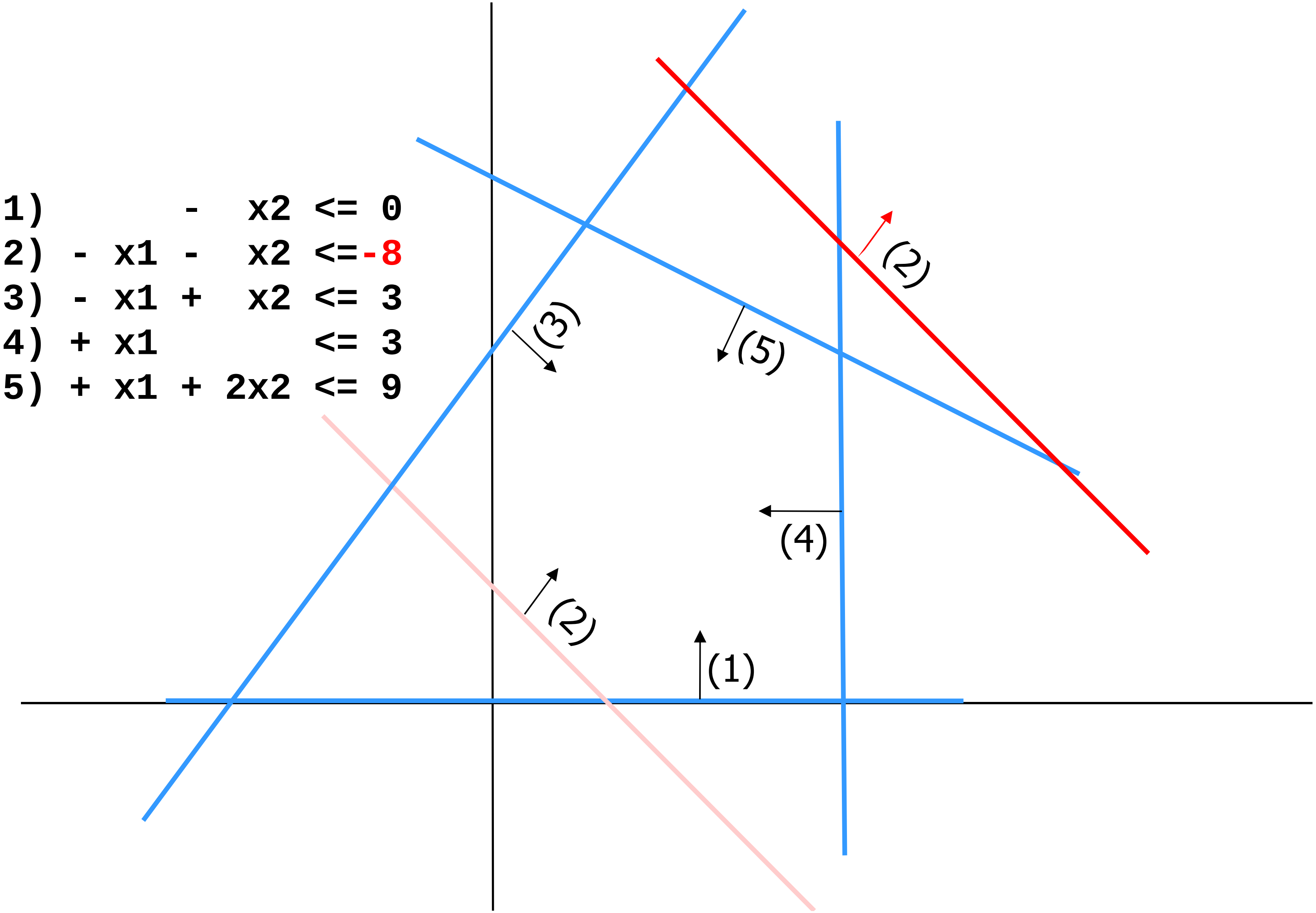
Fourier-Motzkin Elimination: an example


```
(1)         -   x2 <= 0
(2) - x1 -   x2 <=-8
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```



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Fourier-Motzkin Elimination: an example, call of PORTA


```
(1)        -   x2 <= 0
(2) - x1 -   x2 <=-8
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```

```
DIM = 3

INEQUALITIES_SECTION

(1)        -   x2        <= 0
(2) - x1 -   x2        <=-8
(3) - x1 +   x2        <= 3
(4) + x1               <= 3
(5) + x1 + 2x2        <= 9


ELIMINATION_ORDER
1 0
```

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Fourier-Motzkin Elimination: an example, call of PORTA


```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2       <=  0
(2,4) (2) - x2       <=  -5
(2,5) (3) + x2       <=  1
(3,4) (4) + x2       <=  6
(3,5) (5) + x2       <=  4
```

```
DIM = 3

INEQUALITIES_SECTION

(1)         -   x2      <= 0
(2) - x1 -   x2      <=-8
(3) - x1 +   x2      <= 3
(4) + x1              <= 3
(5) + x1 + 2x2      <= 9


ELIMINATION_ORDER
1 0
```

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Fourier-Motzkin Elimination: an example, call of PORTA


```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2      <=  0
(2,4) (2) - x2      <=  -5
(2,5) (3) + x2      <=  1
(3,4) (4) + x2      <=  6
(3,5) (5) + x2      <=  4

ELIMINATION_ORDER

0 1
```

→

```
DIM = 3

INEQUALITIES_SECTION

(2,3) 0 <= -4
```

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Fourier-Motzkin elimination proves the Farkas Lemma

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

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Which LP solvers are used in practice?

- Fourier-Motzkin: hopeless
- Ellipsoid Method: total failure
- primal Simplex Method: good
- dual Simplex Method: better
- Barrier Method: for LPs frequently even better
- For LP relaxations of IPs: dual Simplex Method

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Fourier-Motzkin works reasonably well for polyhedral transformations:

Example: Let a polyhedron be given (as usual in combinatorial optimization implicitly) via:

$$P = \text{conv(V)+cone(E)}$$

Find a non-redundant representation of P in the form:

$$P = \{x \in \mathbf{R}^d \mid Ax \leq b\}$$

Solution: Write P as follows

$$P = \{x \in \mathbf{R}^d \mid Vy + Ez - x = 0, \sum_{i=1}^{d} y_i = 1, y \geq 0, z \geq 0\}$$

and eliminate y und z.

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Relations between polyhedra representations

- Given V and E, then one can compute A und b as indicated above.
- Similarly (polarity): Given A und b, one can compute V und E.
- The Transformation of a ς-representation of a polyhedron P into a H-representation and vice versa requires exponential space, and thus, also exponential running time.
- Examples: Hypercube and cross polytope.
- That is why it is OK to employ an exponential algorithm such as Fourier-Motzkin Elimination (or Double Description) for polyhedral transformations.
- Several codes for such transformations can be found in the Internet, e.g., PORTA at ZIB and in Heidelberg.
- Lecture by Michael Joswig on Polymake! http://www.polymake.org/doku.php

The Schläfli Graph S

Claw-free Graphs VI. Colouring Claw-free Graphs

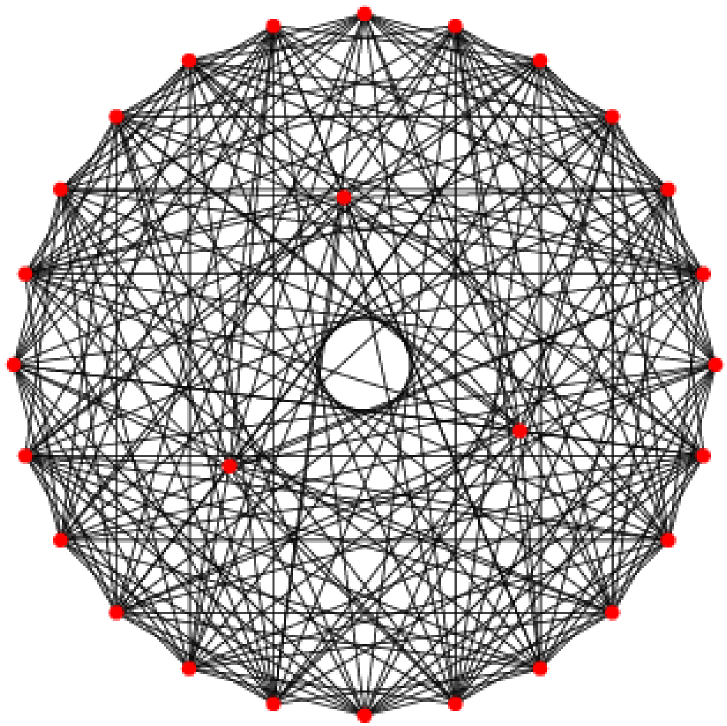
Maria Chudnovsky
Columbia University, New York NY 10027 [1]
and
Paul Seymour
Princeton University, Princeton NJ 08544 [2]

May 27, 2009

Abstract

In this paper we prove that if G is a connected claw-free graph with three pairwise non-adjacent vertices, with chromatic number χ and clique number ω, then $\chi \leq 2\omega$ and the same for the complement of G. We also prove that the choice number of G is at most 2ω, except possibly in the case when G can be obtained from a subgraph of the Schläfli graph by replicating vertices. Finally, we show that the constant 2 is best possible in all cases.

The Schläfli Graph S



Clique and stability number

Maximal cliques in S have size 6.
Maximal stable sets in S have size 3.
S has chromatic number 9 and there are two essentially different ways to color S with 9 colors. The complementary graph has chromatic number 6.

