Nonlinear Mixed-Integer Programming - the MILP perspective

CO@Work Berlin

Armin Fügenschuh

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Berlin Mathematical School

DFG Research Center MATHEON
Mathematics for key technologies

The Beginning of Linear Programming

(G. Dantzig, Linear Programming, Oper Res 50, 42-47, 2002)

- George Dantzig's talk on „Programming in a Linear Structure" (meeting of the Econometric Society, Univ. of Wisconsin in Madison), 1948.
- Harold Hotelling objected „...but we all know the world is nonlinear".
- John von Neumann defended the flustered young Dantzig, saying that „if one has an application that satiesfied the axioms of the model, then it can be used, otherwise not."
- Hotelling was right: The world **is** highly nonlinear.
- But: Systems of linear **<u>in</u>**equalities allow an approximation of most kinds of nonlinear relations encountered in practical applications.

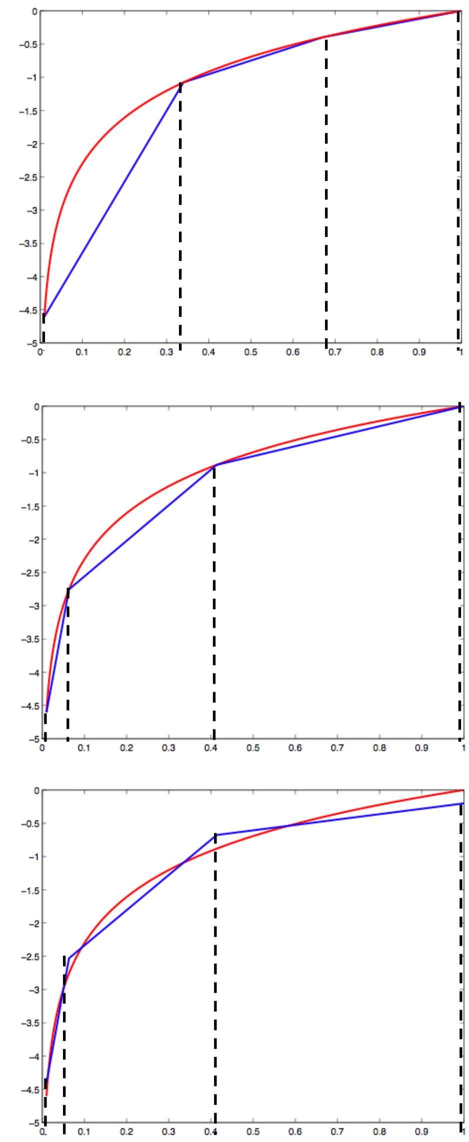
George Dantzig (1913-2005)

Harold Hotelling (1895-1973)

John von Neumann (1903-1957)

Piecewise Linear (Affine) Approximation

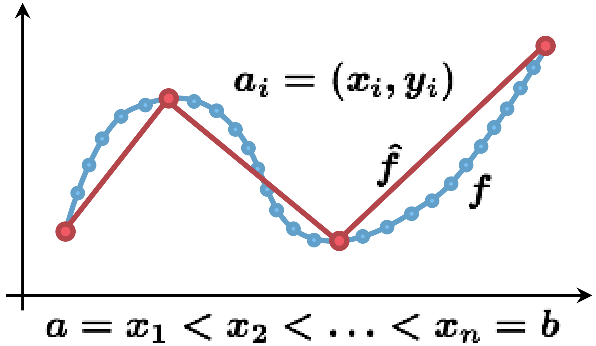
- In one dimension, a nonlinear function can be approximated by a sequence of piecewise linear functions (linear splines).
- The question is where to place the interpolation nodes for a given nonlinear function.
  - Equidistant interpolation.
  - Adaptive interpolation.
  - Adaptive approximation.
- This is a classical well-studied problem in numerical analysis, see for instance the textbook:
  - C. de Boor, A practical guide to splines, Springer, 1978.
- To apply these methods the function has to be „nice“ (i.e., twice continuous differentiable)

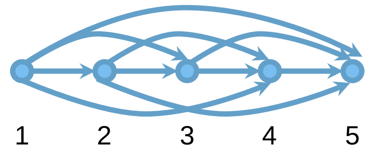


Piecewise Linear Approximation and Shortest Paths

(H. Imai, M. Iri, An optimal algorithm for approximating a piecewise linear function, J Inf Proc 9, 159-162, 1986)

- Assume a continuous function $f : [a,b] \to \mathbb{R}$ is given by a huge table of measurements, and linear interpolation between two measurements.
- We want to replace it by an approximation $\hat{f} : [a,b] \to \mathbb{R}$ using a smaller table.

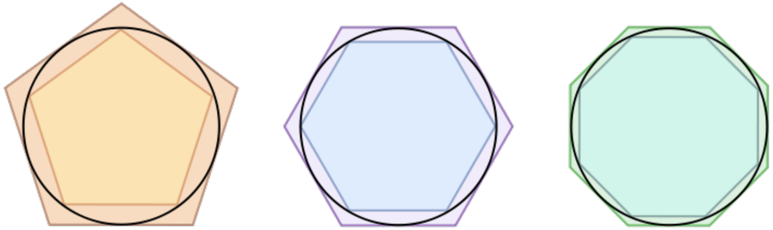




- Weighted digraph $D = (V, A, c)$ with $V = \{1, \ldots, n\}$ and $A = \{(i,j) : i < j\}$
- Select parameters $\alpha, \beta > 0$
  - α : cost per segment
  - β : cost for approximation error
- Define total cost per segment as $c_{i,j} := \alpha + \beta \left(\sum_{k=i}^{j} (f(x_k) - \hat{f}(x_k))^2 \right)$
- A shortest path from a to b in D is the desired approximation.

Piecewise Linear Approximation - An Ancient Idea

- The perhaps first mathematician who used piecewise linear approximation was Archimedes of Syracuse (287 BC-212 BC).
- He considered regular n-gons for an approximative computation of π.

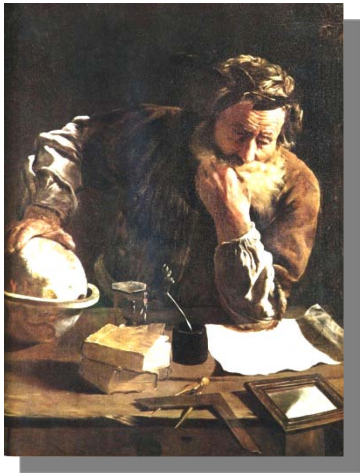


source: wikipedia

- With n = 96 Archimedes achieved the bounds

$$3.1408450 \approx 3\tfrac{10}{71} < \pi < 3\tfrac{10}{70} \approx 3.1428571$$

- Liu Hui (220-280): 192-gon and 3072-gon.
- Zu Chong-Zhi (429-500): $3*2^{12}$ = 12288-gon.
- Jamshid Masud Al-Kashi (1380-1429): $3*2^{28}$-gon.
- 1596 Ludolph van Ceulen computed the first 35 digits of π using Archimedes method on a 2^{62}-gon. It took him 30 years of his life.



Archimedes, 1620
Domenico Fetti (1589-1624)

Ludolph van Ceulen
(1540-1610)

The Incremental Method

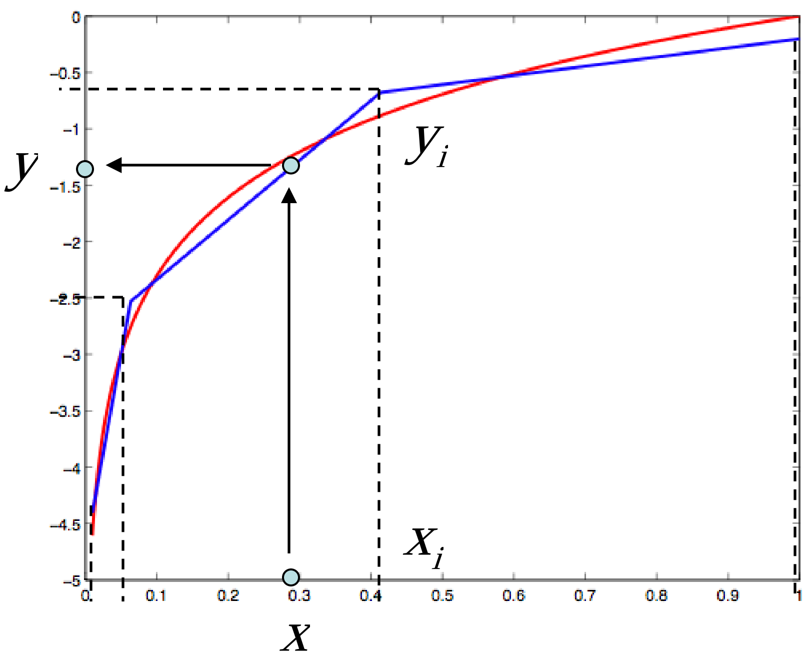
(H. Markowitz, A. Manne, On the solution of discrete programming problems, Econometrica 25, 84-110, 1957)

- Idea: „Filling“ of intervals
- Variables:
  - $w_i \in \{0,1\}$
  - $\delta_i \in [0,1]$
- Constraints:

$$w_i \geq \delta_i \geq w_{i+1}$$

$$x = x_0 + \sum_{i \in I} (x_i - x_{i-1})\delta_i$$

$$y = y_0 + \sum_{i \in I} (y_i - y_{i-1})\delta_i$$



$\delta_1 = 1 \quad \delta_2 = 0.7 \qquad \delta_3 = 0$

$w_1 = 1 \quad w_2 = 1 \qquad w_3 = 0$

The Convex Combination Method

(G. Dantzig, On the significance of solving linear programming problems with some integer variables, Econometrica 28, 30-44, 1960)

- Idea: Selection of exactly one interval
- Variables:
  - $w_i \in \{0,1\}$
  - $\lambda_i \in [0,1]$
- Constraints:

