Introduction to Mixed Integer Nonlinear Programming

CO@Work Berlin

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Humboldt University Berlin

10/08/2009

Recall Monday last week (28/09/2009)...

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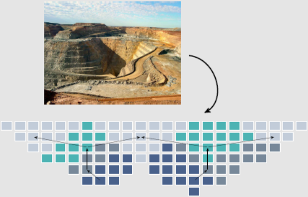
Open Pit Mining Production Scheduling

Input



- ▷ block model with precedences
- ▷ deterministic block content
- ▷ ore prices and production costs
- ▷ equipment capacities
- ▷ possibly aggregation of blocks

Problem

Find an order of excavation with

- ▷ max. net present value

satisfying

- ▷ precedence constrai
- ▷ resource constraints.

Simultaneously: Detern

- ▷ opt. processing decis

for each block.

Ambros M. Gleixner (ZIB) Open Pit Mine Production Scheduling 09/

Open Pit Mine Production Scheduling Problem

- $x_{i,t} \in \{0,1\}$ whether block i is completely mined in period t
- $y_{i,t} \in [0,1]$ fraction of block i mined during period t
- $z_{i,t} \in [0,1]$ fraction of block i processed during period t

$$\max \sum_{t=1}^{T} \left(\frac{1}{1+q}\right)^{t-1} \left(\sum_i \underbrace{c_i z_{i,t}}_{\text{income by sale}} - \underbrace{p_i z_{i,t}}_{\text{processing cost}} - \underbrace{m_i y_{i,t}}_{\text{mining cost}} \right)$$

s.t. (x, y) satisfy precedence constraints

$$\sum_t y_{i,t} \leq 1 \quad \forall i \qquad \text{(mine not more than the whole block)}$$

$$\sum_i \bar{R}_i y_{i,t} \leq M \quad \forall t \qquad \text{(obey mining capacity)}$$

$$\sum_i R_i z_{i,t} \leq P \quad \forall t \qquad \text{(obey processing capacity)}$$

$$0 \leq z_{i,t} \leq y_{i,t} \quad \forall i \forall t \qquad \text{(process not more than mined)}$$

$$\sum_{t' \leq t} y_{i,t'} \geq x_{i,t} \quad \forall i \forall t \qquad \text{(mining turns mined indicator off)}$$

Solve the MIP...

Solve the MIP...

```
Starting Cplex...
Tried aggregator 1 time.
MIP Presolve eliminated 154 rows and 137 columns.
Aggregator did 8 substitutions.
Reduced MIP has 1415 rows, 696 columns, and 6852 nonzeros.
[...]
        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Node    ItCnt     Gap

*     0+    0                          0.0000                     729     ---
      0     0        8.0964    84      0.0000        8.0964       729     ---
      0     0        8.0043    87      0.0000      Fract: 6       779     ---
*     0+    0                          6.9018        8.0043       779   15.97%
*     0+    0                          7.0221        8.0043       779   13.99%
*     0+    0                          7.1293        8.0043       779   12.27%
      0     2        8.0043    87      7.1293        8.0042       779   12.27%
*    20+   16                          7.2666        7.9037      1402    8.77%
    100    45        cutoff              7.2666        7.8249      3115    7.68%
    200    76        7.2772    22      7.2666        7.6182      5411    4.84%
    300    83        cutoff              7.2666        7.4420      7467    2.41%
    400    53        cutoff              7.2666        7.3440      8627    1.07%

Gomory fractional cuts applied:   6
MIP status(101): integer optimal solution
```

Time: 1.2 seconds

CPLEX 12.1	1.2s	7.27
GUROBI 2.0	1.9s	7.27
SCIP(CPX) 1.2.0	2.3s	7.27
CBC 2.3	78.9s	7.27
GLPK 4.39	140.7s	7.27

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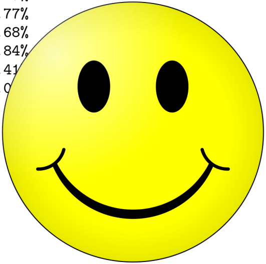


"It work's like a charm..."

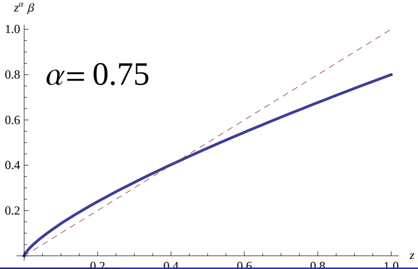
Change the objective a bit...

Consider "Economy of scales"...

$$\max \sum_{t=1}^{T} \left(\frac{1}{1+q}\right)^{t-1} \left(\sum_i \underbrace{c_i \beta z_{i,t}^{\alpha}}_{\text{income by sale}} - \underbrace{p_i z_{i,t}}_{\text{processing cost}} - \underbrace{m_i y_{i,t}}_{\text{mining cost}} \right)$$

s.t. precedence constraints; mining and processing capacity constraints

Parameter $\alpha > 0$, $\quad \beta = 1/(2-\alpha)$

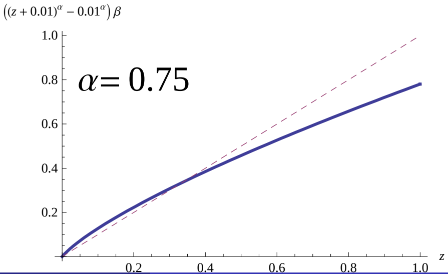


Consider "Economy of scales"...

$$\max \sum_{t=1}^{T} \left(\frac{1}{1+q}\right)^{t-1} \left(\sum_i \underbrace{c_i \beta((z_{i,t}+\varepsilon)^\alpha - \varepsilon^\alpha)}_{\text{income by sale}} - \underbrace{p_i z_{i,t}}_{\text{processing cost}} - \underbrace{m_i y_{i,t}}_{\text{mining cost}} \right)$$

s.t. precedence constraints; mining and processing capacity constraints

Parameter $\alpha > 0$, $\beta = 1/(2-\alpha)$
$\varepsilon > 0$ for differentiability in $z_{i,t} = 0$



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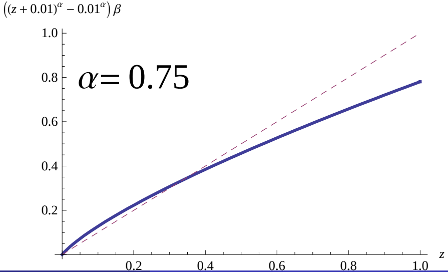
Nonlinear objective!



Solve the MINLP...

Solve the MINLP...

```
Simple B&B       ALFA  20ct09 23.3.0 LNX 12168.13781 LX3 x86/Linux
[...]
 Root node solved locally optimal.
 Resetting optcr to 1.0e-09
   Node  Act. Lev.  Objective  IInf  Best Int.   Best Bound   Gap  (1 secs)
      0     2   0      24.6966 260           -      24.6966        -
      1     3   1      24.6961 256           -      24.6966        -
[...]
   Node  Act. Lev.  Objective  IInf  Best Int.   Best Bound   Gap  (12 secs)
    200   202 200       0.9735   4           -      24.6966        -
    201   203 201       0.9735   3           -      24.6966        -
    202   204 202       0.9735   2           -      24.6966        -
    203   205 203       0.9735   1           -      24.6966        -
*   204   204 204       0.9735   0      0.9735      24.6966  0.960581
[...]
  28269 28176  27      17.1427 161      4.3680      19.3698  0.774496
 Resource limit exceeded
 Statistics:
    Iterations    :          1902560
    NLP Seconds   :      3600.000000
    B&B nodes     :            28270
    MIP solution  :         4.367976 found in node 14747
    Best possible :        19.369812
    Absolute gap  :        15.001836     optca :  0.000000
    Relative gap  :         0.774496     optcr :  0.000000
    Model Status  :                8
    Solver Status :                3
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```

after 1 hour:

solver	primal	dual	gap
SBB	4.37	19.37	77.4%
AlphaECP	11.0	–	–
DICOPT	10.18	–	–
BONMIN-BB	11.21	**16.73**	33.0%
BONMIN-OA	11.44	24.70	53.7%
BONMIN-QG	11.45	24.15	52.6%
BONMIN-Hyb	10.86	24.00	54.7%
SCIP “1.3”	**12.42**	16.92	**26.6%**
BARON	11.72	18.96	38.2%
LINDOGlobal	10.15	19.17	47%
COUENNE	8.89	20.99	58%

Solve the MINLP...