The Schläfli graph is a strongly regular graph on 27 nodes which is the graph complement of the generalized quadrangle $GQ(2, 4)$. It is the unique strongly regular graph with parameters $(27, 16, 10, 8)$ (Godsil and Royle 2001, p. 259).

http://mathworld.wolfram.com/SchlaefliGraph.html

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The Polytope of stable sets of the Schläfli Graph

input file Schlaefli.poi
dimension : 27
number of cone-points : 0
number of conv-points : 208

The incidence vectors of the stable sets of the Schläfli graph

sum of inequalities over all iterations : 527962
maximal number of inequalities : 14230

transformation to integer values
sorting system

number of equations : 0
number of inequalities : 4086

The Polytope of stable sets of the Schläfli Graph

FOURIER - MOTZKIN - ELIMINATION:

iteration	upper bound # ineq	# ineq	max bit-length	long arithmetic	non-zeros in %	mem used in kB	time used in sec
180	29	29	1	n	0.04	522	1.00
179	30	29	1	n	0.04	522	1.00
10	8748283	13408	3	n	0.93	6376	349.00
9	13879262	12662	3	n	0.93	6376	368.00
8	12576986	11877	3	n	0.93	6376	385.00
7	11816187	11556	3	n	0.93	6376	404.00
6	11337192	10431	3	n	0.93	6376	417.00
5	9642291	9295	3	n	0.93	6376	429.00
4	10238785	5848	3	n	0.92	6376	441.00
3	3700762	4967	3	n	0.92	6376	445.00
2	2924601	4087	2	n	0.92	6376	448.00
1	8073	4086	2	n	0.92	6376	448.00

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The Polytope of stable sets of the Schläfli Graph

INEQUALITIES_SECTION

(1) - x1 <= 0

(4086) +2x1+2x2+2x3+ x4+ x5+ x6 + x10+ x11+ x12+ x13+ x14+ x15
+x16+ x17+ x18+ x19+2x20 + x22+2x23 + x25+2x26 <= 3

8 different classes of inequalities found in total, among these, 5 classes have been unknown so far.

Data resources at ZIB, open access

- MIPLIB
- FAPLIB
- STEINLIB
- TSPLIB
- ...

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ZIB offerings

- **PORTA** **-** POlyhedron Representation Transformation Algorithm
- **SoPlex** **-** The **S**equential **o**bject-oriented sim**plex** class library
- **Zimpl** **-** A mathematical modelling language
- **SCIP** **-** Solving constraint integer programs (IP & MIP)

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Contents

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Semi-algebraic Geometry
Real-algebraic Geometry

$\mathrm{f}_i(x), g_j(x), h_k(x)$ are polynomials in d real variables

$$S_{\geq} := \{x \in \mathbf{R}^{d\,\mathbf{d}} : \mathrm{f}_1(x) \geq 0, \ldots, \mathrm{f}_{\mathrm{l}}(x) \geq 0\}$$ basic closed

$$S_{>} := \{x \in \mathbf{R}^{d\,\mathbf{d}} : g_1(x) > 0, \ldots, g_{\mathrm{m}}(x) > 0\}$$ basic open

$$S_{=} := \{x \in \mathbf{R}^{d\,\mathbf{d}} : h_1(x) = 0, \ldots, h_{\mathrm{n}}(x) = 0\}$$

$S := S_{\geq} \bigcup S_{>} \bigcup S_{=}$ is a semi-algebraic set

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Theorem of Bröcker(1991) & Scheiderer(1989) basic closed case

Every basic closed semi-algebraic set of the form

$$S = \{x \in \mathbf{R}^{d}: f_1(x) \geq 0, \ldots, f_l(x) \geq 0\},$$

where $f_i \in \mathbf{R}[x_1, \ldots, x_d], 1 \leq i \leq l,$ are polynomials,

can be represented by at most $d(d+1)/2$

polynomials, i.e., there exist polynomials

such that

$$p_1, \ldots, p_{d(d+1)/2} \in \mathbf{R}[x_1, \ldots, x_d]$$

$$S = \{x \in \mathbf{R}^d : p_1(x) \geq 0, \ldots, p_{d(d+1)/2}(x) \geq 0\}.$$

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Theorem of Bröcker(1991) & Scheiderer(1989) basic open case

Every basic open semi-algebraic set of the form

$$S = \{x \in \mathbf{R}^{d} : f_1(x) > 0, \ldots, f_l(x) > 0\},$$

where $f_i \in \mathbf{R}[x_1, \ldots, x_d], 1 \le i \le l,$ are polynomials,

can be represented by at most d

polynomials, i.e., there exist polynomials

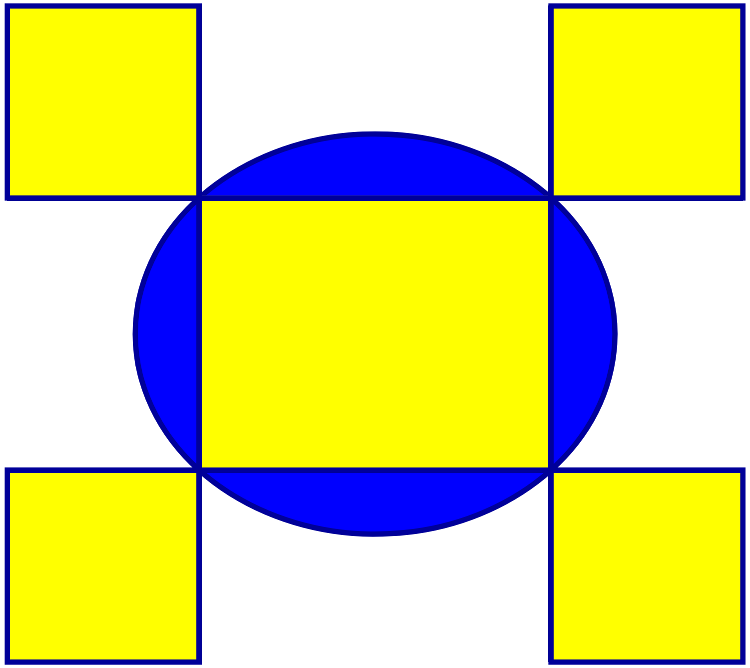
such that

$$p_1, \ldots, p_d \in \mathbf{R}[x_1, \ldots, x_d]$$

$$S = \{x \in \mathbf{R}^{d} : p_1(x) > 0, \ldots, p_d(x) > 0\}.$$

A first constructive result

Bernig [1998] proved that, for $d=2$, every convex polygon can be represented by two polynomial inequalities.

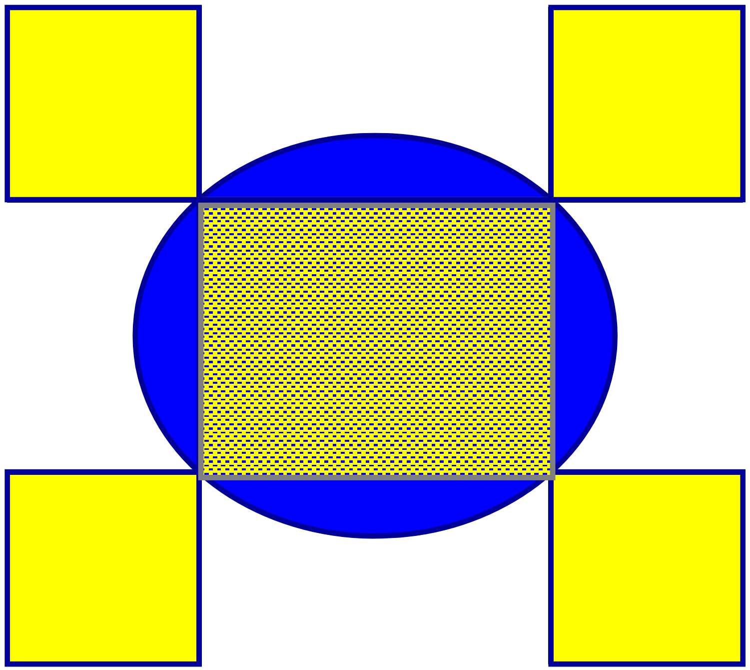


$p(1)$= product of all linear inequalities

$p(2)$= ellipse

A first Constructive Result

Bernig [1998] proved that, for $d=2$, every convex polygon can be represented by two polynomial inequalities.



p(1)= product of all linear inequalities

p(2)= ellipse

Our first theorem

Theorem Let $P \subset \mathbf{R}^n$ be a n-dimensional

polytope given by an inequality representation. Then

$k \le n^n$ polynomials $p_i \in \mathbf{R}[x_1,\ldots,x_n]$

can be constructed such that

$$P = \mathrm{P}\,(p_1,\ldots,p_k).$$

Martin Grötschel, Martin Henk:

The Representation of Polyhedra by Polynomial Inequalities

Discrete & Computational Geometry, 29:4 (2003) 485-504

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Our main theorem

Theorem Let $P \subset \mathbf{R}^n$ be a n-dimensional polytope given by an inequality representation. Then