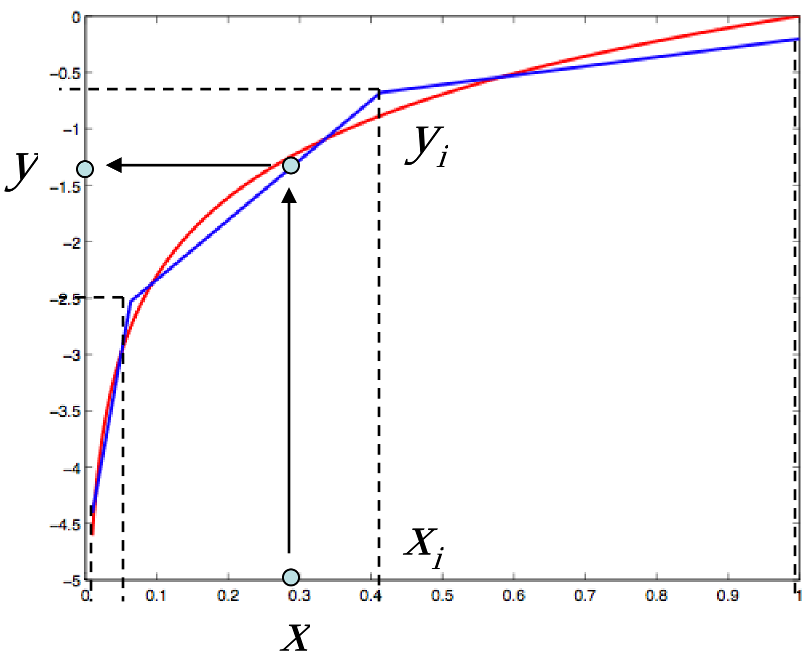
$$\sum_{i \in I} w_i = 1$$

$$\lambda_0 + \sum_{i \in I} \lambda_i = 1$$

$$w_i \leq \lambda_{i-1} + \lambda_i$$

$$x = x_0 \lambda_0 + \sum_{i \in I} x_i \lambda_i$$

$$y = y_0 \lambda_0 + \sum_{i \in I} y_i \lambda_i$$

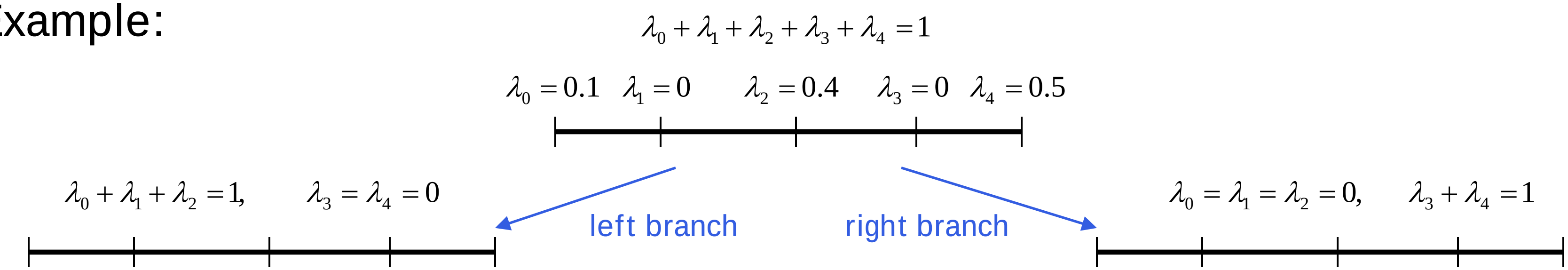


$\lambda_0 = 0$ $\lambda_3 = 0$

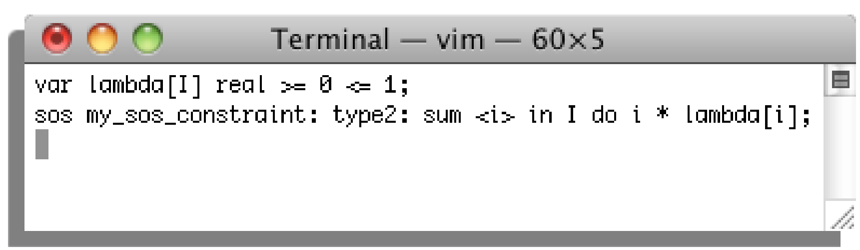
$\lambda_1 = 0.3$ $\lambda_2 = 0.7$

$w_1 = 0$ $w_2 = 1$ $w_3 = 0$

- Binary variables in the convex combination model were only introduced to model a logical relation:
  - At most two lambda variables are nonzero.
  - Nonzero lambda variables must be adjacent.
- Beale and Tomlin (1970) introduce SOS2 to handle this constraint implicitly in a branch-and-bound framework, without an explicit use of binary variables and constraints.
- Example:

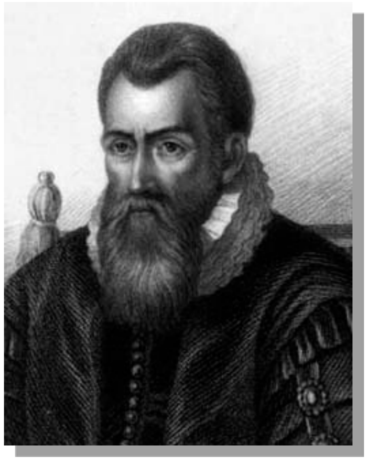


- Modern modeling languages / MILP solvers offer SOS2 as a built-in feature.
- Example (Zimpl):



Dealing with Functions of Several Variables

- Sometimes multivariate functions can be reduced to a sequence of monovariate functions.
- Example: $z = x^a \cdot y^b$.
- Here we can use an idea of John Napier from 1614: „*Mirifici logarithmorum canonis constructio*".
- Assume $x, y > 0$. Then:



John Napier (1550-1617)

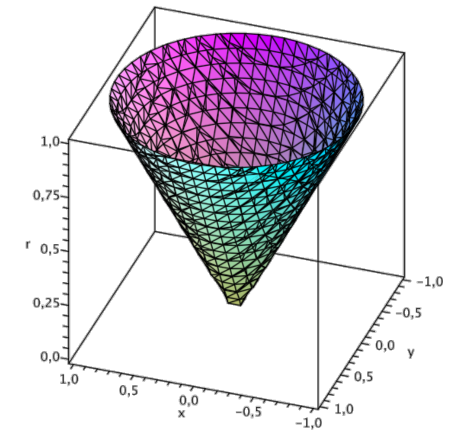
$$z = x^a \cdot y^b$$
$$\Leftrightarrow \quad \ln z = \ln(x^a \cdot y^b) = a \ln x + b \ln y$$

- Instead of one bivariate function we have to approximate three monovariate functions (by one of the methods discussed before):

$$\tilde{z} = \ln z$$
$$\tilde{x} = \ln x$$
$$\tilde{y} = \ln y$$
$$\tilde{z} = a\tilde{x} + b\tilde{y}$$

- Some multivariate functions have a special structure that can be exploited for highly efficient approximations.
- A quadratic cone (also second order cone, coll. „ice-cream cone") is the set described by

$$\sqrt{x_1^2 + x_2^2 + \ldots + x_{n-1}^2} \leq x_n$$



- Many other constraints can be transformed into a SOC:

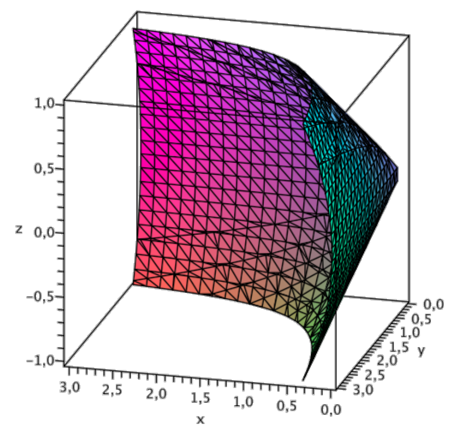
$$a \cdot b \geq c \quad \Leftrightarrow \quad \sqrt{x^2 + y^2} \leq z$$

where

$$x := \tfrac{1}{2}(a-b), \quad y := \sqrt{c}, \quad z := \tfrac{1}{2}(a+b)$$

- Further examples include the cone

$$z^T z \leq xy, \quad x, y \geq 0$$



Linear Approximations of Second Order Cones

(A. Ben-Tal, A. Nemirovski, On polyhedral approximations of the second-order cone, Math Oper Res 26, 193-205, 1998 & F. Glineur, Polyhedral approximation of the second-order cone: computational experiments, Tech Rep, 2000)

- Pure SOCPs can be solved by nonlinear (interior point) methods.
- When also integer constraints are present, linear approximations of the SOC should be considered.
- Clearly one can approximate the SOC in the original space by linear cones (i.e., the unit disc by an n-gon).
- The error is $\varepsilon = \cos(\frac{\pi}{n})^{-1} - 1.$
- For an accuracy of 10^{-4} we would need a 223-gon.
- A much better approximation is the following:
  - Variables: $\alpha_i, \beta_i \in]-\infty, \infty[$
  - Constraints: $\sqrt{x_1^2 + x_2^2} \leq x_3 \quad <\approx>$

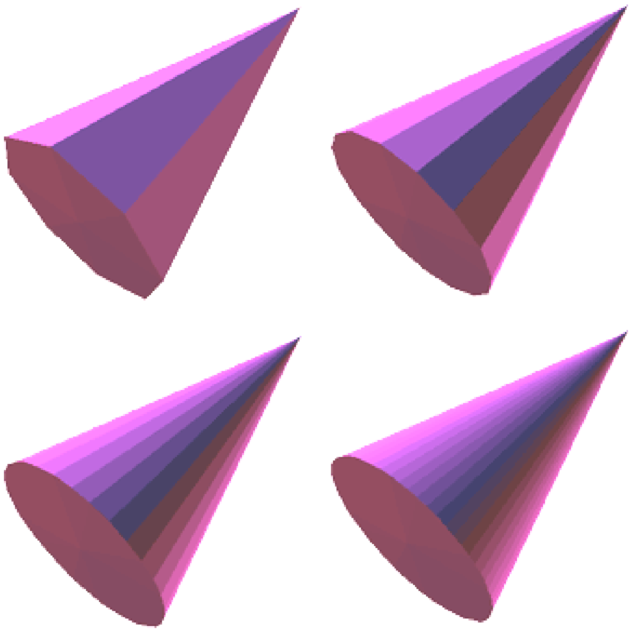
$$\alpha_0 = x_1$$

$$\beta_0 = x_2$$

$$\alpha_{i+1} = \cos(\tfrac{\pi}{2^i})\alpha_i + \sin(\tfrac{\pi}{2^i})\beta_i$$