```
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[...]
 Root node solved locally optimal.
 Resetting optcr to 1.0e-09
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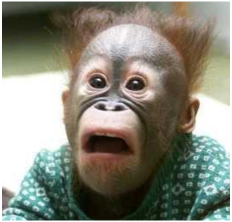

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MINLP is slooooooow!

What does SBB do?

MINLP: Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$

$$
\begin{aligned}
\min\ & f(x) \\
\text{s.t. } & g(x) \leq 0 \\
& x \in [x^L, x^U] \\
& x_i \in \mathbb{Z}, \quad i \in I
\end{aligned}
$$

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NLP based Branch–and–Bound:

- Dual Bounds: Solve NLP relaxation

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. } & g(x) \le 0, \\ & x \in [x^L, x^U], \qquad x_i \in [\ell_i, u_i], \quad i \in I \end{aligned}$$

- Primal Bounds: integer feasible solution of NLP relaxation
- Branching: On integer variables with fractional value in NLP relax.

Solvers that implement NLP based Branch–and–Bound

SBB = Simple Branch and Bound

[ARKI Consulting & Development A/S and GAMS Inc., 2002]

- developed by A. Drud (ARKI Consulting)
- NLP-based Branch-and-Bound
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- COIN-OR solver developed by P. Bonami, A. Wächter, et.al. (CMU, IBM, Bologna)
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MINLPBB = Mixed Integer NonLinear Programming Branch and Bound

[Leyffer, 2001]

- developed by S. Leyffer
- based on FilterSQP and bqpd

Solve the MINLP again...

Solve the MINLP again...

```
List of user-set options:
                                    Name   Value                used
                        bonmin.algorithm = B-OA                  yes

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Common Public License (CPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

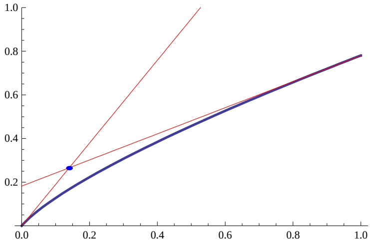
NLP0012I
              Num      Status      Obj             It       time
NLP0013I     1        OPT        -24.69658700869476      31       0.420026
OA0003I New best feasible of -11.3798 found after 93.0778 sec.
OA0010I After   184.4 seconds, upper bound   -11.3797, lower bound   -24.0727
OA0003I New best feasible of -11.442 found after 184.492 sec.
OA0010I After   321.3 seconds, upper bound   -11.4418, lower bound   -24.0708
OA0010I After   565.1 seconds, upper bound   -11.4418, lower bound   -24.0665
OA0010I After   830.0 seconds, upper bound   -11.4418, lower bound   -24.0318
OA0010I After  1064.1 seconds, upper bound   -11.4418, lower bound   -24.0262
OA0010I After  1804.6 seconds, upper bound   -11.4418, lower bound   -23.9881
OA0010I After  2983.6 seconds, upper bound   -11.4418, lower bound    -23.948
OA0010I After  3600.1 seconds, upper bound   -11.4418, lower bound   -23.9747
Time limit exceeded.

Bonmin finished. Found feasible point. Objective function = -11.441962.

MINLP solution:         11.44196163   (0 nodes, 3600.37 seconds)
Best possible:          24.69658706
Absolute gap:                13.255
Relative gap:                0.5367
```

What does Bonmin (B-OA) do?

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } g(x) \leq 0 \\ &\quad x \in [x^L, x^U] \\ &\quad x_i \in \mathbb{Z}, \quad i \in I \end{aligned}$$



Outer Approximation Algorithm:

1. Solve MIP relaxation

$$\min z \quad \text{s.t. } f(\hat{x}) + \nabla f(\hat{x})(x - \hat{x}) \leq z, \quad \hat{x} \in S,$$
$$g(\hat{x}) + \nabla g(\hat{x})(x - \hat{x}) \leq 0, \quad \hat{x} \in S,$$
$$x^L \leq x \leq x^U, \qquad x_i \in \mathbb{Z}, i \in I$$

[Duran and Grossmann, 1986, Fletcher and Leyffer, 1994]

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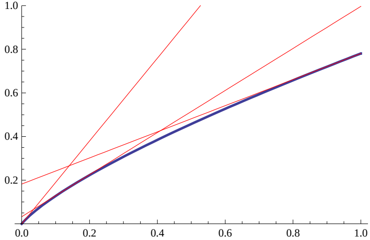

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2. If solution x^* satisfies $f(x^*) = z$, $g(x^*) \le 0$, then finish.
3. Otherwise, add x^* to $S \Rightarrow$ cut off x^* from relaxation and repeat.

[Duran and Grossmann, 1986, Fletcher and Leyffer, 1994]

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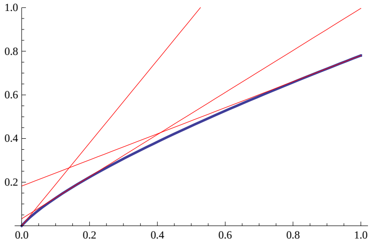

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3. Otherwise, add x^* to $S \Rightarrow$ cut off x^* from relaxation and repeat.

Primal Solutions: Solve NLP

$\min\{f(x) : g(x) \leq 0,\ x \in [x^L, x^U],\ x_i = x_i^*,\ i \in I\}$ and add solution to S.

[Duran and Grossmann, 1986, Fletcher and Leyffer, 1994]

Solvers that implement Outer Approximation Algorithm

DICOPT = Discrete and Continuous Optimizer

[Duran and Grossmann, 1986]

- developed by J. Viswanathan and I.E. Grossmann (CMU)
- pure GAMS solver

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AlphaECP = α Extended Cutting Plane

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Solve the MINLP a third time...

Solve the MINLP a third time...

```
SCIP version 1.1.0.11 [precision: 8 byte] [memory: block] [mode: optimized] [LP solver: CPLEX 11.21]
Copyright (c) 2002-2009 Konrad-Zuse-Zentrum fuer Informationstechnik Berlin (ZIB)

Original problem has 1576 linear constraints and 1 nonlinear constraints.
original problem has 841 variables (280 bin, 0 int, 0 impl, 561 cont) and 1577 constraints
presolving:
(round 1) 7 del vars, 8 del conss, 0 chg bounds, 0 chg sides, 0 chg coeffs, 1 upgd conss, 1916 impls, 0 clqs
[...]
presolving (8 rounds):
 28 deleted vars, 52 deleted constraints, 2 tightened bounds, 0 added holes, 0 changed sides, 0 changed coeffic
 14761 implications, 0 cliques
presolved problem has 1093 variables (263 bin, 0 int, 0 impl, 830 cont) and 1805 constraints
    115 constraints of type <varbound>
   1182 constraints of type <linear>
    229 constraints of type <logicor>
    279 constraints of type <nonlinear>
Presolving Time: 1.28
Initializing expression interpreter CppAD 20090303.0
Oracle initializes expression interpreter CppAD 20090303.0

 time | node  | left  |LP iter|frac |cols |rows |cuts |confs|strbr|  dualbound   | primalbound  |  gap
  1.4s|     1 |     0 |  1369 | 261 |1930 |3758 |   0 |   0 |   0 | 3.520238e+01 |      --      |   Inf
  1.9s|     1 |     0 |  2732 | 261 |1930 |3972 | 214 |   0 |   0 | 2.952667e+01 |      --      |   Inf
[...]
X12.6s|     1 |     0 |  4492 | 242 |1930 |5515 |1757 |   0 |   0 | 2.461582e+01 |-7.578685e+02 |   Inf
[...]
 3600s| 86900 | 10242 | 15252k|   - |1930 |4748 |6663k|   3 |6969 | 1.691782e+01 | 1.241630e+01 |  26.61%
[...]
Constraints        :     Number  #Separate #Propagate    #EnfoLP    #EnfoPS    DomReds       Cuts   Children
  integral         :          0          0          0      52468          0       1351          0     102882
  nonlinear        :        279      90608      52890        243          1     292358    6663347          0
```

What does SCIP do?

$$\min f(x) \quad \text{s.t.} \quad g(x) \leq 0,\, x \in [x^L, x^U],\, x_i \in \mathbb{Z}, \quad i \in I$$

Integrate Outer Approximation into LP based Branch–and–Bound:

- Dual Bounds: Solve LP relaxation

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If solution x^* violates $g(x^*) \leq 0$, $f(x^*) = z$, add x^* to S.