2n polynomials $\mathrm{p}_i \in \mathbf{R}[x_1, \ldots, x_n]$

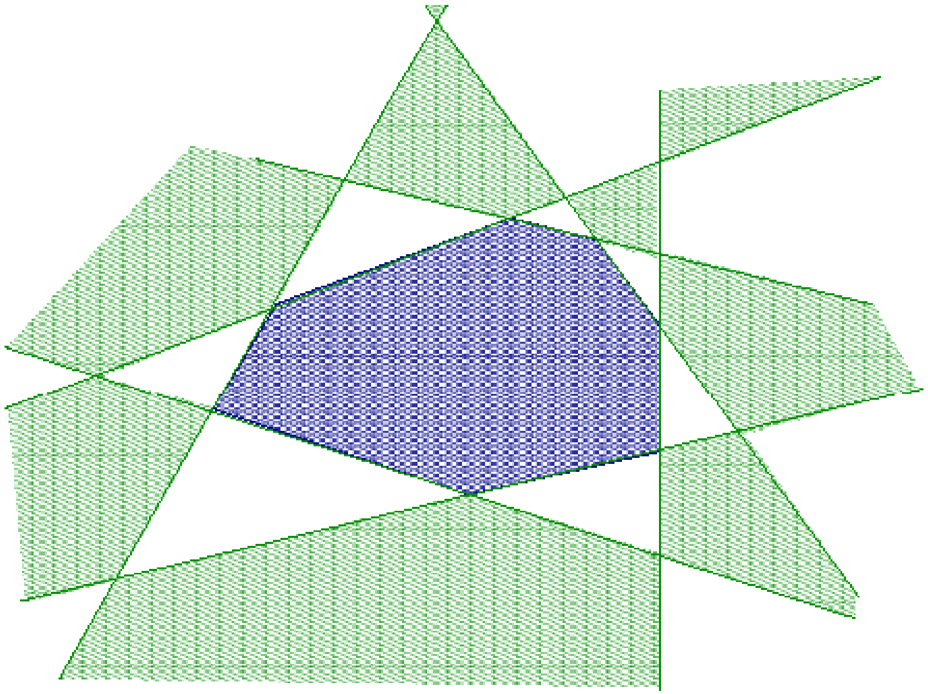
can be constructed such that

$$P = \mathrm{P}\,(\mathrm{p}_1, \ldots, \mathrm{p}_{2n}).$$

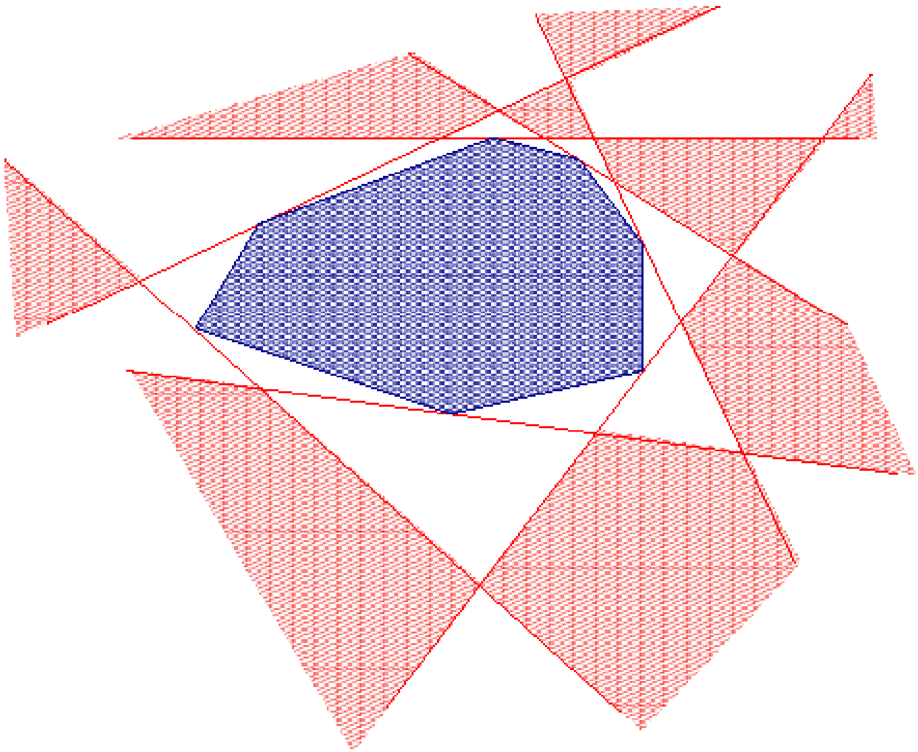
Hartwig Bosse, Martin Grötschel, Martin Henk:
Polynomial inequalities representing polyhedra
Mathematical Programming 103 (2005)35-44

http://www.springerlink.com/index/10.1007/s10107-004-0563-2

The construction in the 2-dimensional case

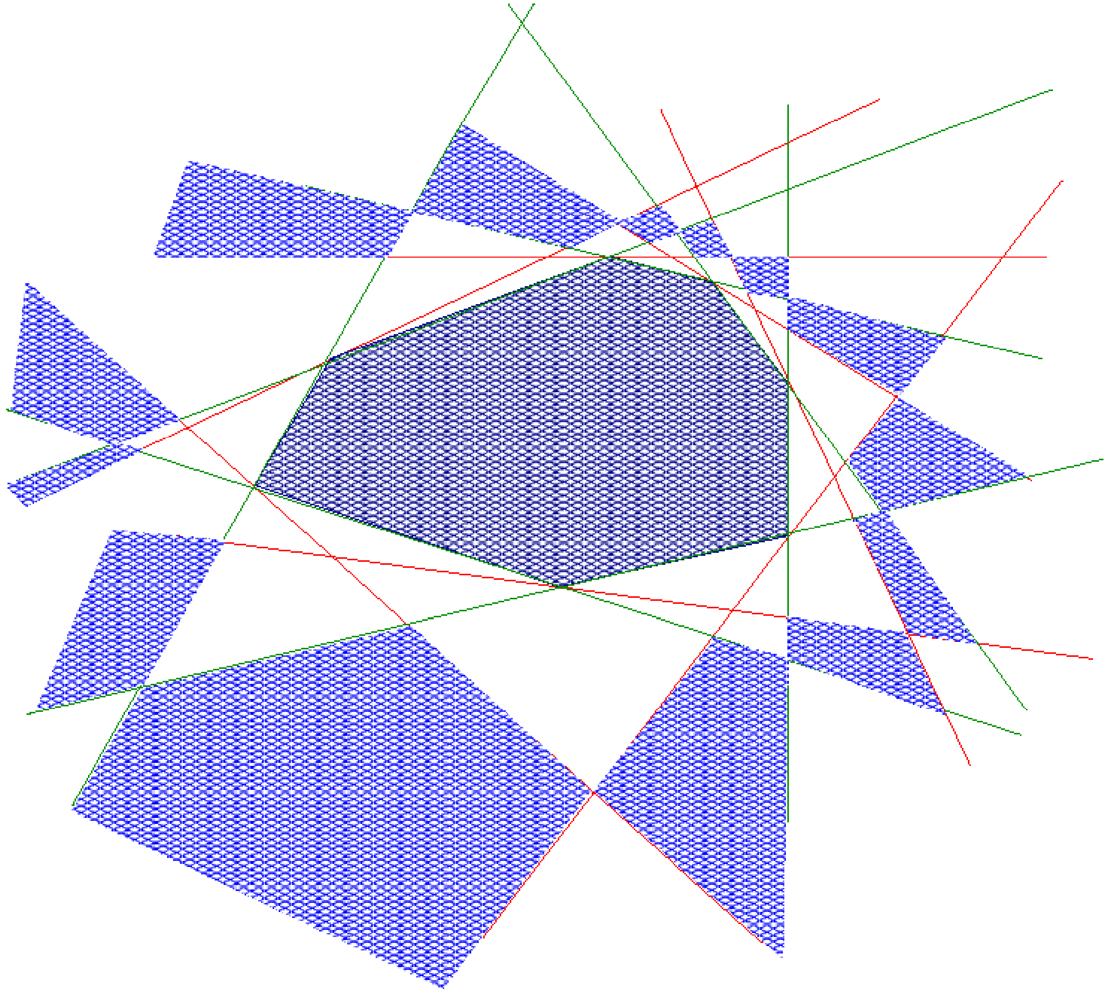


$\{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0\}$



$\{x \in \mathbb{R}^d : \mathfrak{p}_0(x) \geq 0\}$

The construction in the 2-dimensional case



$$\{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0 \text{ and } \mathfrak{p}_0(x) \geq 0\}$$

d polynomials suffice

Minimal polynomial descriptions of polyhedra and special semialgebraic sets

Gennadiy Averkov, Ludwig Bröcker

(Submitted on 4 Feb 2010)

We show that a d-dimensional polyhedron S in $\real^d$ can be represented by d-polynomial inequalities, that is, $S = \{x \in \real^d : p_0(x) \ge 0, >..., p_{d-1}(x) \ge 0 \}$, where $p_0, \ldots, p_{d-1}$ are appropriate polynomials. Furthermore, if an elementary closed semialgebraic set S is given by polynomials $q_1, \ldots, q_k$ and for each $x \in S$ at most s of these polynomials vanish in x, then S can be represented by $s + 1$ polynomials (and by s polynomials under the extra assumption that the number of points $x \in S$ in which s q_i's vanish is finite).

Subjects: **Algebraic Geometry (math.AG)**; Metric Geometry (math.MG)
MSC classes: 14P05; 52B11; 14Q99; 52A20
Cite as: arXiv:1002.0921 **[math.AG]**
(or arXiv:1002.0921v1 **[math.AG]** for this version)

Submission history

From: Gennadiy Averkov [view email]
[v1] Thu, 4 Feb 2010 08:36:55 GMT (13kb)

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Contents

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Faces etc. (extremely short)

- Important concept: dimension

- face
- vertex
- edge
- (neighbourly polytopes)
- ridge = subfacet
- facet

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Linear Programming: The DualityTheorem

CO@W

The most important and influential theorem in optimization.

$$\min\{wx \mid Ax \geq b\} = \max\{yb \mid y \geq 0, yA = w\}$$

A good research idea is to try to mimic this result:

$$\min\{something\} = \max\{something\}$$

A relation of this type is called min-max result.