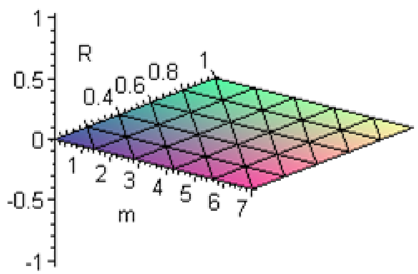
$$\beta_{i+1} \geq \left|\sin(\tfrac{\pi}{2^i})\alpha_i - \cos(\tfrac{\pi}{2^i})\beta_i\right|$$

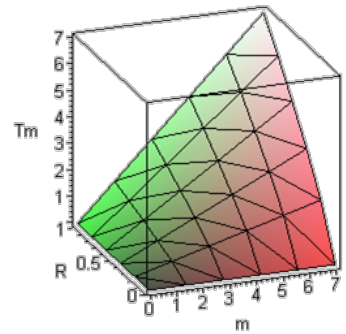
$$\cos(\tfrac{\pi}{2^I})\alpha_I + \sin(\tfrac{\pi}{2^I})\beta_I = x_3$$



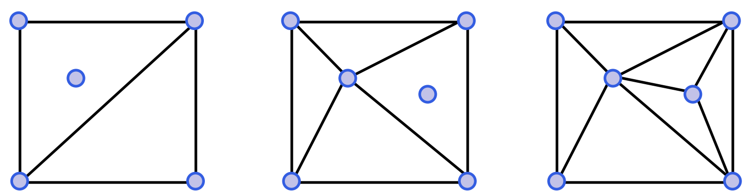
- Accuracy is $\varepsilon = \cos(\frac{\pi}{2^n})^{-1} - 1.$

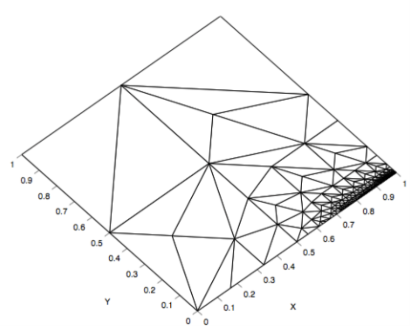
- If the function does not fit in that catagories, one can apply a higher dimensional analogon of 1-d piecewise linear approximation.
- Prerequisite: Introduce a triangulation (in 2-d) or, in general, a decomposition of the domain in simplices.
  - Equidistant:

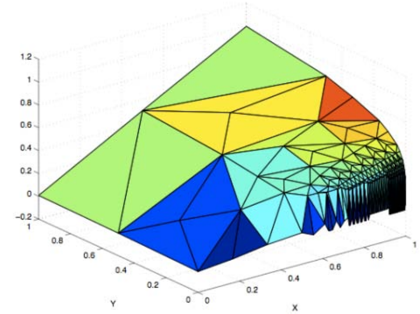




  - Adaptive:
    - Delauney triangulation of the domain.
    - Find the point with maximum error.
    - Introduce a new node there.
    - Compute a refined Delauney triangulation.







The Convex Combination Method in Higher Dimensions

- Idea: Selection of exactly one ~~interval~~ triangle
- Variables:
  - $w_i \in \{0,1\}$
  - $\lambda_i \in [0,1]$
- Constraints:

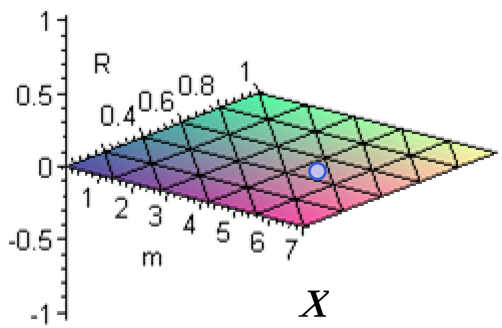
$$\sum_{i\in I} w_i = 1$$

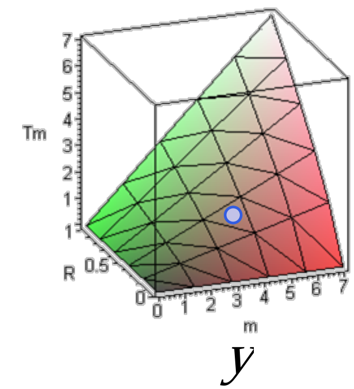
$$\lambda_0 + \sum_{i\in I} \lambda_i = 1$$

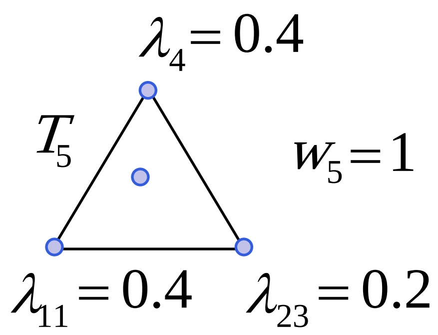
$$\cancel{w_i \le \lambda_{i-1} + \lambda_i} \qquad w_i \le \sum_{j\in T_i} \lambda_j$$

$$x = x_0\lambda_0 + \sum_{i\in I} x_i\lambda_i$$

$$y = y_0\lambda_0 + \sum_{i\in I} y_i\lambda_i$$



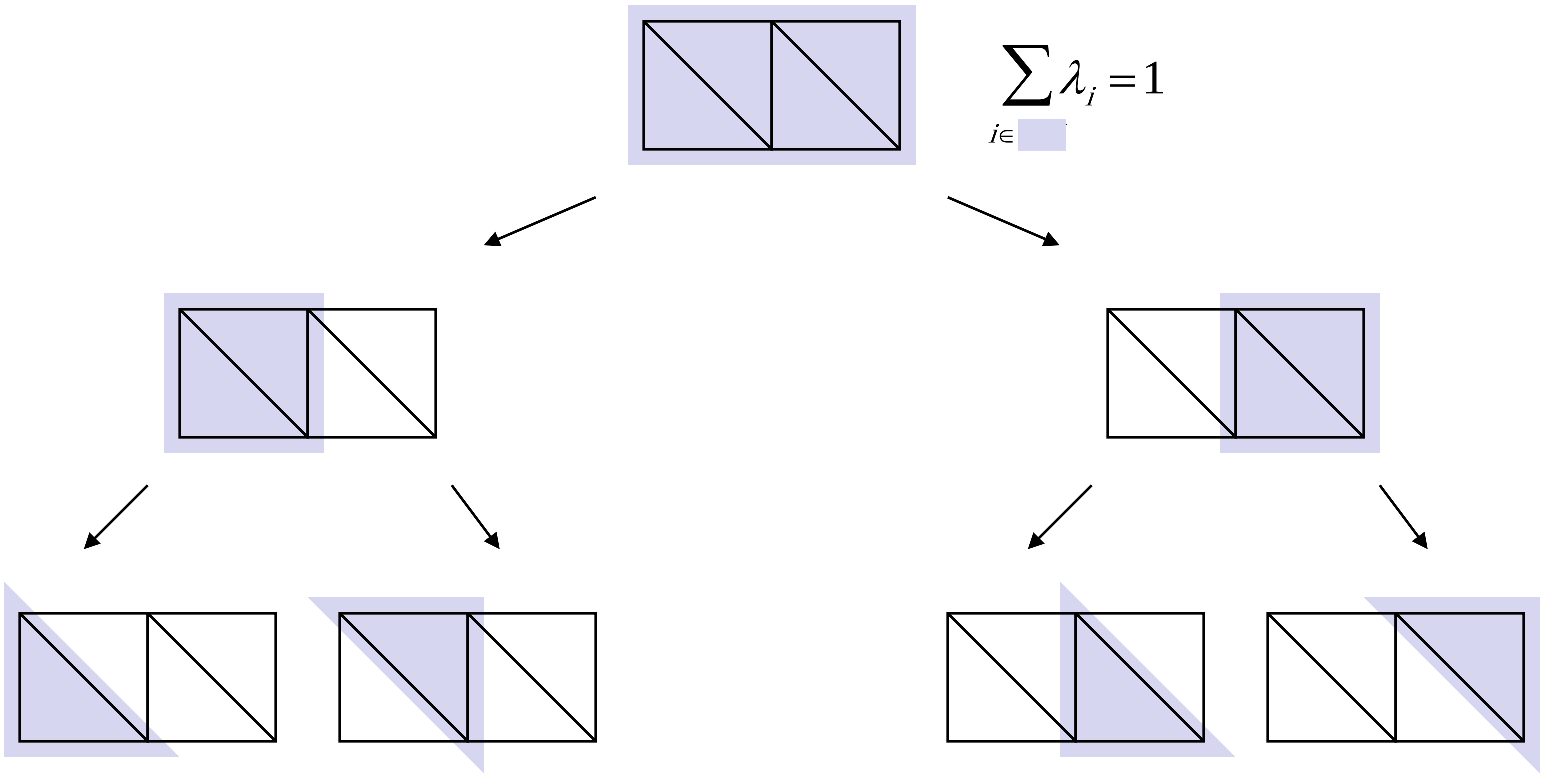




SOS Branching in Higher Dimensions

(A. Martin, M. Möller, S. Moritz, Mixed Integer Models for the Stationary Case of Gas Network Optimization, Math Prog B 105, 563-582, 2006)

- Similar to the 1-d case it is possible to remove the auxiliary binary variables and handle the SOS property within the branching.