[Quesada and Grossmann, 1992]

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$$\begin{aligned} & \min f(x) \\ & \text{s.t. } g(x) \leq 0, \\ & \quad x \in [x^L, x^U], \qquad x_i = x_i^*, \quad i \in I \end{aligned}$$

[Quesada and Grossmann, 1992]

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- Branching: On integer variables with fractional value in LP relax.

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Solvers that implement LP-based Branch–and–Bound

FilMINT = Filter-Mixed INTeger optimizer

[Abhishek et al., 2006]

- developed by K. Abhishek, J.T. Linderoth (Lehigh), and S. Leyffer (Argonne)
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[Achterberg, 2007, Berthold et al., 2009]

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Bonmin = Basic Open-source Nonlinear Mixed Integer programming

- COIN-OR solver developed by P. Bonami, A. Wächter et.al.
- option B-QG = Quesada-Grossmann LP–based B&B
- option B-Hyb = solve NLP (instead of LP) relaxation at some nodes

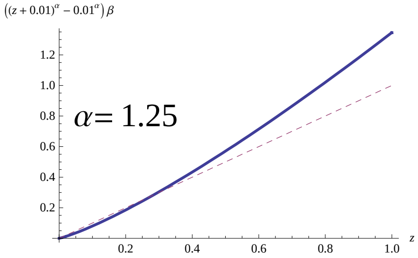
Let's try $\alpha = 1.25$...

$$\max \sum_{t=1}^{T} \left(\frac{1}{1+q}\right)^{t-1} \left(\sum_i \underbrace{c_i \beta((z_{i,t}+\varepsilon)^\alpha - \varepsilon^\alpha)}_{\text{income by sale}} - \underbrace{p_i z_{i,t}}_{\text{processing cost}} - \underbrace{m_i y_{i,t}}_{\text{mining cost}} \right)$$

s.t. (x, y) satisfy precedence constraints

obey mining and processing capacities

$\dots$



$\alpha = 1.25$: Try SBB again

$\alpha = 1.25$: Try SBB again – Ups...

```
Simple B&B       ALFA  20ct09 23.3.0 LNX 12168.13781 LX3 x86/Linux

 841 columns (280 discrete), 1577 rows, 7951 nonzeros
C O N O P T 3    ALFA  20ct09 23.3.0 LNX 13062.13781 LX3 x86/Linux


    C O N O P T 3   version 3.14T
    Copyright (C)   ARKI Consulting and Development A/S
                    Bagsvaerdvej 246 A
                    DK-2880 Bagsvaerd, Denmark

 Using default options.

 Reading Data

   Iter Phase Ninf   Infeasibility   RGmax    NSB   Step InItr MX OK
      0   0        4.4408920985E-16 (Input point)
                               Pre-triangular equations:        0
                               Post-triangular equations:       1
      1   0        0.0000000000E+00 (After pre-processing)
      2   0        0.0000000000E+00 (After scaling)

 ** Feasible solution. Value of objective =    4.440892098501E-16

   Iter Phase Ninf     Objective     RGmax    NSB   Step InItr MX OK
      4   3        4.4408920985E-16 0.0E+00     0

 ** Optimal solution. There are no superbasic variables.

 Root node solved locally optimal.
 Relaxed problem gives integer solution.
 Terminating.
```

$\alpha = 1.25$: Multistart for root node NLP

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```
Simple B&B       ALFA  20ct09 23.3.0 LNX 12168.13781 LX3 x86/Linux