Max-flow min-cut theorem
(Ford & Fulkerson, 1956)

Let $D=(V,A)$ be a directed graph, let $r,s\in V$ and let $c:A\to\mathbb{R}_+$ be a capacity function. Then the maximum value of an $r{-}s$-flow subject to the capacity c is equal to the minimum capacity of an $r{-}s$-cut.

If all capacities are integer, there exists an integer optimum flow.

Here an $r{-}s$-flow is a vector $x:A\to\mathbb{R}$ such that

(1) (i) $x(a)\geq 0 \quad \forall a\in A$

(ii) $x(\delta^-(v))=x(\delta^+(v)) \quad \forall v\in V,\ r\neq v\neq s$

The value of the flow is the net amount of flow leaving r, i.e., is

(2) $x(\delta^+(r))-x(\partial^-(r))$

(which is equal to the net amount of flow entering s). The flow x is *subject to* c if $x(a)\leq c(a)$ for all a in A.

Ford-Fulkerson animation

- http://www.cse.yorku.ca/~aaw/Wang/MaxFlowStart.htm

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Flow Algorithms

- The Ford-Fulkerson Algorithm
  The grandfather of augmenting paths algorithms
- The Dinic-Malhorta-Kumar-Maheshwari Algorithm
- Preflow (Push-Relabel) Algorithms

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Complexity survey

from Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, 2003 Springer

10.8b. Complexity survey for the maximum flow problem

Complexity survey ($*$ indicates an asymptotically best bound in the table):

$O(n^2mC)$	Dantzig [1951a] simplex method
$O(nmC)$	Ford and Fulkerson [1955,1957b] augmenting path
$O(nm^2)$	Dinits [1970], Edmonds and Karp [1972] shortest augmenting path
$O(n^2m \log nC)$	Edmonds and Karp [1972] fattest augmenting path
$O(n^2m)$	Dinits [1970] shortest augmenting path, layered network
$O(m^2 \log C)$	Edmonds and Karp [1970,1972] capacity-scaling
$O(nm \log C)$	Dinits [1973a], Gabow [1983b,1985b] capacity-scaling
$O(n^3)$	Karzanov [1974] (preflow push); cf. Malhotra, Kumar, and Maheshwari [1978], Tarjan [1984]
$O(n^2\sqrt{m})$	Cherkasskiĭ [1977a] blocking preflow with long pushes
$O(nm \log^2 n)$	Shiloach [1978], Galil and Naamad [1979,1980]
$O(n^{5/3}m^{2/3})$	Galil [1978,1980a]

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Complexity survey

from Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, 2003 Springer

continued

	$O(nm \log n)$	Sleator [1980], Sleator and Tarjan [1981,1983a] dynamic trees
*	$O(nm \log(n^2/m))$	Goldberg and Tarjan [1986,1988a] push-relabel+dynamic trees
	$O(nm + n^2 \log C)$	Ahuja and Orlin [1989] push-relabel + excess scaling
	$O(nm + n^2\sqrt{\log C})$	Ahuja, Orlin, and Tarjan [1989] Ahuja-Orlin improved
*	$O(nm \log((n/m)\sqrt{\log C} + 2))$	Ahuja, Orlin, and Tarjan [1989] Ahuja-Orlin improved + dynamic trees
*	$O(n^3/\log n)$	Cheriyan, Hagerup, and Mehlhorn [1990,1996]
	$O(n(m + n^{5/3} \log n))$	Alon [1990] (derandomization of Cheriyan and Hagerup [1989,1995])
	$O(nm + n^{2+\varepsilon})$	(for each $\varepsilon > 0$) King, Rao, and Tarjan [1992]
*	$O(nm \log_{m/n} n + n^2 \log^{2+\varepsilon} n)$	(for each $\varepsilon > 0$) Phillips and Westbrook [1993,1998]
*	$O(nm \log_{\frac{m}{n \log n}} n)$	King, Rao, and Tarjan [1994]
*	$O(m^{3/2} \log(n^2/m) \log C)$	Goldberg and Rao [1997a,1998]
*	$O(n^{2/3} m \log(n^2/m) \log C)$	Goldberg and Rao [1997a,1998]

Here $C := \|c\|_\infty$ for integer capacity function c. For a complexity survey for unit capacities, see Section 9.6a.

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Complexity survey

from Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, 2003 Springer

Research problem: Is there an $O(nm)$-time maximum flow algorithm?

For the special case of *planar* undirected graphs:

	$O(n^2 \log n)$	Itai and Shiloach [1979]
	$O(n \log^2 n)$	Reif [1983] (minimum cut), Hassin and Johnson [1985] (maximum flow)
	$O(n \log n \log^* n)$	Frederickson [1983b]
*	$O(n \log n)$	Frederickson [1987b]

For *directed* planar graphs:

	$O(n^{3/2} \log n)$	Johnson and Venkatesan [1982]
	$O(n^{4/3} \log^2 n \log C)$	Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]
*	$O(n \log n)$	Weihe [1994b,1997b]

Min-cost flow

Let $D=(V,A)$ be a directed graph, let $r,s\in V$, let $c:A\to\mathbb{R}_+$ be a capacity function, $w:A\to\mathbb{R}$ a cost function, and f a flow value. Find a flow x of value f subject to c with minimum value $w^T x$.

$$\min \sum_{a\in A} w(a)x(a)$$

$$0\le x(a)\le c(a)\quad \forall a\in A$$

$$x\left(\delta^+(v)\right)-x\left(\partial^-(v)\right)=0\quad \forall r\ne v\ne s$$

$$x\left(\delta^+(r)\right)-x\left(\partial^-(r)\right)=f$$

There is a similarly large number of algorithms with varying complexity, see Schrijver (2003).

Min-Max Results

König´s Matching Theorem (1931) (Frobenius, 1912)

The maximum size of a matching in a bipartite graph is equal to the minimum number of vertices covering all edges, i. e.,

$$\nu\left(G\right)=\tau\left(G\right)$$

for bipartite graphs G.

Tutte-Berge Formula (Tutte(1947), Berge(1958))

$$\max\left\{|M|: M \subseteq E \text{ } matching\right\} = \min_{W\subseteq V} \frac{1}{2}(|V|+|W|-O(G-W))$$

where G=(V,E) is an arbitrary graph.

Total unimodularity

A matrix A is called *totally unimodular* if each square submatrix of A has determinant 0, +1 or –1. In particular, each entry of A is 0, +1 or –1.

The interest of totally unimodular matrices for optimization was discovered by the following theorem of Hoffman and Kruskal (1956):

If A is totally unimodular and b and w are integer vectors, then both sides of the LP-duality equation

$$\max\{wx \mid Ax \le b\} = \min\{yb \mid y \ge 0, yA = w\}$$

have integer optimum solutions.

Total unimodularity

There have been many characterizations of totally unimodular matrices:

Ghouila-Houri (1962)

Camion (1965)

Padberg (1976)

Truemper(1977)

....

Full understanding was achieved by establishing a link to regular matroids, Seymour (1980). This connection also yields a polynomial time algorithm to recognize totally unimodular matrices.

Min-Max Results

Dilworth's theorem (1950)

The maximum size of an antichain in a partially ordered set $(P, <)$ is equal to the minimum number of chains needed to cover P.

Fulkerson's optimum branching theorem (1974)

Let $D=(V,A)$ be a directed graph, let $r \in V$ and let $l : A \to R_+$ be a length function. Then the minimum length of an r-arborescence is equal to the maximum number t of r-cuts $C_1, \ldots, C_t$ (repetition allowed) such that no arc a is in more than $l(a)$ of the C_i.

Edmonds' disjoint branching theorem (1973)

Let $D=(V,A)$ be a directed graph, and let $r \in V$. Then the maximum number of pairwise disjoint r-arborescences is equal to the minimum size of an r-cut.

Min-Max Results

Edmonds' matroid intersection theorem (1970)

Let $M_1 = (S, J_1)$ and $M_2 = (S, J_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the maximum size of a set in $J_1 \cap J_2$ is equal to

$$\min_{S' \subseteq S} \left(r_1(S') + r_2(S \setminus S') \right).$$

Min-Max Results and Polyhedra

- Min-max results almost always provide polyhedral insight and can be employed to prove integrality of polyhedra.
- For instance, the matroid intersection theorem can be used to prove a theorem on the integrality of the intersection of two matroid polytopes.

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Min-Max Results and Polyhedra

Let M=(E, $\mathcal{I}$) be a matroid with rank function *r*.

Define IND($\mathcal{I}$):=conv{x^I | I is an Element of $\mathcal{I}$}.

IND($\mathcal{I}$) is called matroid polytope. Let

$$P(\mathcal{I}) := \left\{ x \in \mathbf{R}^E : \sum_{e \in F} x_e \le r(F) \ \forall\ F \subseteq E,\ x_e \ge 0 \ \forall\ e \in E \ \right\}$$

Theorem: P($\mathcal{I}$) = IND($\mathcal{I}$).

Theorem: Let M_1=(E, $\mathcal{I}_1$) and M_2=(E, $\mathcal{I}_2$) be two matroids with rank functions r_1 and r_2, respectively. Then

IND($\mathcal{I}_1 \mathrm{I} \mathcal{I}_2$) = P($\mathcal{I}_1$)IP($\mathcal{I}_2$)

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Min-Max Results and Polyhedra

In other words, if $M_1=(E, \mathcal{I}_1)$ and $M_2=(E, \mathcal{I}_2)$ are two matroids on the same ground set E with rank functions r_1 and r_2, respectively, and if c_e is a weight for all elements e of E, then a set that is independent in M_1 and M_2 and has the largest possible weight can be found via the following linear program

$$\max \sum_{e \in E} c_e x_e$$

$$\sum_{e \in F} x_e \leq r_1(F) \ \forall \ F \subseteq E$$

$$\sum_{e \in F} x_e \leq r_2(F) \ \forall \ F \subseteq E$$

$$x_e \geq 0 \ \forall \ e \in E$$

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An Excursion into Matroid Theory

- If time permits

Matroids and Independence Systems

CO@W

Let E be a finite set, $\mathcal{I}$ a subset of the power set of E.