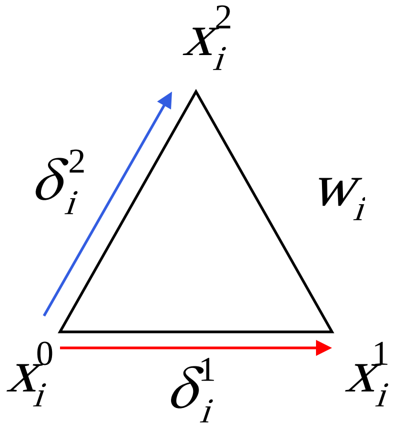
- Idea: „Filling“ of triangles
- Variables:
  - $w_i \in \{0,1\}$
  - $\delta_i^1, \delta_i^2 \in [0,1]$
- Constraints:

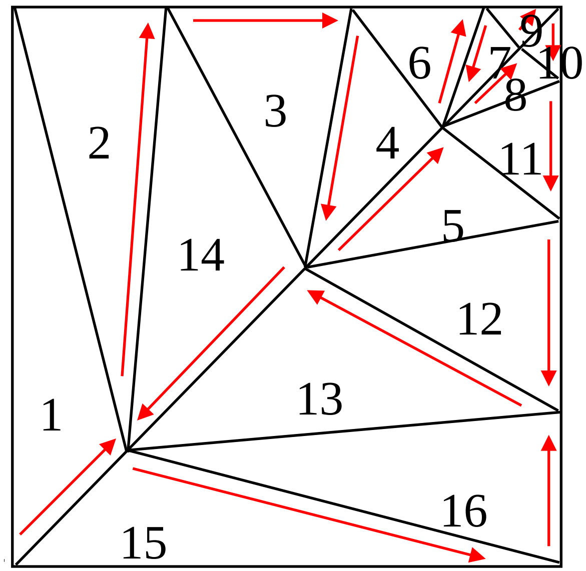
$$\delta_i^1 \geq w_i \geq \delta_{i+1}^1 + \delta_{i+1}^2$$

$$x = x_0^0 + \sum_{i \in I} \left[(x_i^1 - x_i^0)\delta_i^1 + (x_i^2 - x_i^0)\delta_i^2 \right]$$

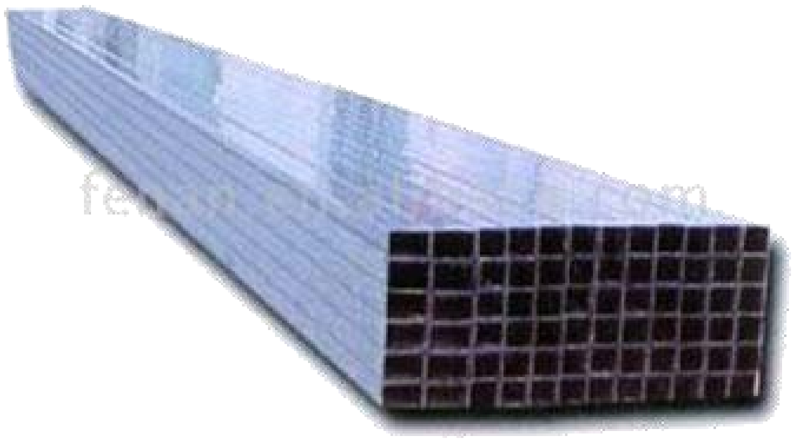
$$y = y_0^0 + \sum_{i \in I} \left[(y_i^1 - y_i^0)\delta_i^1 + (y_i^2 - y_i^0)\delta_i^2 \right]$$

- Prerequisite: Ordering of the main direction





- Design of square-tube conduits
  - Several square-shaped channels, surrounded by metal
  - Given cross section areas
  - Further engineering constraints
  - Design goals
    - <u>Minimal material usage</u>
    - Minimal deflection
    - Minimal torsion



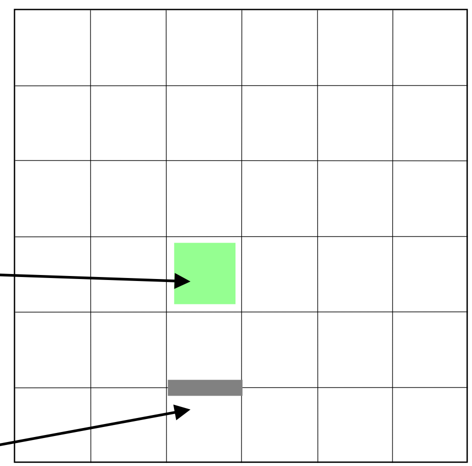
A Linear Mixed-Integer Model

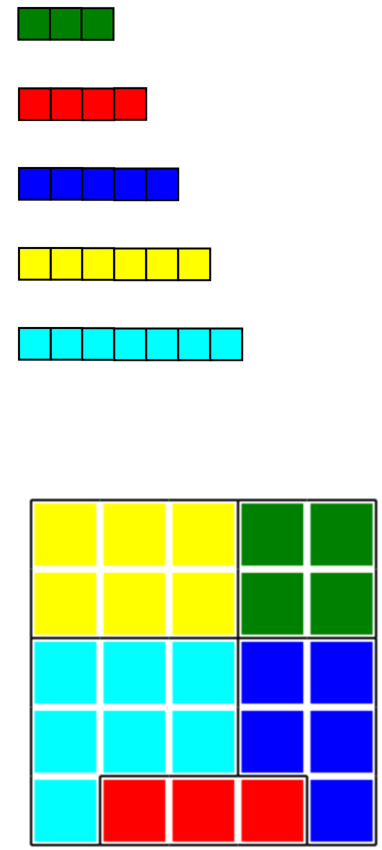
(A. Fügenschuh, M. Fügenschuh, Integer Linear Programming Models for Topology Optimization in Sheet Metal Design, Math Meth Oper Res 68, 313-331, 2008)

- Discretize the design envelope.
- Place channels on pixels and metal on the boundaries.

$$\delta_i^t \in \{0,1\}$$

$$\mu_{i,j} \in \{0,1\}$$



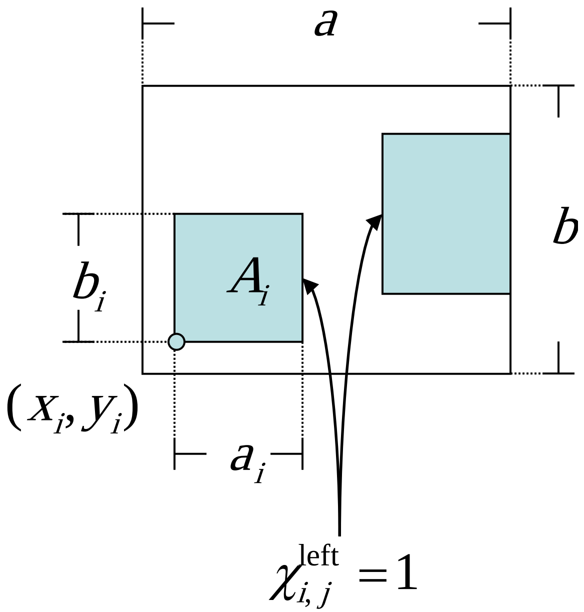


An Nonlinear Nonconvex Model

(A. Fügenschuh, W. Hess, L. Schewe, A. Martin, S. Ulbrich, Verfeinerte Modelle zur Topologie- und Geometrie-Optimierung von Blechprofilen mit Kammern, 2nd Proc SFB 666, 17-28, 2008)

- Given areas $a_i \cdot b_i = A_i$
- Given total area $a \cdot b = \sum_i A_i$
- Packing without overlapping

$$x_i + a_i \le x_j + M \cdot (1 - \chi_{i,j}^{\text{left}})$$
$$x_j + a_j \le x_i + M \cdot (1 - \chi_{i,j}^{\text{right}})$$
$$y_i + b_i \le y_j + M \cdot (1 - \chi_{i,j}^{\text{below}})$$
$$y_j + b_j \le y_i + M \cdot (1 - \chi_{i,j}^{\text{above}})$$
$$\chi_{i,j}^{\text{left}} + \chi_{i,j}^{\text{right}} + \chi_{i,j}^{\text{below}} + \chi_{i,j}^{\text{above}} \ge 1$$

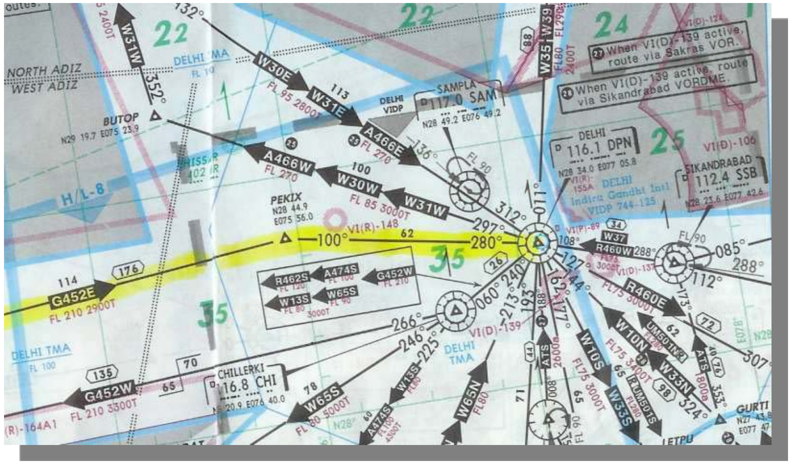


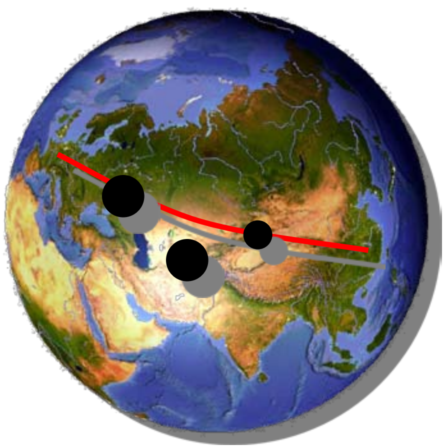
- Boundary conditions $x_i + a_i \le a, \quad y_i + b_i \le b$
- Minimize total border length $\min a + b + \sum_i (a_i + b_i)$

Exercise

- Try the various linearization techniques for this problem
- Basic model: template.zpl
- Contains everything but width*height = Area
- Data set: conduit6.dat
- Interpolation of log function: log-approx <lb> <ub> <error>
- Compute sin/cos values for SOC: trigonometrics <n>
- Solve problem with SCIP
- Write solution to file (wr sol my.sol)
- Visualisation tool: makesvg <my.sol>

- Past/present: air traffic network (ATN)
- Future: free flight
- Main obstacles:
  - Restricted airspaces
  - Weather (wind, temp.)
  - Overflight costs
  - ETOPS
  - ...