Reading parameter(s) from "/nfs/optimi/kombadon/bzfviger/talk_coatwork/sbb.opt"
>>  rootsolver oqnlp.1
Finished reading from "/nfs/optimi/kombadon/bzfviger/talk_coatwork/sbb.opt"
Root node solver: oqnlp.1
 841 columns (280 discrete), 1577 rows, 7951 nonzeros
OQNLP MStart     ALFA  20ct09 23.3.0 LNX 13463.13781 LX3 x86/Linux
Reading parameter(s) from "/nfs/optimi/kombadon/bzfviger/talk_coatwork/oqnlp.opt"
>>  nlpsolver conopt
>>  stage1_iterations 10
>>  max_solver_calls 10
Finished reading from "/nfs/optimi/kombadon/bzfviger/talk_coatwork/oqnlp.opt"
[...]
   Itn     Penval   Merit     Merit     Dist     Best       Solver     Term     Sinf
                    Filter Threshold  Filter    Obj          Obj      Code
     0   +1.000e+30      -1.000e+30      -0.000e+00  -0.000e+00 KTC  +0.000e+00
    80   +3.289e+04 REJ  +2.748e+04  ACC +6.751e+00
   100   +3.683e+04 REJ  +2.525e+04  ACC +2.587e+01
   158   +9.643e+03 ACC  +2.635e+04  ACC +2.962e+01  +2.862e+01 KTC  +1.104e-13
[...]
--- Exiting OQNLP
 Root node solved locally optimal.
 Resetting optcr to 1.0e-09
   Node  Act. Lev.  Objective  IInf  Best Int.    Best Bound   Gap  (1 secs)
      0     2   0      29.6202  19            -      29.6202          -
[...]
     11    13  11      29.7968   1            -      29.7968          -
*    12    11  12      29.7968   0      29.7968      29.7968  0.000000
 Integer solution
 Terminating.
```

$\alpha = 1.25$: Results for all solvers

solver	time	primal	dual	gap
SBB	0s	0	0	0%
SBB+MS	1s	29.8	29.8	0%
AlphaECP	1s	10.3	–	–
DICOPT	1s	0	–	–
BONMIN-BB	3600s	–	26.3	∞
BONMIN-OA	7s	21.1	21.1	0%
BONMIN-QG	2s	12.6	12.6	0%
BONMIN-Hyb	9s	19.3	19.3	0%
SCIP "1.3"	3600s	31.8	33.3	4.4%
BARON	3600s	**32.9**	33.3	1.2%
LINDOGlobal	3600s	31.3	34.0	7.7%
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Wow - MINLP can be fast! ;-)

Why the different results?

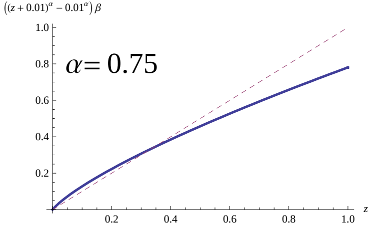
Recall ...

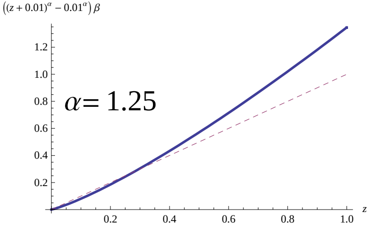
$$\max \sum_{t=1}^{T} \left(\frac{1}{1+q}\right)^{t-1} \left(\sum_i \underbrace{c_i((z_{i,t}+\varepsilon)^\alpha - \varepsilon^\alpha)}_{\text{income by sale}} - \underbrace{p_i z_{i,t}}_{\text{processing cost}} - \underbrace{m_i y_{i,t}}_{\text{mining cost}} \right)$$

s.t. (x, y) satisfy precedence constraints; obey mining and processing capacities

$\ldots$

All we did was changing α a bit...





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$\ldots$

$\alpha = 1.25 \Rightarrow$ Objective is convex $\oplus$ Problem is maximization

NONCONVEX MINLP

- NLP relaxation may have several local (non-global) minima
- linearizations $f(\hat{x}) + \nabla f(\hat{x})(x - \hat{x}) \leq 0$ etc. are not "outer-approximating"

Why the different results?

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PANIC NOW!

The table again...

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$$\min f(x) \quad \text{s.t.} \quad g(x) \leq 0,\, x \in [x^L, x^U],\, x_i \in \mathbb{Z}, \quad i \in I$$

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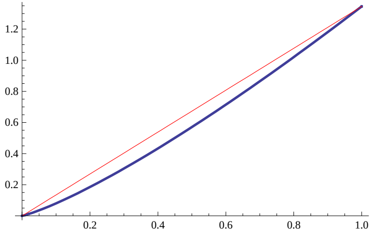
- Dual Bounds: Solve a LP relaxation

$$\begin{aligned} &\min z \\ &\text{s.t. } Ax + A'z \leq b, \\ &\quad x \in [x^L, x^U], \\ &\quad x_i \in [\ell_i, u_i], \quad i \in I \end{aligned}$$



[Tawarmalani and Sahinidis, 2002, 2004]

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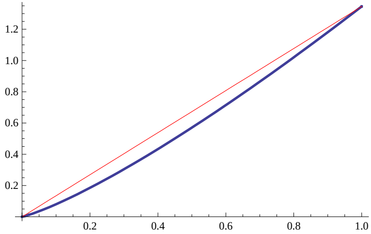

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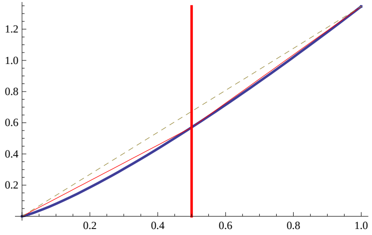

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- Branching: On integer variables with fractional value in LP relax. And on continuous variables in nonconvex terms.

[Tawarmalani and Sahinidis, 2002, 2004]

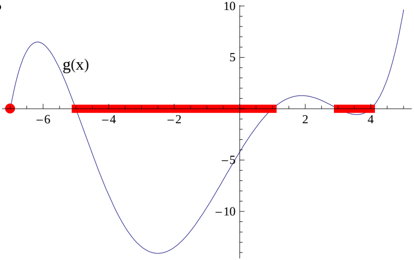
The linear outer–approximation of a nonconvex MINLP

Given: A function $g : \mathbb{R}^n \to \mathbb{R}$ defining the constraint

$$g(x) \leq 0, \qquad x \in [x^L, x^U]$$

Seek: A (linear) outer-approximation of

$$\{x \in [x^L, x^U] : g(x) \leq 0\}$$



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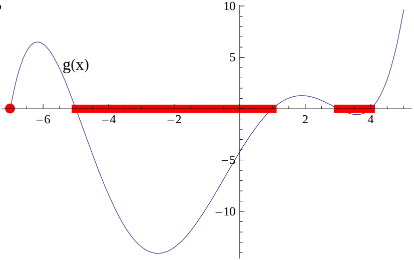
$\Rightarrow$ too difficult in general



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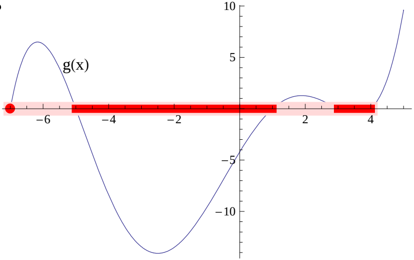


Seek: A (linear) outer-approximation of

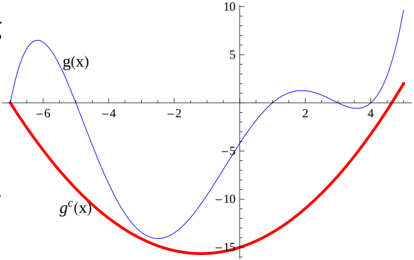
$$\operatorname{conv}\{x \in [x^L, x^U] : g(x) \leq 0\}$$

$\Rightarrow$ too difficult in general

Given: A function $g : \mathbb{R}^n \to \mathbb{R}$ defining the constraint

$$g(x) \leq 0, \qquad x \in [x^L, x^U]$$

Seek: A convex function $g^c(\cdot)$ that underestimates $g(\cdot)$ on $[x^L, x^U]$:



$$\{x \in [x^L, x^U] : g(x) \leq 0\} \subseteq \{x \in [x^L, x^U] : g^c(x) \leq 0\}$$

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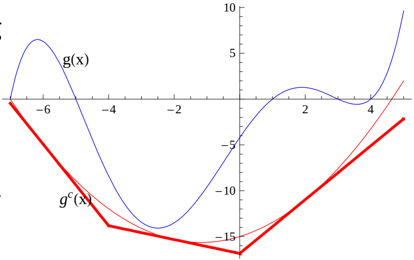

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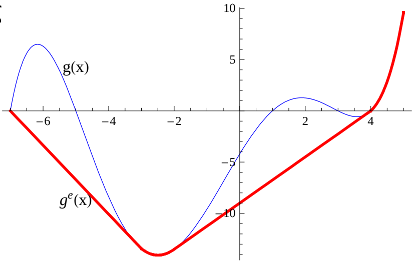
$$\subseteq \{x \in [x^L, x^U] : g^c(\hat{x}) + \nabla g^c(\hat{x})(x - \hat{x}) \leq 0,\ \hat{x} \in S\}$$

The linear outer–approximation of a nonconvex MINLP

Given: A function $g : \mathbb{R}^n \to \mathbb{R}$ defining the constraint

$$g(x) \leq 0, \qquad x \in [x^L, x^U]$$

Seek: The tightest convex function $g^c(\cdot)$ that underestimates $g(\cdot)$ on $[x^L, x^U]$:



The convex envelope $g^e : [x^L, x^U] \to \mathbb{R}$ is a function such that

- analytically:
  - $g^e(x)$ is convex on $[x^L, x^U]$
  - $g^e(x) \leq g(x)$ for all $x \in [x^L, x^U]$
  - $\forall h : [x^L, x^U] \to \mathbb{R}$, $h(\cdot)$ convex, $h(\cdot) \leq g(\cdot)$: $h(\cdot) \leq g^e(\cdot)$

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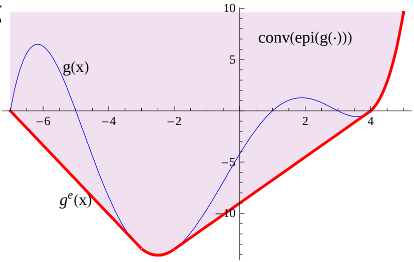

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  - $\forall h : [x^L, x^U] \to \mathbb{R}$, $h(\cdot)$ convex, $h(\cdot) \leq g(\cdot)$: $h(\cdot) \leq g^e(\cdot)$
- geometrically: $\mathrm{epi}(g^e(\cdot)) = \mathrm{conv}(\mathrm{epi}(g(\cdot)))$

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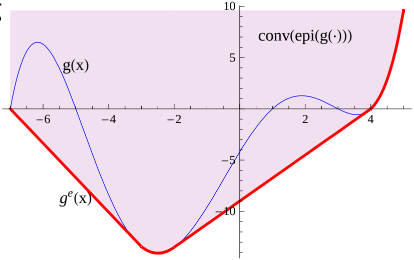

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In general: Convex envelopes are hard to find.

(finding the convex envelope of a function is as hard as finding its global min.)

Convex envelopes for concave functions

Convex envelopes for concave functions

Theorem 1 (of 1): Let $v_1, \ldots, v_k$ be the vertices of a polytope P. The convex envelope $g^e(x)$ of a concave function $g(x)$ over P can be expressed as