The pair $(E,\mathcal{I})$ is called independence system on E if the following axioms are satisfied:

(I.1) The empty set is in $\mathcal{I}$.

(I.2) If J is in $\mathcal{I}$ and I is a subset of J then
I belongs to $\mathcal{I}$.

Let $(E,\mathcal{I})$ satisfy in addition:

(I.3) If I and J are in $\mathcal{I}$ and if J is larger than I then
there is an element j in J, j not in I, such that
the union of I and j is in $\mathcal{I}$.

Then $M=(E,\mathcal{I})$ is called a matroid.

Notation

Let $(E,\mathcal{I})$ be an independence system.

- Every set in $\mathcal{I}$ is called independent.
- Every subset of E not in $\mathcal{I}$ is called dependent.
- For every subset F of E, a basis of F is a subset of F that is independent and maximal with respect to this property.

The rank r(F) of a subset F of E is the cardinality of a largest basis of F. The lower rank $r_u(F)$ of F is the cardinality of a smallest basis of F.

The Largest Independent Set Problem

Problem:

Let $(E, \mathcal{I})$ be an independence system with weights on the elements of E. Find an independent set of largest weight.

We may assume w.l.o.g. that all weights are nonnegative (or even positive), since deleting an element with nonpositive weight from an optimum solution, will not decrease the value of the solution.

The Greedy Algorithm

Let $(E, \mathcal{I})$ be an independence system with weights c(e) on the elements of E. Find an independent set of largest weight.

The Greedy Algorithm:

1. Sort the elements of E such that $c_1 \geq c_2 \geq \ldots \geq c_n \geq 0$.
2. Let $I_{\text{greedy}} := \emptyset$.
3. FOR i=1 TO n DO:

   IF $I_{\text{greedy}} \cup \{i\} \in \mathcal{I}$ THEN $I_{\text{greedy}} := I_{\text{greedy}} \cup \{i\}$.

4. OUTPUT I_{greedy}.

A key idea is to interprete the greedy solution as the solution of a linear program.

Polytopes and LPs

Let M=(E,$\mathcal{I}$) be an independence system

with weights c(e) on the elements of E.

$$\text{IND(M)} = conv\left\{x^{\mathrm{I}} \in \mathbf{R}^E \mid \mathrm{I} \in \mathcal{I}\right\}$$

$$= conv\left\{x \in \mathbf{R}^E \,\middle|\, \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E\right\}$$

The LP relaxation

$$\min c^T x \qquad \text{s.t. } \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,$$

$$x_e \ge 0 \qquad \forall\ e \in E$$

The dual LP

$$\min \sum_{F \subseteq E} y_F r(F) \quad \text{s.t. } \sum_{F \ni e} y_F \ge c_e \qquad \forall\ e \in E,$$

$$y_F \ge 0 \qquad \forall\ F \subseteq E$$

The Dual Greedy Algorithm

CO@W

Let $(E,\mathcal{I})$ be an independence system with weights c(e) for all e.

After sorting the elements of E so that

$c_1 \geq c_2 \geq \ldots \geq c_n \geq 0,\ c_{n+1} := 0$ set

$$E_i := \{1, 2, \ldots, i\},\ i=1, 2, \ldots, n \text{ and}$$

$$y_{E_i} := c_i - c_{i+1}, \qquad i=1, 2, \ldots, n.$$

Then $y_{E_i} = c_i - c_{i+1}$, i=1, 2, ..., n

is a feasible solution of the dual LP

$$\min \sum_{F \subseteq E} y_F r_u(F),$$

$$\text{s.t.} \sum_{F \ni e} y_F \geq c_e \quad \forall\ e \in E,$$

$$y_F \geq 0 \quad \forall\ F \subseteq E$$

Observation

Let (E,$\mathcal{I}$) be an independence system with weights c(e) for all e.

After sorting the elements of E so that

$c_1 \geq c_2 \geq \ldots \geq c_n \geq 0,\ c_{n+1} := 0$

we can express every greedy and optimum solution as follows

$$c(\mathrm{I}_{\text{greedy}}) = \sum_{i=1}^{n} (c_i - c_{i+1}) \left| \mathrm{I}_{\text{greedy}} \cap E_i \right|$$

$$c(\mathrm{I}_{\text{opt}}) = \sum_{i=1}^{n} (c_i - c_{i+1}) \left| \mathrm{I}_{\text{opt}} \cap E_i \right|$$

Rank Quotient

Let $(E,\mathcal{I})$ be an independence system with weights c(e) for all e.

$$q := \min_{\substack{F \subseteq E \\ r(F) > 0}} \frac{r_u(F)}{r(F)}$$

The number q is between 0 and 1 and is called rank quotient of $(E,\mathcal{I})$.

Observation: q = 1 iff $(E,\mathcal{I})$ is a matroid.

The General Greedy Quality Guarantee

$$\max \sum_{e \in E} c_e x_e, \text{s.t.} \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E$$

$$\ge \max \sum_{e \in E} c_e x_e, \text{s.t.} \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E,\ x \text{ integral}$$

$$= c(\mathrm{I}_{opt}) \ge c(\mathrm{I}_{greedy}) = \sum_{i=1}^{n} (c_i - c_{i+1}) \left| \mathrm{I}_{greedy} \cap E_i \right| \ge \sum_{i=1}^{n} (c_i - c_{i+1}) r_u(E_i)$$

$$= \sum_{i=1}^{n} y_{E_i} r_u(E_i)$$

$$\ge \min \sum_{\mathrm{F} \subseteq \mathrm{E}} y_F r_u(F), \text{s.t.} \sum_{\mathrm{F} \ni \mathrm{e}} \mathrm{y_F} \ge c_e\ \forall\ e \in E, y_F \ge 0\ \forall\ F \subseteq E$$

$$\ge q \min \sum_{\mathrm{F} \subseteq \mathrm{E}} y_F r(F), \text{s.t.} \sum_{\mathrm{F} \ni \mathrm{e}} \mathrm{y_F} \ge c_e\ \forall\ e \in E, y_F \ge 0\ \forall\ F \subseteq E$$

$$= q \max \sum_{e \in E} c_e x_e, \text{s.t.} \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E$$

$$\ge q \max \sum_{e \in E} c_e x_e, \text{s.t.} \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E,\ x \text{ integral}$$

$$= q\, c(\mathrm{I}_{opt})$$ a quality guarantee

Consequences

Let M=(E,$\mathcal{I}$) be an independence system with weights c(e) on the elements of E.

$$\text{IND(M)} = conv\left\{x^{\text{I}} \mid \text{I} \in \mathcal{I}\right\}$$

$$\text{P(M)} = \left\{x \in \mathbf{R}^E \,\middle|\, \sum_{e \in F} x_e \le r(F) \;\forall\; F \subseteq E,\; x_e \ge 0 \;\forall\; e \in E\right\}$$

Theorem: (a) P(M) = IND(M) if and only if M is a matroid

(b) If M is a matroid then all optimum vertex solutions of the primal LP

$$\max c^T x \qquad \text{s.t.} \sum_{e \in F} x_e \le r(F) \;\forall\; F \subseteq E,\; x_e \ge 0 \quad \forall\; e \in E$$

are integral. If the weights are integral then the dual LP

$$\min \sum_{F \subseteq E} y_F r(F) \quad \text{s.t.} \sum_{F \ni e} y_F \ge c_e \quad \forall\; e \in E,\; y_F \ge 0 \quad \forall\; F \subseteq E$$

also has integral optimum solutions,

i.e., the system is totally dual integral.

Min-Max Results: Challenges

Despite all the beautiful min-max results mentioned before (and the not mentioned far reaching generalizations such as submodular flows or matroid matching), there is still a great challenge:

understand integral duality.

Where and when does it occur?

Why?....

Contents

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Combinatorial optimization

Given a finite set E and a subset $\mathcal{I}$ of the power set of E (the set of feasible solutions). Given, moreover, a value (cost, length,…) c(e) for all elements e of E. Find, among all sets in $\mathcal{I}$, a set I such that its total value c(I) (= sum of the values of all elements in I) is as small (or as large) as possible.

The parameters of a combinatorial optimization problem are: (E, $\mathcal{I}$, c).

$$\min \left\{ c(\mathrm{I}) = \sum_{e \in \mathrm{I}} c(e) \mid \mathrm{I} \in \mathcal{I} \right\}, \ where\ \mathcal{I} \subseteq 2^E and\ \ E\ \ finite$$

An important issue: How is $\mathcal{I}$ given?

The travelling salesman problem

Given n „cities" and „distances" between them. Find a tour (roundtrip) through all cities visiting every city exactly once such that the sum of all distances travelled is as small as possible. (TSP)

The TSP is called symmetric (STSP) if, for every pair of cities i and j, the distance from i to j is the same as the one from j to i, otherwise the problem is called asymmetric (ATSP).