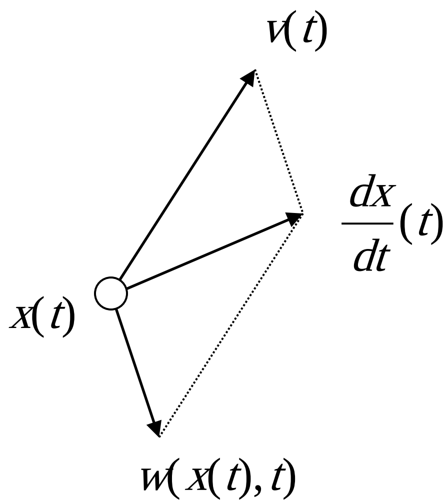


A Free Flight Model Based on ODE

- Equations of motion

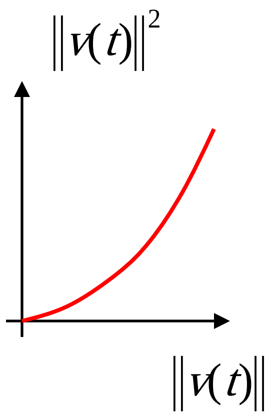
$$\frac{dx_1}{dt}(t) = v_1(t) + w_1(x_1(t), x_2(t), t)$$

$$\frac{dx_2}{dt}(t) = v_2(t) + w_2(x_1(t), x_2(t), t)$$



- Minimize fuel consumption:

$$\min \int_{t=0}^{T} \|v(t)\|^2 \, dt$$

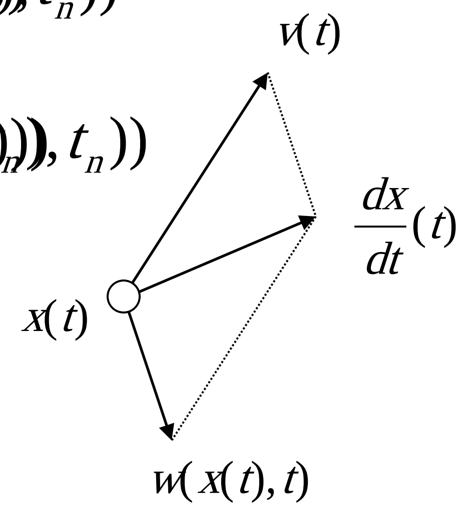


t	time
$x(t)$	position of aircraft at time t
$v(t)$	aircraft velocity at time t
$w(x(t),t)$	wind velocity at time t in $x(t)$

- Equations of motion

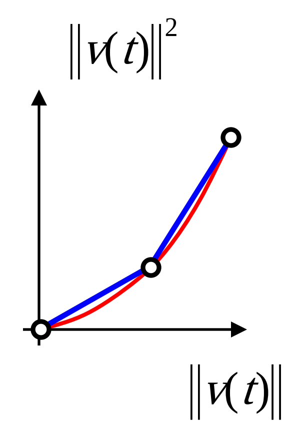
$$\frac{x_1(t+\Delta t) - x_1(t)}{\Delta t} \approx \frac{dx_1}{dt}(t) \qquad x_1(t_{n+1}) = x_1(t_n) + \Delta t\,(v_1(t_n) + w_1(x_1(t_n), x_2(t_n), t_n))$$

$$\frac{x_2(t+\Delta t) - x_2(t)}{\Delta t} \approx \frac{dx_2}{dt}(t) \qquad x_2(t_{n+1}) = x_2(t_n) + \Delta t\,(v_2(t_n) + w_2(x_1(t_n), x_2(t_n), t_n))$$



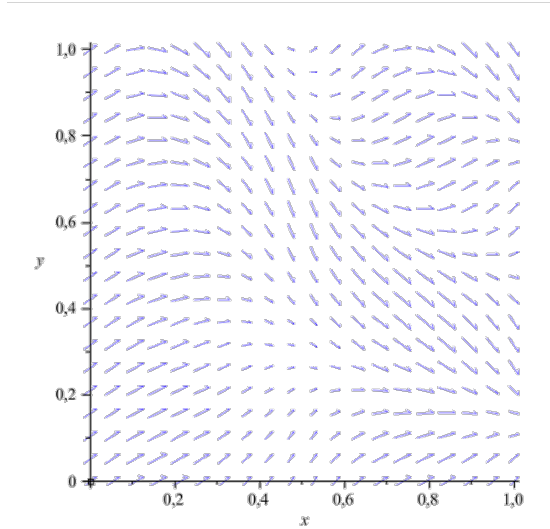
- Minimize fuel consumption:

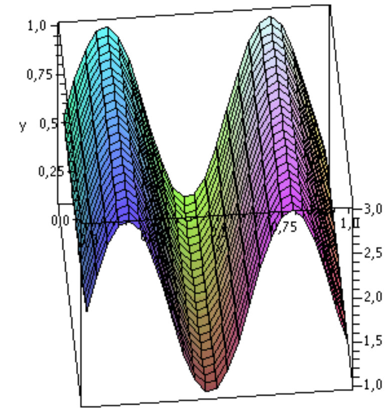
$$\min \sum_{n=0}^{N} \|v(t_n)\|^2 \Delta t \qquad \min \int_0^T \|v(t)\|^2 dt \qquad \sum_{k=1}^{K} \alpha_k v_n^k$$

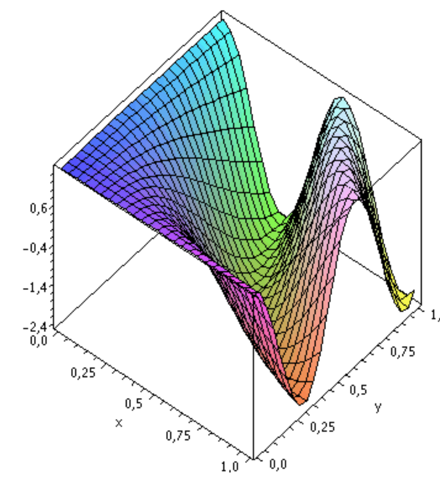


t	time
$x(t)$	position of aircraft at time t
$v(t)$	aircraft velocity at time t
$w(x(t),t)$	wind velocity at time t in $x(t)$