$$g^e(x) = \min_{\alpha} \sum_{i=1}^{k} \alpha_i g(v_i)$$
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[Tawarmalani and Sahinidis, 2002]

Convex envelopes for concave functions

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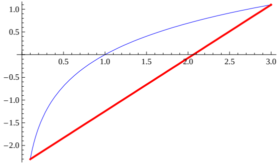
  $\Rightarrow \log(x)$, $-\exp(x)$, $-x^{2k}$, $\sqrt{x}$, ...



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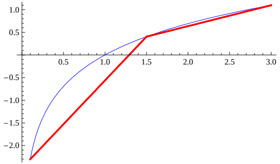
- In Branch–and–Bound: tighten by branching



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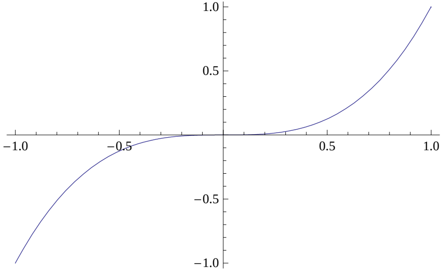
Convex envelope for monomials of odd degree

$$g(x) = x^{2k+1}, \qquad k \in \mathbb{N}$$



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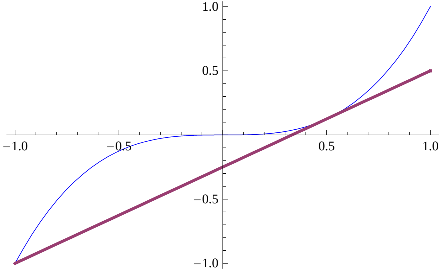

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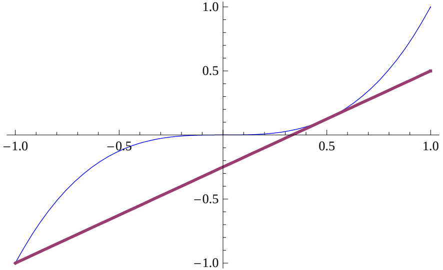

$\Rightarrow$ c is the positive real root of the polynomial

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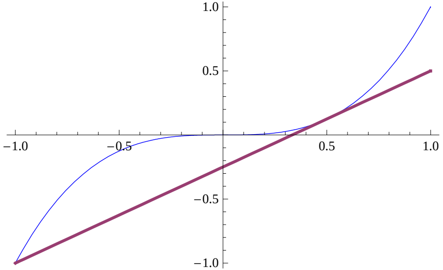

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k	c (for $x^L = -1$)
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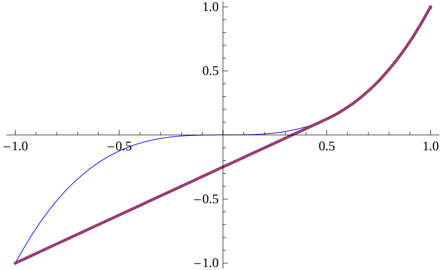

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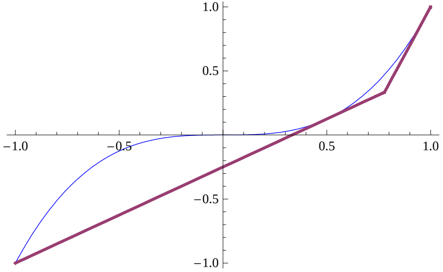

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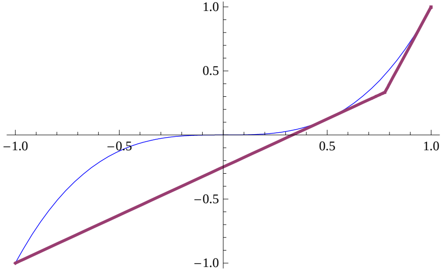

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- obtain convex envelope and linear approximation
- easily generalized to $x|x|^{n-1} = \text{sign}(x)|x|^n$, $n \geq 1$ (application for flow of water and gas in pipes)

[Liberti and Pantelides, 2003]

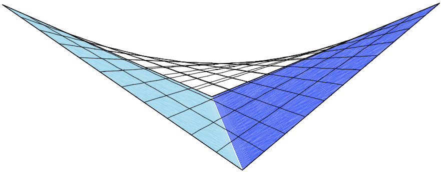
Convex envelope for $x \cdot y$

McCormick [McCormick, 1976]:
The convex envelope of

$$f(x) = xy$$

for $x \in [x^L, x^U]$, $y \in [y^L, y^U]$ is

$$xy \geq \max \left\{ \begin{array}{c} x^U y + y^U x - x^U y^U \\ x^L y + y^L x - x^L y^L \end{array} \right\}$$



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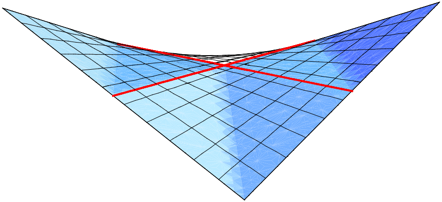

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- apply McCormick underestimators to (nonconvex) quadratic forms

$$f(x) = \sum_{i,j} a_{i,j} x_i x_j$$

- branching on x and y changes variable bounds
  $\Rightarrow$ tighter underestimator $\Rightarrow$ improving dual bounds

Reformulation for the general (factorable) case

Let $g : \mathbb{R}^m \to \mathbb{R}$ be factorable, i.e., a recursive sum and product of univariate functions

Examples: $g(x) = x_1 x_2$, $g(x) = x_1/x_2$, $g(x) = \sqrt{\exp(x_1 x_2 + x_3 \ln x_4) x_3^3}$

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Reformulation: replace products of functions or variables by new variables

$$g = \sqrt{\exp(y_1) y_2}$$
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Convex outer-approximation by

- estimating univariate functions by known convex envelopes
- estimating bilinear terms by McCormick underestimators

[Tawarmalani and Sahinidis, 2004]

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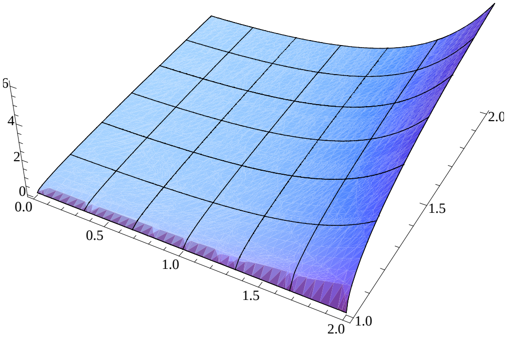
$$g(x) = \sqrt{\exp(x_1^2)\ln(x_2)}$$

$$x_1 \in [0,2], \quad x_2 \in [1,2]$$



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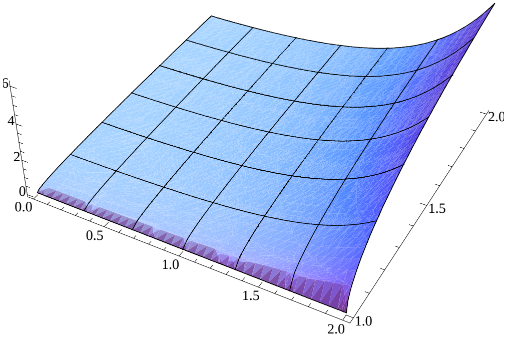

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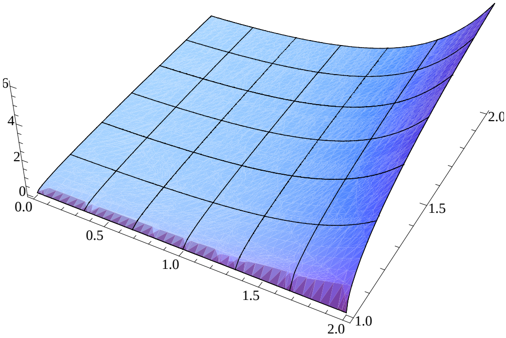

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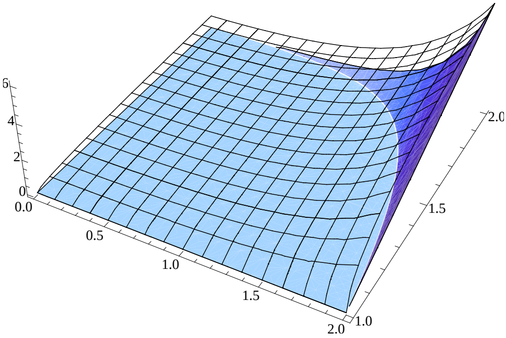

Convex relaxation:

$$\sqrt{y_1^L} + \frac{y_1 - y_1^L}{\sqrt{y_1^U} + \sqrt{y_1^L}} \le g \le \sqrt{y_1}; \quad \ln x_1^L + (x_2 - x_2^L)\frac{\ln x_2^U - \ln x_2^L}{x_2^U - x_2^L} \le y_3 \le \ln(x_2)$$

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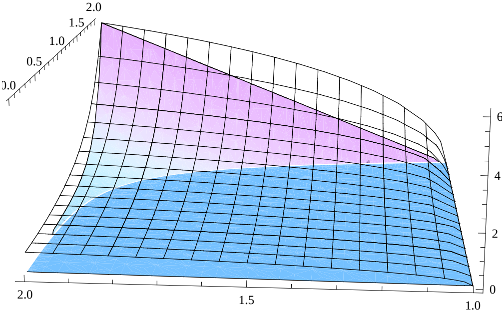

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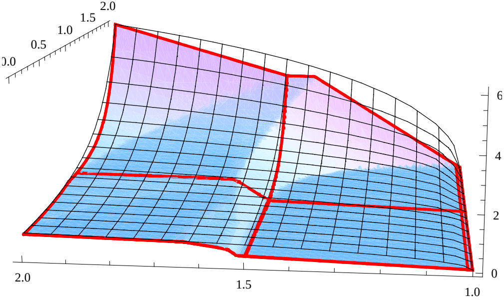

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Adding a stockpile to open pit mining production

- stockpile allows to store mined material before processing

[Bley et al., 2009a,b]

Adding a stockpile to open pit mining production

- stockpile allows to store mined material before processing
- BUT: materials of different quality (metal/ore ratio) are mixed

$$\frac{(\text{metal send from stockpile to processing})_t}{(\text{ore send from stockpile to processing})_t} = \frac{(\text{metal in stockpile at})_{t-1}}{(\text{ore in stockpile at})_{t-1}}$$

[Bley et al., 2009a,b]

Ask SCIP to solve it...

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```
SCIP version 1.1.0.11 [precision: 8 byte] [memory: block] [mode: optimized] [LP solver: CPLEX 11.21]
Copyright (c) 2002-2009 Konrad-Zuse-Zentrum fuer Informationstechnik Berlin (ZIB)

Original problem has 1856 linear constraints and 280 nonlinear constraints.
original problem has 1688 variables (280 bin, 0 int, 0 impl, 1408 cont) and 2136 constraints
[...]
presolved problem has 1431 variables (263 bin, 0 int, 0 impl, 1168 cont) and 2004 constraints
    115 constraints of type <varbound>
   1416 constraints of type <linear>
    229 constraints of type <logicor>
    244 constraints of type <quadratic>
Presolving Time: 0.20
There are nonconvex quadratic constraints.
[...]
SCIP Status        : problem is solved [optimal solution found]
Solving Time (sec) : 9.78
Solving Nodes      : 418
Primal Bound       : +7.32628041419470e+00 (24 solutions)
Dual Bound         : +7.32628041419470e+00
Gap                : 0.00 %
[...]
Constraints        :     Number  #Separate #Propagate    #EnfoLP    DomReds       Cuts      Conss   Children
  branchnonlinear  :          0          0          0         35          0          0          0         38
  integral         :          0          0          0        315         90          0          0        470
  varbound         :        115          2       4316         16         23          0          0          0
  linear           :       1416          2       4316         16       7222          0          0          0
  logicor          :        229          2       2437         16       1812          0          0          0
  quadratic        :        244        229       2107         41         54         66          0          0
  countsols        :          0          0          0         16          0          0          0          0
```