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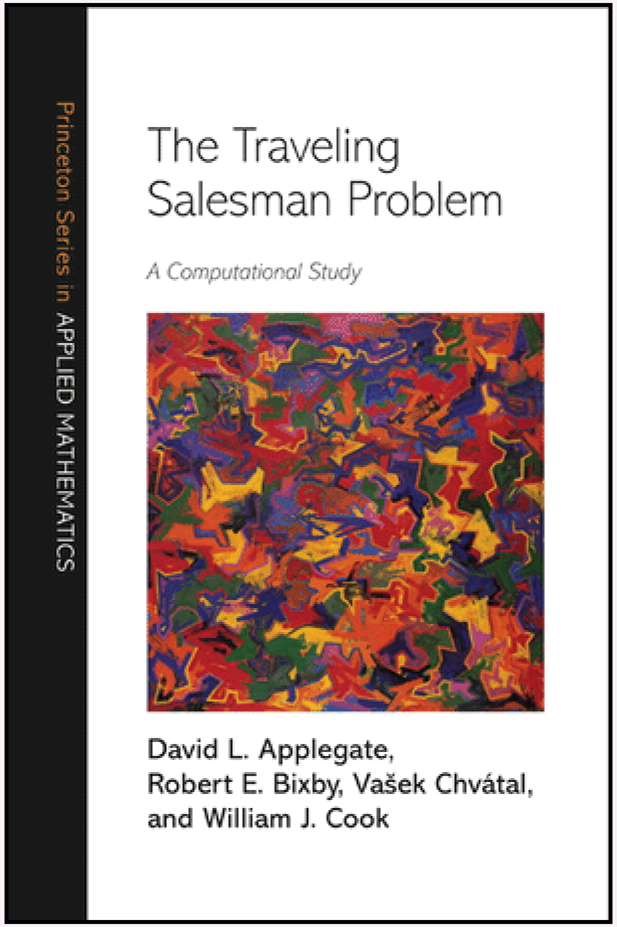
THE TSP book

suggested reading for everyone interested in the TSP



The Traveling Salesman Problem:
A Computational Study

David L. Applegate, Robert E. Bixby, Vasek Chvátal & William J. Cook

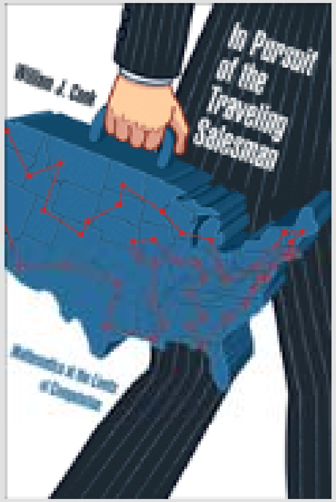
Princeton University Press 2006, 606 pp.

Our primary concern in this book is to describe a method and computer code that have succeeded in solving a wide range of large-scale instances of the TSP. Along the way we cover the interplay of applied mathematics and increasingly more powerful computing platforms, using the solution of the TSP as a general model in computational science.

- Table of Contents
- Links to Bookstores

Cover illustration by Julian Lethbridge, *Traveling Salesman 4*, 1995, oil on linen, 72 x 72 inches, The Robert and Jane Meyerhoff Collection, photograph by Adam Reich.

Another recommendation
Bill Cook‘s new book



Google Preview

In Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation
William J. Cook

Cloth | 2012 | **$27.95** / £19.95 | ISBN: 9780691152707
248 pp. | 6 x 9 | 113 color illus. 19 halftones. 19 line illus. 2 tables.

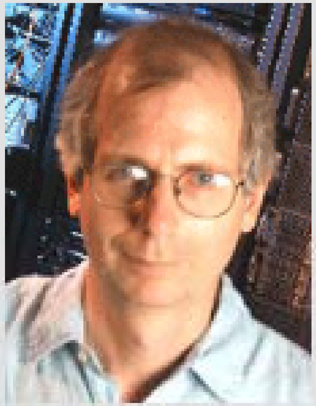
eBook | ISBN: 9781400839599 | **Where to buy this ebook**

Shopping Cart | Reviews | Table of Contents
Chapter 1

Google full text of this book:

GO

Further explorations with the TSP



(Princeton L & L Lecture)

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The Travelling Salesman Problem and Some of its Variants

- The symmetric TSP
- The asymmetric TSP
- The TSP with precedences or time windows
- The online TSP
- The symmetric and asymmetric m-TSP
- The price collecting TSP
- The Chinese postman problem (undirected, directed, mixed)
- Bus, truck, vehicle routing
- Edge/arc & node routing with capacities
- Combinations of these and more

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The travelling salesman problem

Two mathematical formulations of the TSP

1. *Version:*

Let $K_n = (V, E)$ *be the complete graph or digraph with n nodes and let* c_e *be the length of* $e \in E$. *Let* H *be the set of all hamiltonian cycles (tours) in* K_n. *Find*

$$\min\{c(T) \mid T \in \mathrm{H}\}.$$

2. *Version:*

Find a cyclic permutation π *of* $\{1, \ldots, n\}$ *such that*

$$\sum_{i=1}^{n} c_{i\pi(i)}$$

is as small as possible.

Does that help solve the TSP?

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Polyhedral Theory (of the TSP)

STSP-, ATSP-,TSP-with-side-constraints-

Polytope:= Convex hull of all incidence vectors of feasible tours

To be investigated:

- Dimension
- Equation system defining the affine hull
- Facets
- Separation algorithms

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The symmetric travelling salesman polytope

$$Q_T^n := conv\{\chi^T \in \mathbf{Z}^E \mid T \ tour \ in \ K_n\} \qquad (\chi_{ij}^T = 1 \ if \ ij \in T, \ else = 0)$$

$$\begin{aligned} \subseteq \{x \in \mathbf{R}^E \mid x(\delta(i)) &= 2 && \forall i \in V \\ x(E(W)) &\leq |W|-1 && \forall W \subset V \setminus \{1\}, 3 \leq |W| \leq n-3 \\ 0 \leq x_{ij} &\leq 1 && \forall ij \in E\} \end{aligned}$$

$$\begin{aligned} &\min c^T x \\ &x(\delta(i)) = 2 && \forall i \in V \\ &x(E(W)) \leq |W|-1 && \forall W \subset V \setminus \{1\}, 3 \leq |W| \leq n-3 \\ &x_{ij} \in \{0,1\} && \forall ij \in E \end{aligned}$$

The LP relaxation is solvable in polynomial time

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General cutting plane theory: Gomory Mixed-Integer Cut

- Given $y, x_j \in ¢_+$, and

$$y + \sum a_{ij} x_j = d = \lfloor d \rfloor + f,\ f > 0$$

- Rounding: Where $a_{ij} = \lfloor a_{ij} \rfloor + f_j,$ define

$$t = y + \sum \left(\lfloor a_{ij} \rfloor x_j : f_j \le f \right) + \sum \left(\lceil a_{ij} \rceil x_j : f_j > f \right) \in ¢$$

- Then

$$\sum \left(f_j x_j : f_j \le f \right) + \sum \left(f_j - 1 \right) x_j : f_j > f = d - t$$

- Disjunction:

$$t \le \lfloor d \rfloor \Rightarrow \sum \left(f_j x_j : f_j \le f \right) \ge f$$

$$t \ge \lceil d \rceil \Rightarrow \sum \left(\left(1 - f_j \right) x_j : f_j > f \right) \ge 1 - f$$

- Combining

$$\sum \left(\left(f_j / f \right) x_j : f_j \le f \right) + \sum \left(\left[\left(1 - f_j \right) / \left(1 - f \right) \right] x_j : f_j > f \right) \ge 1$$

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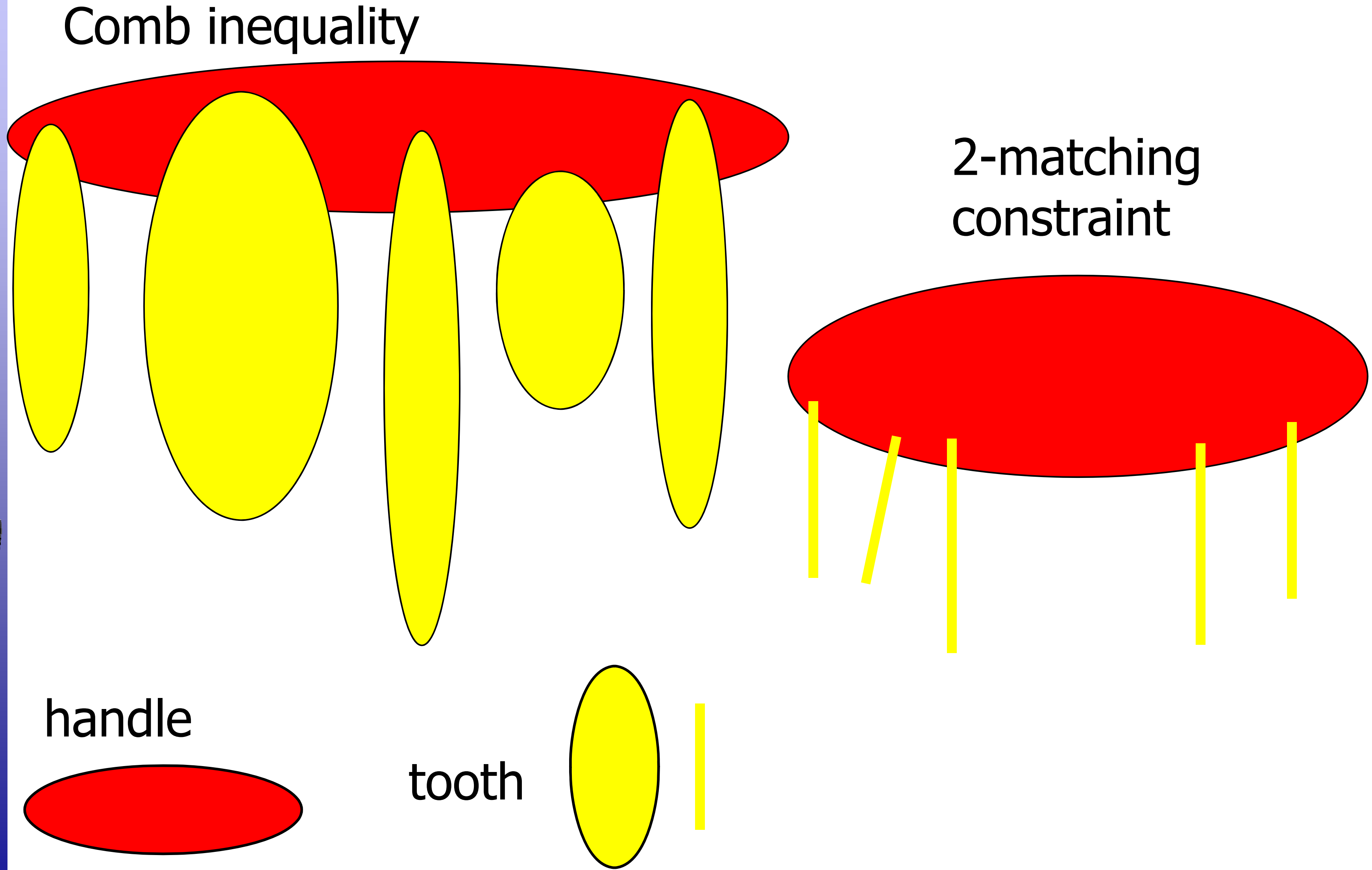
clique trees