- 2-d vectorfield
- 2 nonlinear functions in 2-d

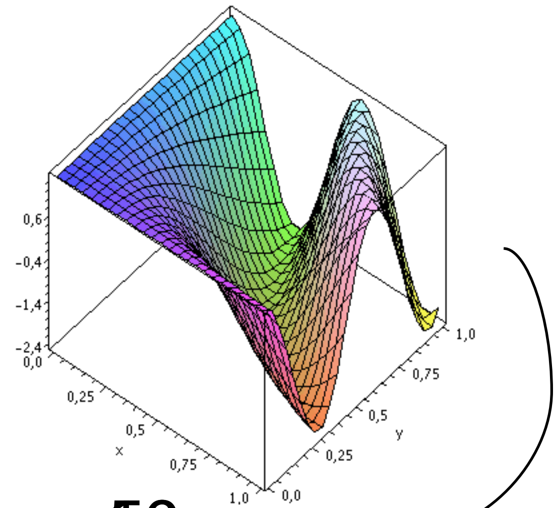


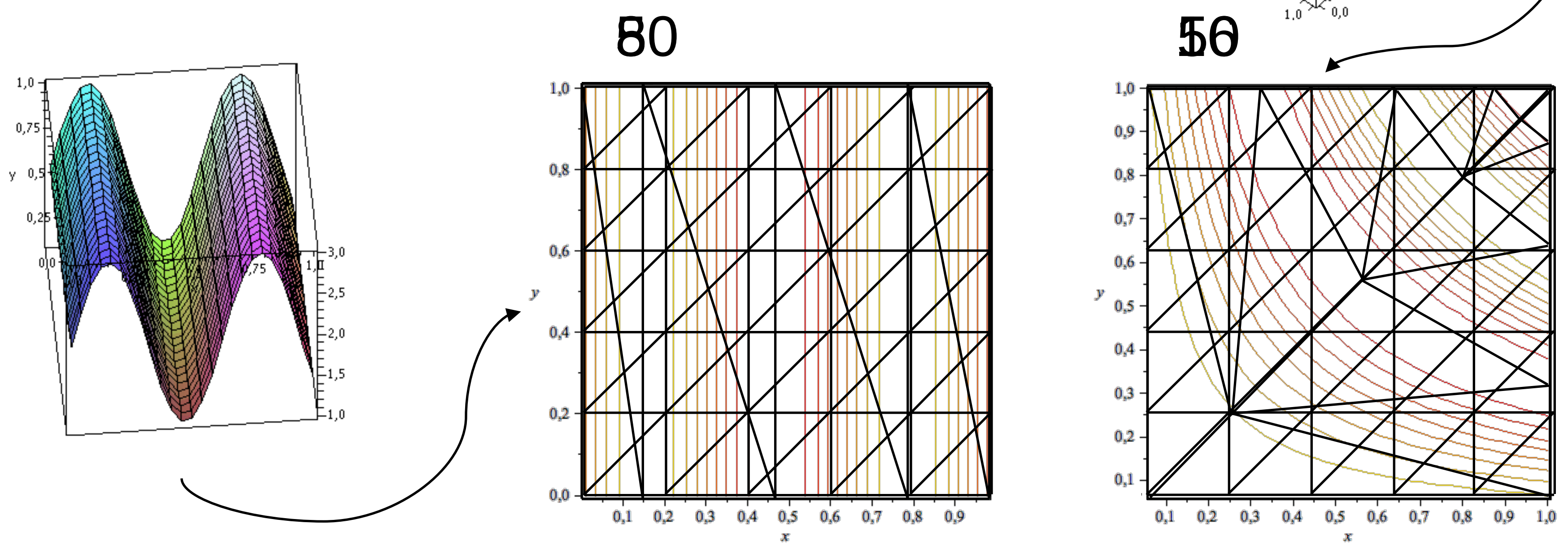




Piecewise Linear Approximation of a 2-D Nonlinear Function

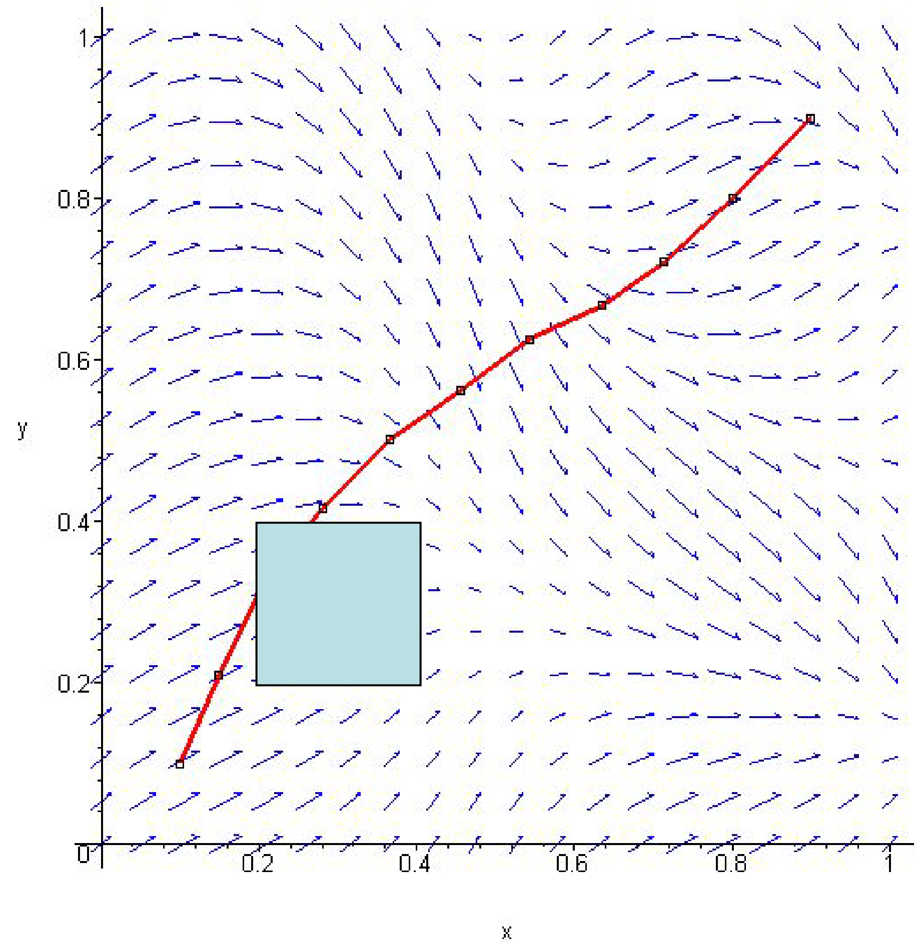
- Contour lines
- Interpolation (equidistant)
- Interpolation (adaptive)





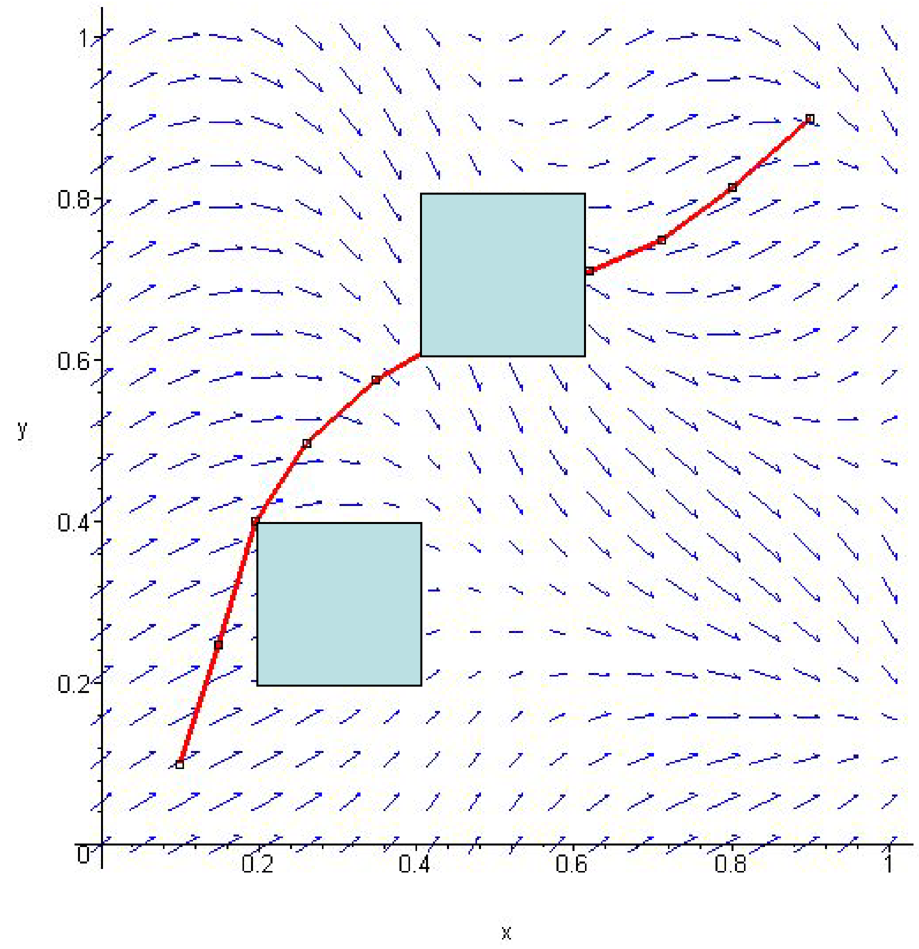
Free Flight: Results

- Incremental method in 2-d (CPlex10)
  - Equidist. 1: 891 sec.
  - Equidist. 2: 40 sec.
  - Adaptive: 5 sec.
- Restricted air-space
  - Blue area forbidden from $t = 0$ to $t = 5$



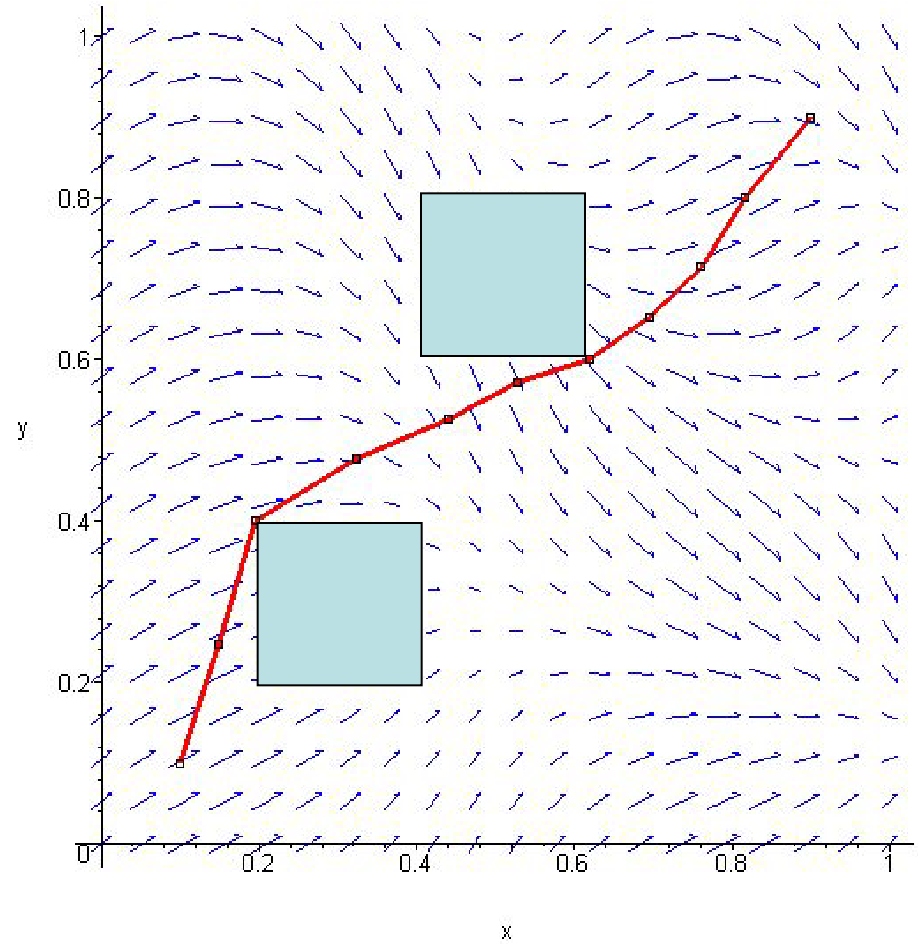
Free Flight: Restricted Airspaces

- Incremental method in 2-d (CPlex10)
  - Equidist. 1: 6600 sec.
  - Equidist. 2: 238 sec.
  - Adaptive: 78 sec.