Ask SCIP to solve it...

```
SCIP version 1.1.0.11 [precision: 8 byte] [memory: block] [mode: optimized] [LP solver: CPLEX 11.21]
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Original problem has 1856 linear constraints and 280 nonlinear constraints.
original problem has 1688 variables (280 bin, 0 int, 0 impl, 1408 cont) and 2136 constraints
[...]
presolved problem has 1431 variables (263 bin, 0 int, 0 impl, 1168 cont) and 2004 constraints
    115 constraints of type <varbound>
   1416 constraints of type <linear>
    229 constraints of type <logicor>
    244 constraints of type <quadratic>
Presolving Time: 0.20
There are nonconvex quadratic constraints.
[...]
SCIP Status         : problem is solved [optimal solution found]
Solving Time (sec) : 9.78
Solving Nodes       : 418
Primal Bound        : +7.32628041419470e+00 (24 solutions)
Dual Bound          : +7.32628041419470e+00
Gap                 : 0.00 %
[...]
Constraints         :     Number  #Separate #Propagate    #EnfoLP    DomRe
  branchnonlinear   :          0          0          0         35
  integral          :          0          0          0        315
  varbound          :        115          2       4316         16
  linear            :       1416          2       4316         16       72
  logicor           :        229          2       2437         16       18
  quadratic         :        244        229       2107         41
  countsols         :          0          0          0         16
```