- A clique tree is a connected graph C=(V,E), composed of cliques satisfying the following properties

(1) The cliques are partitioned into two sets, the set of *handles* and the set of *teeth*.
(2) No two teeth intersect.
(3) No two handles intersect.
(4) Each tooth contains at least two and at most $n-2$ vertices and at least one vertex not belonging to any handle.
(5) The number of teeth that each handle intersects is odd and at least three.
(6) If a tooth T and a handle H have a nonempty intersection, then $H \cap T$ is an articulation set of the clique tree.

Polyhedral Theory of the TSP



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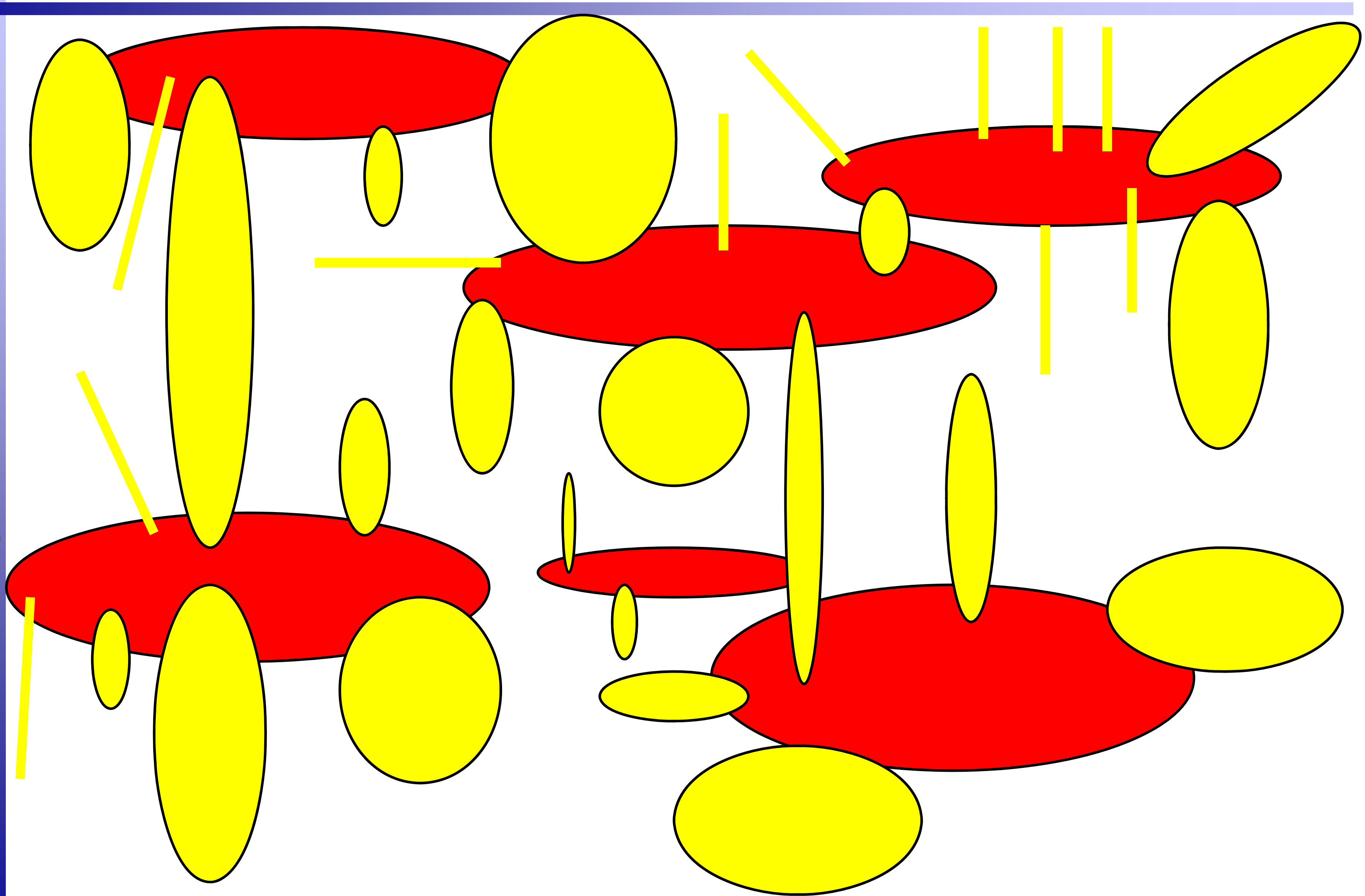
Clique Tree Inequalities



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Clique Tree Inequalities

http://www.zib.de/groetschel/pubnew/paper/groetschelpulleyblank1986.pdf

CO@W

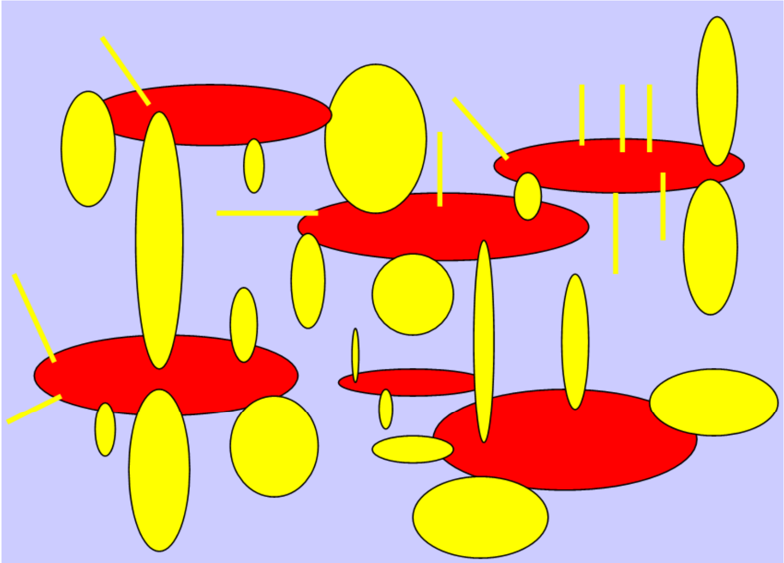
$$\sum_{i=1}^{h} x(\partial(H_i)) + \sum_{j=1}^{t} x(\partial(T_j)) \geq \sum_{i=1}^{h} |H_i| + h + 2t$$

$$\sum_{i=1}^{h} x(E(H_i)) + \sum_{j=1}^{t} x(E(T_j)) \leq \sum_{i=1}^{h} |H_i| + \sum_{i=1}^{t} (|T_j| - t_j) - \frac{t+1}{2}$$

H_i, i=1,…,h are the handles

T_j, j=1,…,t are the teeth

t_j is the number of handles that tooth T_j intersects



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Valid Inequalities for STSP

Trivial inequalities
Degree constraints
Subtour elimination constraints
2-matching constraints, comb inequalities
Clique tree inequalities (comb)
Bipartition inequalities (clique tree)
Path inequalities (comb)
Star inequalities (path)
Binested Inequalities (star, clique tree)
Ladder inequalities (2 handles, even # of teeth)
Domino inequalities
Hypohamiltonian, hypotraceable inequalities
etc.