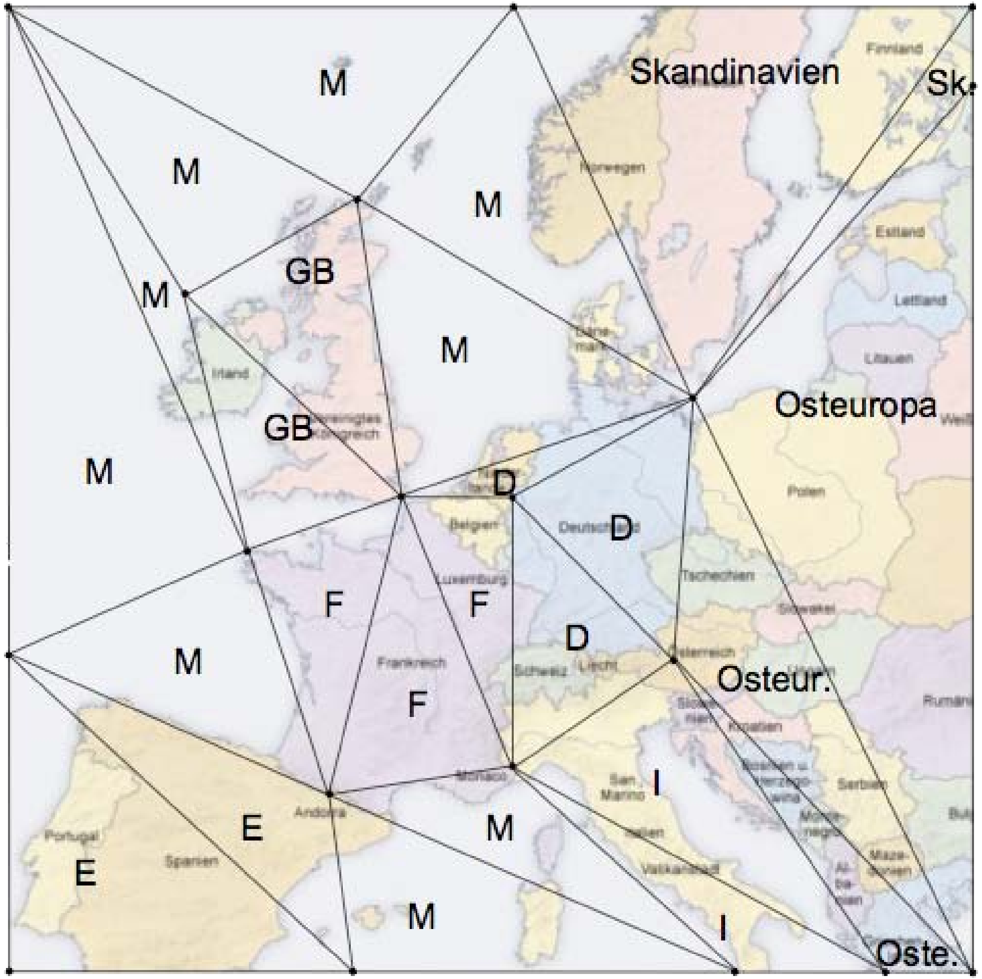


Free Flight: Resticted Airspaces

- Incremental method in 2-d (CPlex10)
  - Equidist. 1: 36579 sec.
  - Equidist. 2: 900 sec.
  - Adaptive: 618 sec.



Free Flight: Overflight Costs



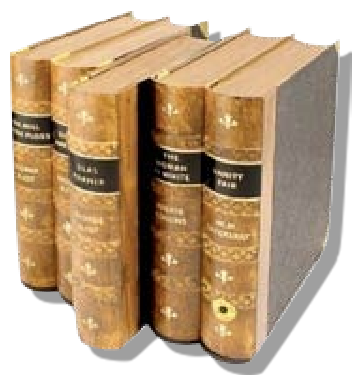
Free Flight: Overflight Costs

Free Flight: Overflight Costs

Some Facts about Paper and Recycling

(sources: Valkama, 2007 & Wikipedia)

- Newspapers, journals, books, packing material, hygienic articles,... are all made of paper and carton.
- Per year Germany consumes 21 million tons of paper and carton. That is, every person consumes ~250kg paper/year.
- Paper is one of the best-recycled products: 15.5 millions tons are reused.
- An increasing rate of today 67% of the fibers come from these sources.

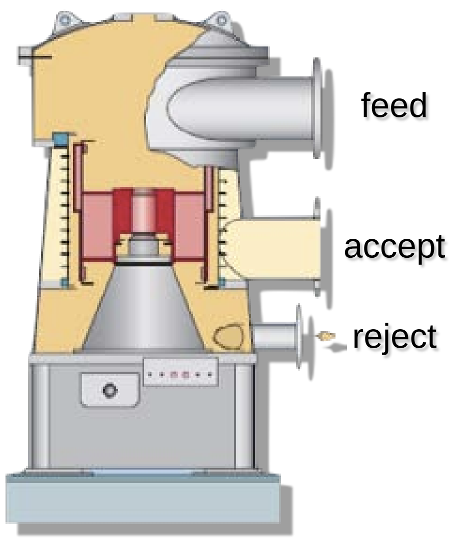


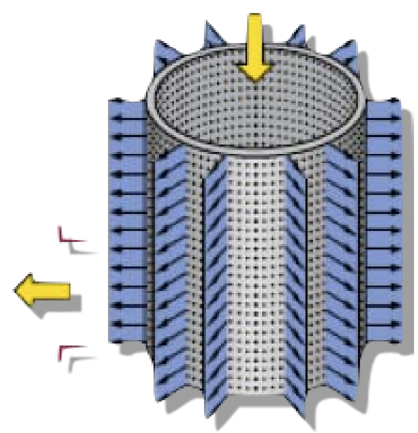
Steps in the Recovered Paper Production

(Valkama, 2007)

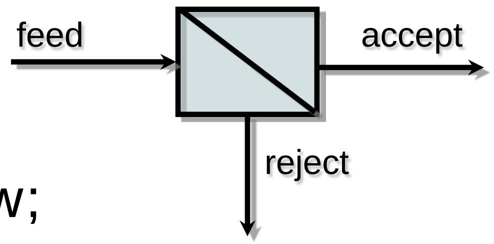
- Recycling fibres from waste paper consists of several steps:
  - Manual removal of contaminent materials.
  - Hackle paper into small pieces.
  - Resolve pieces in water and obtain pulp.
  - **Clean the pulp from** paper clips, plastic materials, and **stickies**.
  - De-ink the pulp.
  - The recovered paper suspension (fibres) is layed on grids and dried.
  - New paper rolls can now be produced.
- Too many stickies reduce the quality of the recovered paper, and can even break the rolls during production.
- Estimated production loss due to stickies: 265 mill. €.

- Sorters (screeners, separators) come in various types and sizes.
- Differences:
  - Capacity (amount of pulp per time).
  - Sieves (size, slot type and width).
  - Max. admissible operating pressure.

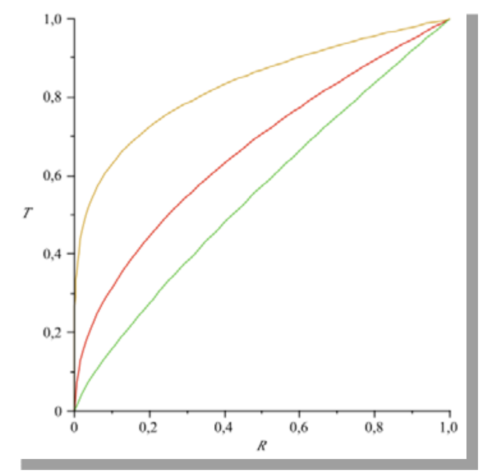




The Plug Flow Model

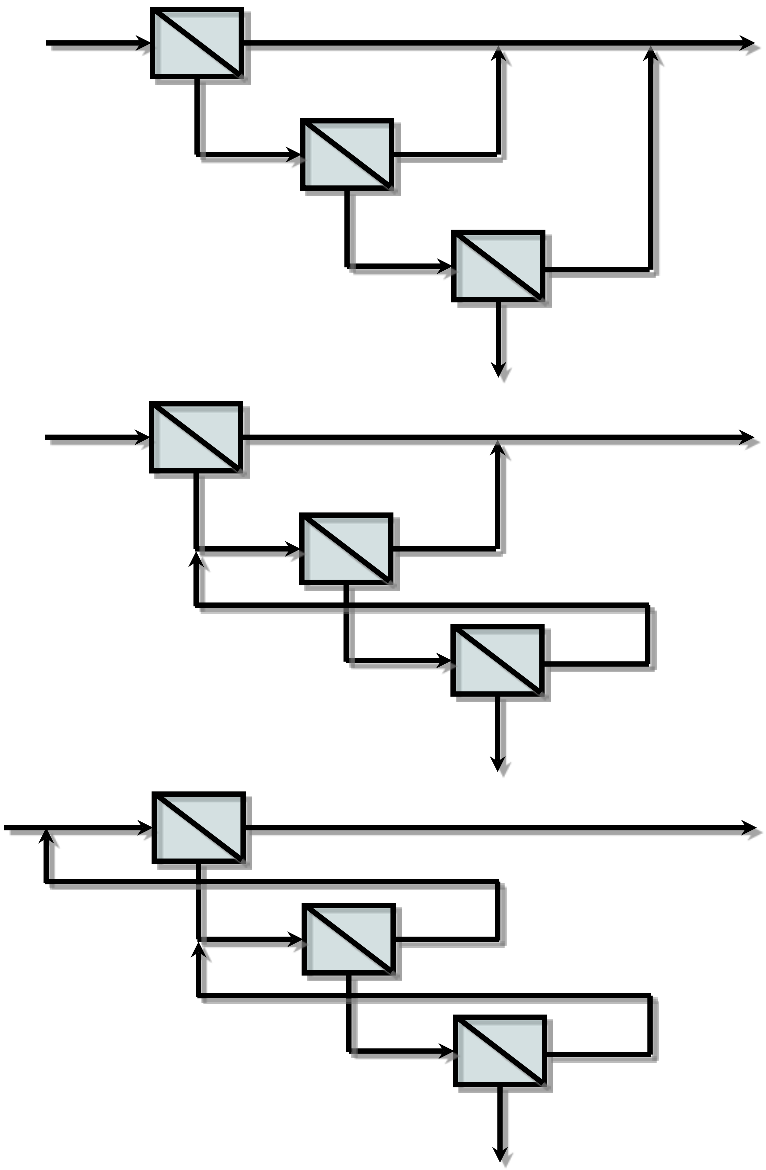


- Each sorter has one inflow feed, and two outflows, accept and reject.
- Mass is conserved: $m^{in} = m^{acc} + m^{rej}$.
- Several components (~12) are in the pulp flow; we restrict here to two, fibers and stickies.
- The separation efficiency for component k is $T_k = \dfrac{m_k^{rej}}{m_k^{in}}$.
- The total mass reject loss (the reject rate) is $R = \dfrac{m^{rej}}{m^{in}}$.
- Kubát and Steenberg developed in the 1950‘s the plug flow model. According to their model the coupling $T_k = R^{\beta_k}$ holds for each k. Parameters β_k depend on the sorter and the component. They are obtained by measurements.