MINLP can be fun!

Another convexification technique: α-underestimators

Given $g \in C^2(\mathbb{R}^n, \mathbb{R})$. Let

$$\alpha \leq \lambda_1(\nabla^2 f(x)) \qquad \forall x \in [x^L, x^U].$$

Then

$$g(x) + \frac{1}{2}\alpha(x - x^L)^\top (x^U - x)$$

is a convex underestimator of $g(x)$.

[Adjiman and Floudas, 1997]

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How to find α?

1. Use Interval-Arithmetic to compute Interval-Hessian:

$$\nabla^2 g([x^L, x^U]) = \begin{pmatrix} [g^L_{1,1}, g^U_{1,1}] & \cdots & [g^L_{1,n}, g^U_{1,n}] \\ \vdots & & \vdots \\ [g^L_{n,1}, g^U_{n,1}] & \cdots & [g^L_{n,n}, g^U_{n,n}] \end{pmatrix}.$$

   That is, $\frac{\partial^2 g}{\partial x_i \partial x_j}(x) \in [g^L_{i,j}, g^U_{i,j}]$ for all $x \in [x^L, x^U]$.

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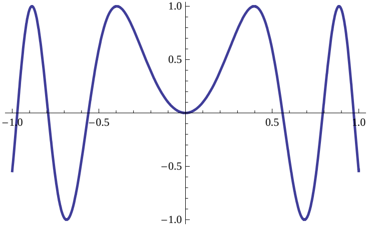
2. Estimate α as minimal eigenvalue over 2^{n-1} vertex matrices of $\nabla^2 g([x^L, x^U])$. Or: use Gershgorin circles [Adjiman and Floudas, 1997]

(Bad) Example for α-underestimator

$$g(x) = \sin(10x^2), \qquad x \in [-1, 1]$$



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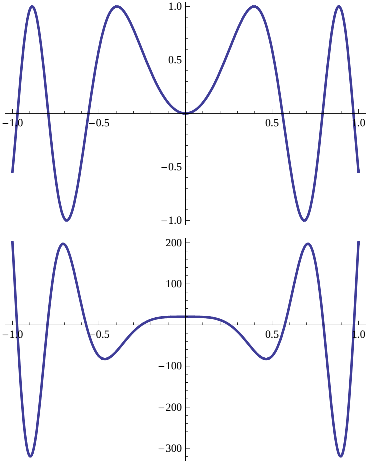
$$g(x) = \sin(10x^2), \qquad x \in [-1, 1]$$

$$\nabla^2 g(x) = 20\cos(10x^2) - 400x^2 \sin(10x^2)$$



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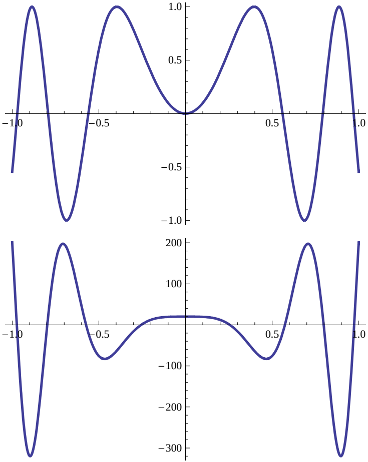
Bounds on $\lambda_1(\nabla^2 g(x))$:

- Tight:

$$\lambda_1(\nabla^2 g(x)) \geq -320$$

- By Intervalarithmetic:

$$\lambda_1(\nabla^2 g(x)) \geq -420$$



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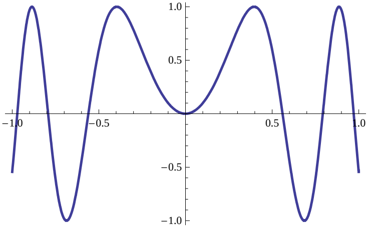

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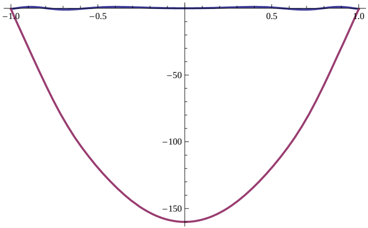


Convex underestimator:

$$\sin(10x^2) - \frac{320}{2}(x+1)(1-x) = \sin(10x^2) + 160x^2 - 160$$

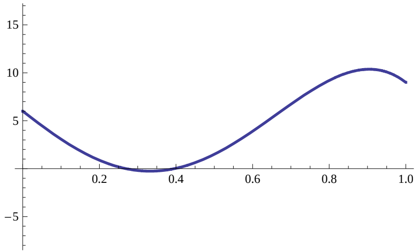
Underestimators for polynomials

$$f(x) = \sum_{i=0}^{\ell} a_i x^i$$

a multivariate polynomial over $[0, 1]$

(i and ℓ are multiindices).

$f(x) = 6 - 31x + 6x^2 + 124x^3 - 96x^4$



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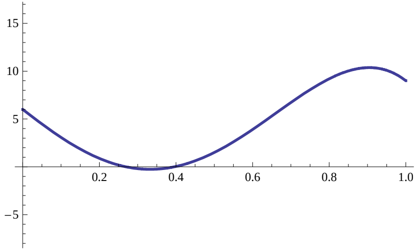

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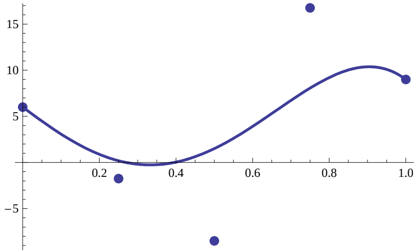

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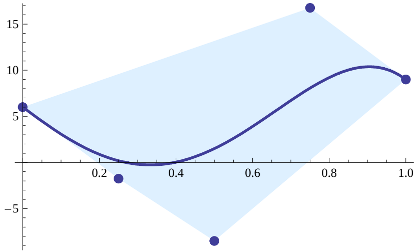

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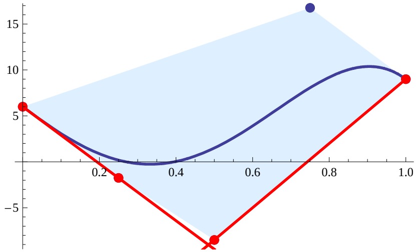

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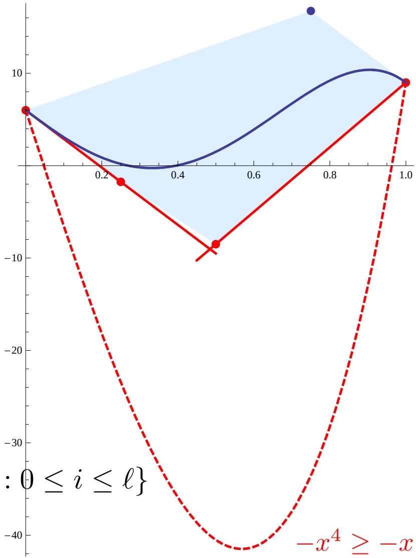

Solvers for nonconvex MINLP

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MINOTAUR = Mixed-Integer Nonconvex Optimization Toolbox – Algorithms, Underestimators, Relaxations

- J.T. Linderoth, S. Leyffer, T. Munson et.al.
- in development

Heuristics in MINLP

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  - (partially) solve sub-MIQCP, including presolve

The Feasibility Pump for MINLP

$$\min\{c^\top x : g(x) \leq 0, \quad x \in [x^L, x^U], \quad x_i \in \mathbb{Z}, i \in I\}$$

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Basic Idea: Alternating projection on

MIP relaxation

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- Convergence for convex problems: finds feasible point ($\bar{x}^j$) or proofs infeasibility
- Heuristic for nonconvex problems

[Bonami et al., 2009]

Domain Propagation (Bound Tightening)

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$$\Rightarrow \quad y \leq \sum_{i\in\mathcal{P}} a_i x_i^U + \sum_{i\in\mathcal{N}} a_i x_i^L$$

$$y \geq \sum_{i\in\mathcal{P}} a_i x_i^L + \sum_{i\in\mathcal{N}} a_i x_i^U$$

Interval Propagation on DAGs

Represent algebraic structure of problem in one directed acyclic graph:

- nodes: variables, operations, constraints
- arcs: flow of computation

[Schichl and Neumaier, 2005, Vu et al., 2008]

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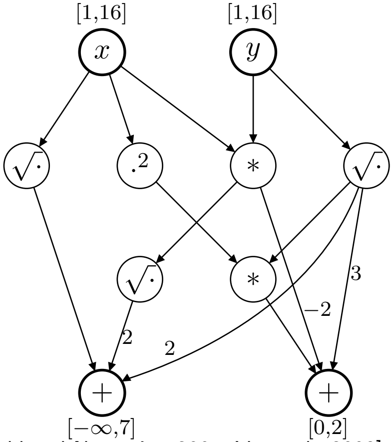
Example:

$$\sqrt{x} + 2\sqrt{xy} + 2\sqrt{y} \in [-\infty, 7]$$
$$x^2\sqrt{y} - 2xy + 3\sqrt{y} \in [0, 2]$$
$$x, y \in [1, 16]$$



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Represent algebraic structure of problem in one directed acyclic graph:

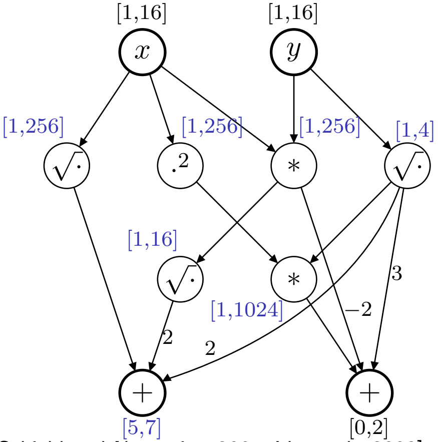
- nodes: variables, operations, constraints
- arcs: flow of computation

Example:

$$\sqrt{x} + 2\sqrt{xy} + 2\sqrt{y} \in [-\infty, 7]$$
$$x^2\sqrt{y} - 2xy + 3\sqrt{y} \in [0, 2]$$
$$x, y \in [1, 16]$$

Forward propagation:

- compute bounds on intermediate nodes (top-down)



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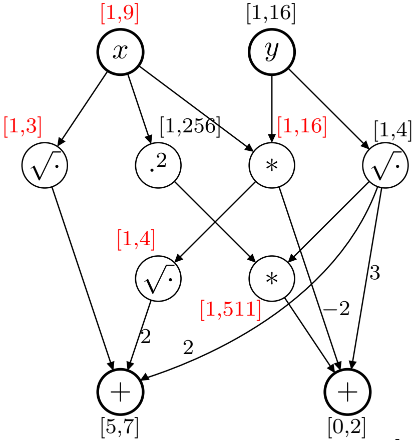

[Schichl and Neumaier, 2005, Vu et al., 2008]

Forward propagation:

- compute bounds on intermediate nodes (top-down)

Backward propagation:

- reduce bounds using reverse operations (bottom-up)

Reformulation: Power transformations for signomial terms

For $c, p_i \in \mathbb{R}$, $i = 1, \ldots, n$, let $$f(x) = c \prod_{i=1}^{n} x_i^{p_i}.$$

[Lundell et al., 2009]

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For $c, p_i \in \mathbb{R}$, $i = 1, \ldots, n$, let $$f(x) = c \prod_{i=1}^{n} x_i^{p_i}.$$

$f(x)$ is convex if

$$p_i \leq 0 \quad \forall i$$

or $$\sum_{i=1}^{n} p_i \geq 1 \ \wedge \ \exists k \, \forall i \neq k : p_i \leq 0.$$

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Introduce transformations $\quad x_i = X_i^{Q_i}$. $\quad$ Then

$$c \prod_{i=1}^{n} X_i^{p_i Q_i} \quad \text{is convex/concave if}$$

$$\begin{cases} Q_i \geq 1, & \text{if } p_i > 0, i = k, \\ Q_i < 0, & \text{if } p_i > 0, i \neq k, \\ Q_i = 1 & \text{if } p_i < 0. \end{cases} \sum_{i=1}^{n} p_i Q_i \geq 1 \qquad \begin{cases} 0 < Q_i \leq 1, & \text{if } p_i > 0, \\ Q_i < 0, & \text{if } p_i < 0, \end{cases} \sum_{i=1}^{n} p_i Q_i \leq 1$$

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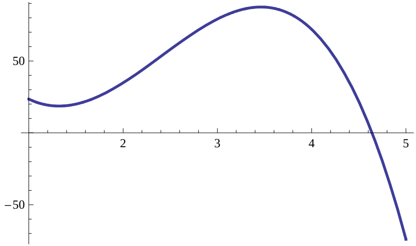
Example: $f(x) = x^4 + 79.5x^2 - 170x + 120 - 7x^{3.5}, \quad x \in [1, 6]$

$\Rightarrow \quad f(X) = x^4 + 79.5x^2 - 170x + 120 - 7X$ and $x = X^{3.5}$

[Lundell et al., 2009]



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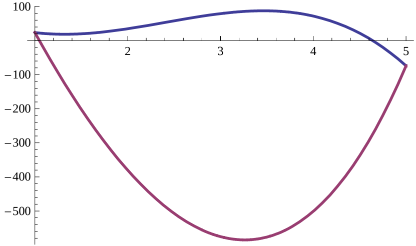
approximating X piecewise linear with 1 breakpoint

[Lundell et al., 2009]



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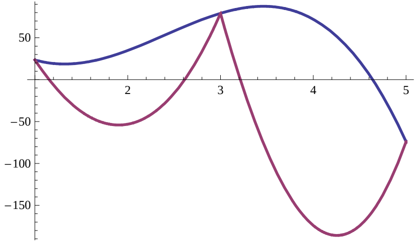

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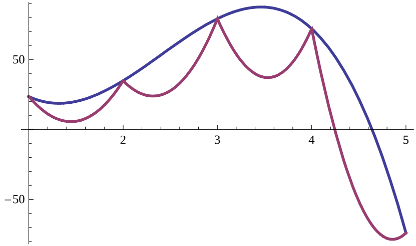
[Lundell et al., 2009]



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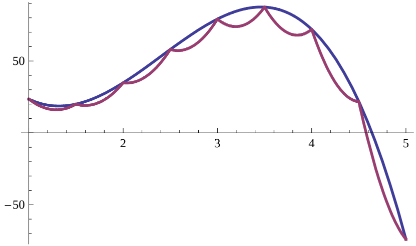

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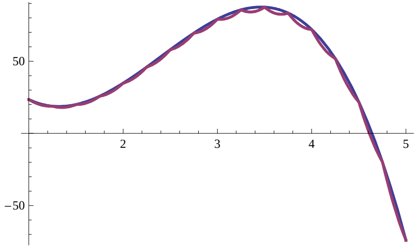

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Reformulation: Elimination of bilinear terms

Idea: Reveal hidden linearities in intersections of bilinear and linear constraints.

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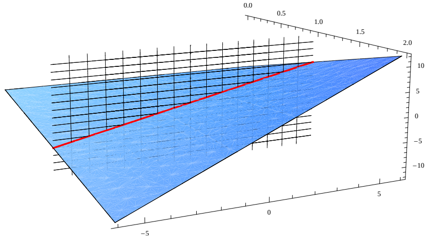
Simple example:

$A = \{(w, x, y) : w = xy \;\wedge\; x = 1\}$



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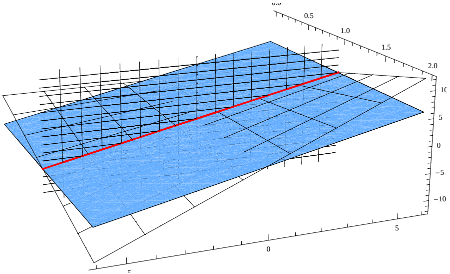

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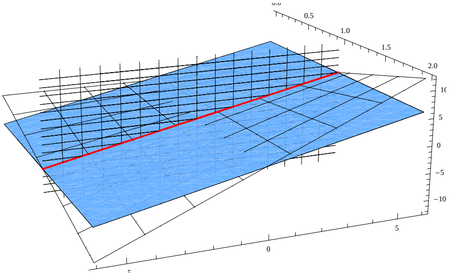

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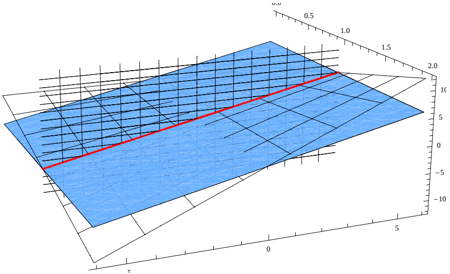

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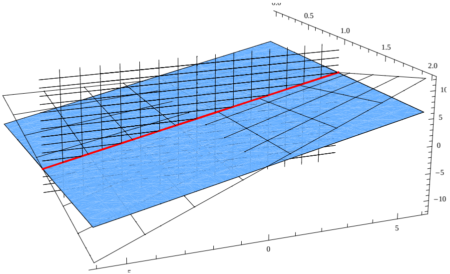

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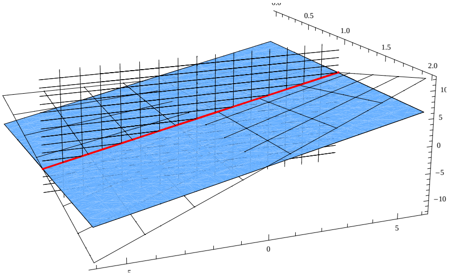

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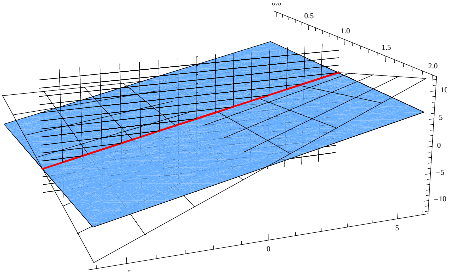

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End

Let's stop here (I'M HUNGRY!)...