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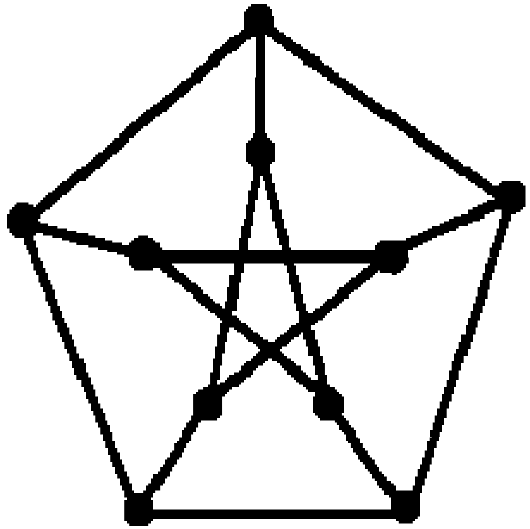
A very special case



Petersen graph, G = (V, F),

the smallest hypohamiltonian graph

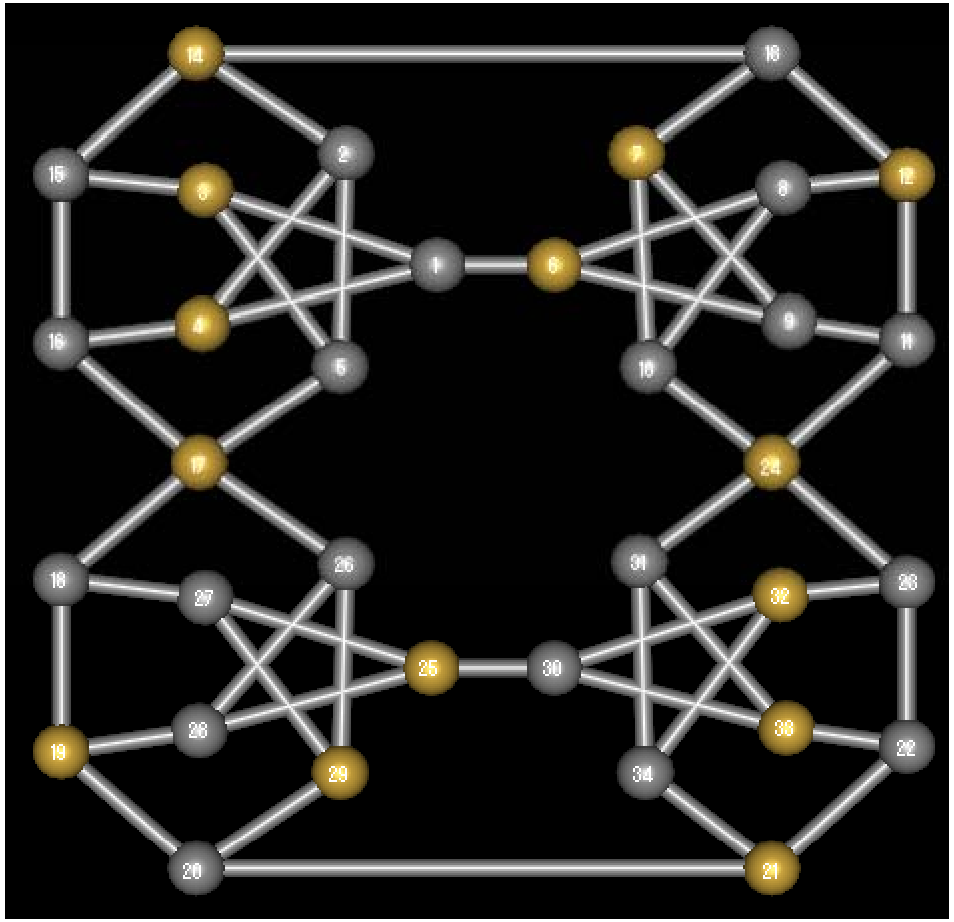
$$x(F) \leq 9 \;\; \text{defines a facet of } Q_T^{10}$$

$$\text{but not a facet of } Q_T^n, n \geq 11$$

M. Grötschel & Y. Wakabayashi

Hypotraceable graphs and the STSP

On the right is the smallest known hypotraceable graph (Thomassen graph, 34 nodes).



Such graphs have no hamiltonian path, but when any node is deleted, the remaining graph has a hamiltonian path.

How do such graphs induce inequalities valid for the symmetric travelling salesman polytope?

For further information see:
http://www.zib.de/groetschel/pubnew/paper/groetschel1980b.pdf

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“Wild facets of the asymmetric travelling salesman polytope”

- Hypohamiltonian and hypotraceable directed graphs also exist and induce facets of the polytopes associated with the asymmetric TSP.

Theorem 4.11. *There are hypohamiltonian digraphs of order* n *if and only if* $n \geq 6$. ◻

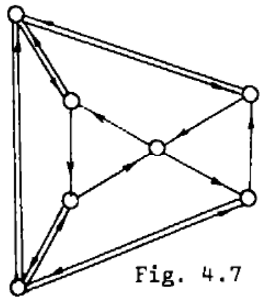

Fig. 4.7

Theorem 1. There exists a hypotraceable digraph of order n iff $n \geq 7$. Furthermore, for each $k \geq 1$ there exist infinitely many hypotraceable digraphs with precisely k strong components.

- Information “hypohamiltonian” and “hypotraceable” inequalities can be found in
  http://www.zib.de/groetschel/pubnew/paper/groetschelwakabayashi1981a.pdf
  http://www.zib.de/groetschel/pubnew/paper/groetschelwakabayashi1981b.pdf

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Valid and facet defining inequalities for STSP: Survey articles

M. Grötschel, M. W. Padberg (1985 a, b)

M. Jünger, G. Reinelt, G. Rinaldi (1995)

D. Naddef (2002)

The TSP book (ABCC, 2006)

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Counting Tours and Facets

n	# tours	# different facets	# facet classes
3	1	0	0
4	3	3	1
5	12	20	2
6	60	100	4
7	360	3,437	6
8	2520	194,187	24
9	20,160	42,104,442	192
10	181,440	>= 52,043,900,866	>=15,379

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Separation Algorithms

- Given a system of valid inequalities (possibly of exponential size).
- Is there a polynomial time algorithm (or a good heuristic) that,
  - given a point,
  - checks whether the point satisfies all inequalities of the system, and
  - if not, finds an inequality violated by the given point?

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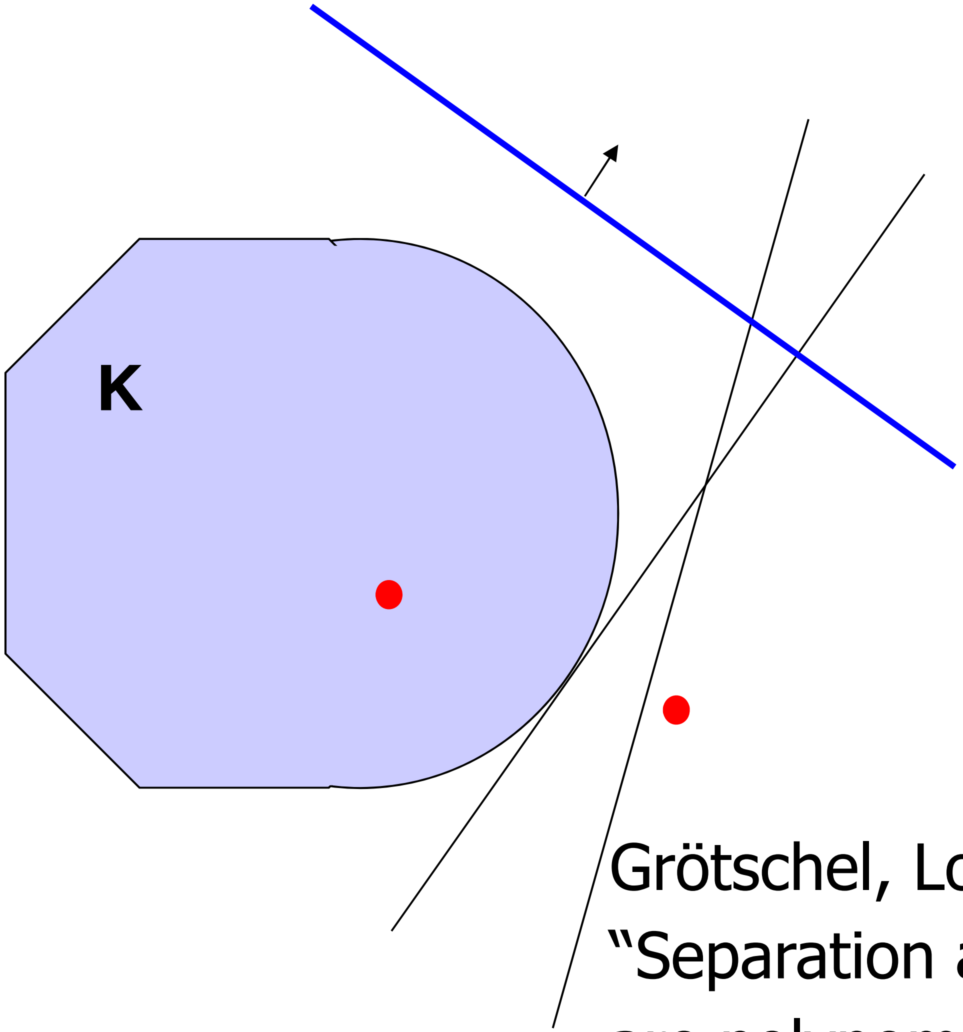
Separation



Grötschel, Lovász, Schrijver (GLS):
"Separation and optimization
are polynomial time equivalent."

Separation Algorithms

- There has been great success in finding exact polynomial time separation algorithms, e.g.,
  - for subtour-elimination constraints
  - for 2-matching constraints (Padberg&Rao, 1982)
- or fast heuristic separation algorithms, e.g.,
  - for comb constraints
  - for clique tree inequalities
- and these algorithms are practically efficient

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Polyhedral Combinatorics

- This line of research has resulted in powerful cutting plane algorithms for combinatorial optimization problems.
- They are used in practice to solve exactly or approximately (including branch & bound) large-scale real-world instances.

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Some TSP World Records

2006
pla 85,900
solved
3,646,412,050
variables

number of cities
2000x
increase

4,000,000
times
problem size
increase

in 52
years

year	authors	# cities	# variables
1954	DFJ	42/49	820/1,146
1977	G	120	7,140
1987	PR	532	141,246
1988	GH	666	221,445
1991	PR	2,392	2,859,636
1992	ABCC	3,038	4,613,203
1994	ABCC	7,397	27,354,106
1998	ABCC	13,509	91,239,786
2001	ABCC	15,112	114,178,716
2004	ABCC	24,978	311,937,753
2005	W. Cook, D. Epsinoza, M. Goycoolea	33,810	571,541,145

Basics of polyhedral theory, flows and networks

CO@W Berlin

The End

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