From Single Sorters to Systems of Sorters

- The sticky-sorter facility consists of 3-5 sorters and pipelines.
- Several examples of such systems are known:
  - feed forward,
  - partial cascade, and
  - full cascade.
- The pulp flow is sent through pipelines from one sorter to the next.
- The amount per commodity in the total inflow is known.
- The system has a total accept and a total reject.
- Goal: maximize stickies in total reject and fibers in total accept.



A Nonlinear Mathematical Model (NLP)

- Sets: pipes P, sorters S, components K.
- Parameters
  - Component $k \in K$ inflow mass: $m_k^{in} \geq 0$.
  - Pipe from accept/reject of sorter s_1 to inflow of s_2? $p_{s_1 s_2}^{acc}, p_{s_1 s_2}^{rej} \in \{0, 1\}$.
  - Gain/loss per unit of k in total accept/reject: $c_k^{acc}, c_k^{rej} \in \mathbb{R}$.
  - Sorter's beta parameter vector: $\beta_{s,k} \in]0, 1[$.
- Variables
  - Mass flow of k into/out of sorter s: $m_{s,k}^{in}, m_{s,k}^{acc}, m_{s,k}^{rej} \geq 0$.
  - Mass flow to total accept/reject: $m_k^{acc}, m_k^{rej} \geq 0$.
  - Reject rate of sorter s: $R_s \in [l_s, u_s]$.
- Constraints
  - Mass conservation: $m_{s,k}^{in} = m_{s,k}^{rej} + m_{s,k}^{acc}$.
  - Plug flow: $m_{s,k}^{rej} = R_s^{\beta_{s,k}} \cdot m_{s,k}^{in}$.
  - Network topology: $m_{s_2,k}^{in} = \sum_{s_1:(s_1 s_2) \in P} (p_{s_1 s_2}^{acc} \cdot m_{s_1,k}^{acc} + p_{s_1 s_2}^{rej} \cdot m_{s_2,k}^{rej})$.
- Objective: $\sum_{k \in K} (c_k^{acc} \cdot m_k^{acc} + c_k^{rej} \cdot m_k^{rej}) \rightarrow \max$.

Including the Topology

- There are many ways to connect the sorters.

#sorters	#topologies
1	1
2	8
3	318
4	26,688
5	3,750,240

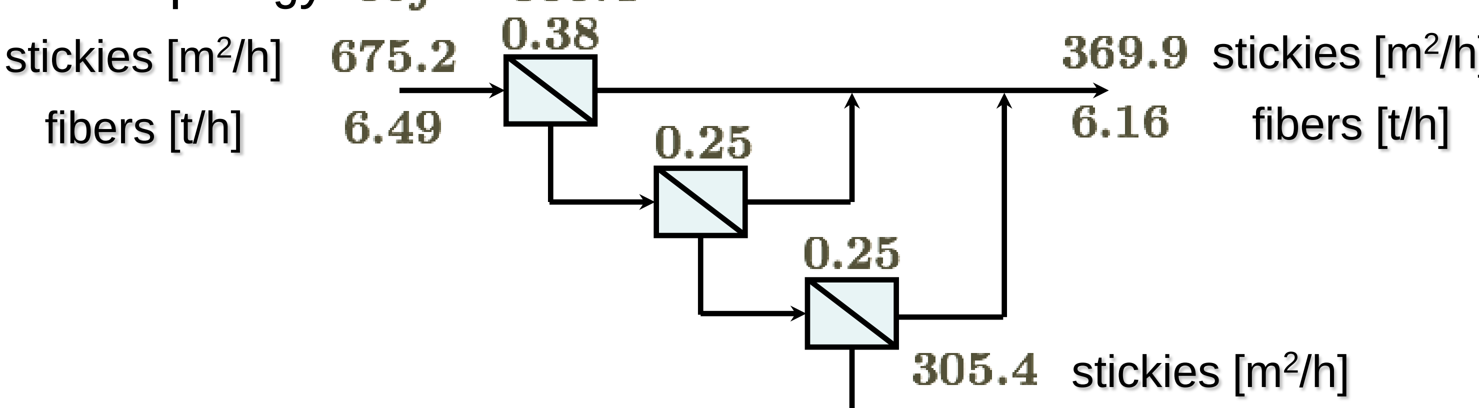
- Topological decisions can be taken into the model.
- Instead of parameters $p^{rej}_{s_1 s_2}$ and $p^{acc}_{s_1 s_2}$ we introduce a binary variables.
- Expressions $p^{acc}_{s_1 s_2} \cdot m^{acc}_{s_1,k}$ and $p^{rej}_{s_1 s_2} \cdot m^{rej}_{s_2,k}$ then are also nonlinear.
- They have to be linearized again.
- See also Floudas (1987, 1995), Nath, Motard (1981), Nishida, Stephanopoulos, Westerberg (1981), Friedler, Tarjan, Huang, Fan (1993), Grossmann, Caballero, Yeomans (1999), and many more.

Linearizing the Topology Constraints

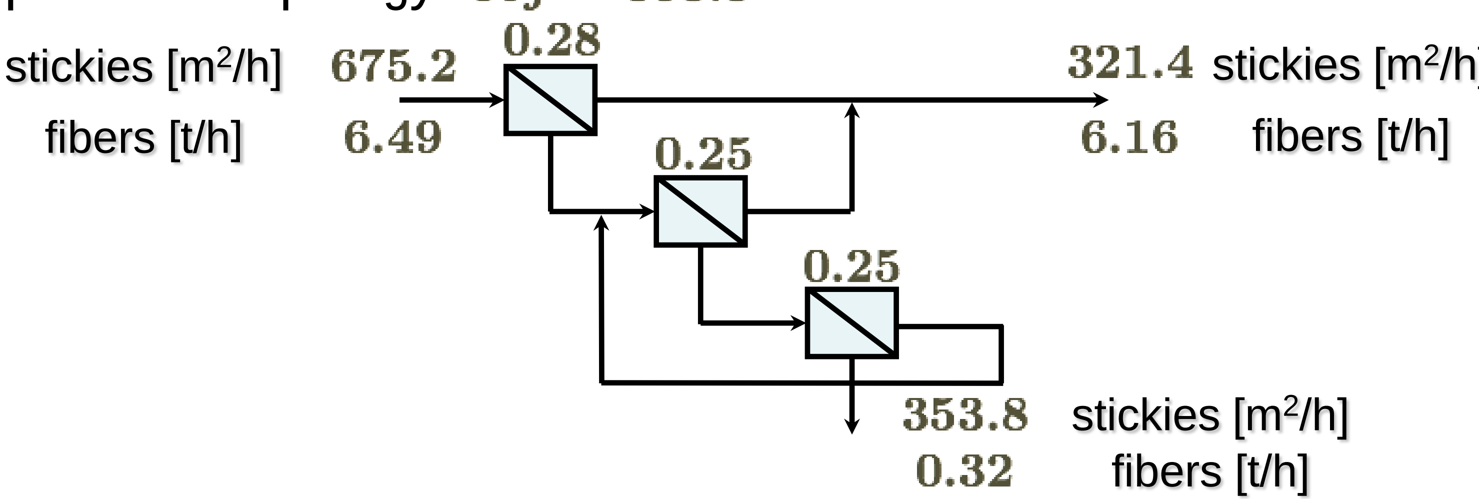
- Remember: $m^{in}_{s_2,k} = \sum_{s_1:(s_1 s_2)\in P} (p^{acc}_{s_1 s_2} \cdot m^{acc}_{s_1,k} + p^{rej}_{s_1 s_2} \cdot m^{rej}_{s_2,k})$.
- Introduce new variables $\mu^{acc}_{s_1 s_2,k}, \mu^{rej}_{s_1 s_2,k} \geq 0$ for potential mass flowing from accept/reject of sorter s_1 to the input of sorter s_2.
- Constraints
  - Mass flow only in pipes (M a sufficiently large constant):
    $$\mu^{acc}_{s_1 s_2,k} \leq M \cdot p^{acc}_{s_1 s_2}, \qquad \mu^{rej}_{s_1 s_2,k} \leq M \cdot p^{rej}_{s_1 s_2}.$$
  - Exactly one pipe is selected:
    $$\sum_{s_2:(s_1 s_2)\in P} p^{acc}_{s_1 s_2} = \sum_{s_2:(s_1 s_2)\in P} p^{rej}_{s_1 s_2} = 1.$$
  - Coupling of mass and potential mass:
    $$m^{in}_{s_2,k} = \sum_{s_1:(s_1 s_2)\in P} (\mu^{acc}_{s_1 s_2,k} + \mu^{rej}_{s_1 s_2,k}),$$
    $$m^{acc}_{s_1,k} = \sum_{s_2:(s_1 s_2)\in P} \mu^{acc}_{s_1 s_2,k},$$
    $$m^{rej}_{s_1,k} = \sum_{s_2:(s_1 s_2)\in P} \mu^{rej}_{s_1 s_2,k}.$$

Computational Results

- Objective: maximize stickies within reject.
- Additional constraint: fiber-loss (i.e., fibers in reject) at most 5%.
- Given topology: $obj = 305.4$



- Optimized topology: $obj = 353.8$



Thank you for your attention!

Questions?