Binpacking

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Set Covering Formulation

Packing

- A packing s is an assignment vector $\lambda_s \in \{0,1\}^n$.
- Defining which items belong to packing s.
- Item $i \in \{1,\dots,n\}$ belongs to packing s if $(\lambda_s)_i$ is one.

Feasible Packing

- Capacity restriction

Binary variables

- $x_s = 1$ if packing s is used

Set Covering

$$\begin{aligned} \min \quad & \sum_{s \in \mathcal{S}} x_s & \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}} (\lambda_s)_i \, x_s \geq 1 & \forall i \in \mathcal{I} \\ & x_s \in \{0,1\} & \forall s \in \mathcal{S} \end{aligned}$$

LP relaxation

Set Covering

$$\begin{aligned} \min \quad & \sum_{s \in \mathcal{S}} x_s \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}} (\lambda_s)_i \, x_s \geq 1 && \forall i \in \mathcal{I} \\ & x_s \in \{0, 1\} && \forall s \in \mathcal{S} \end{aligned}$$

LP Relaxation

$$\begin{aligned} \min \quad & \sum_{s \in \mathcal{S}} x_s \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}} (\lambda_s)_i \, x_s \geq 1 && \forall i \in \mathcal{I} \\ & x_s \geq 0 && \forall s \in \mathcal{S} \\ & \cancel{x_s \leq 1} && \cancel{\forall s \in \mathcal{S}} \end{aligned}$$

Duality Theory for LPs

Primal (P)

$$\begin{aligned} \min \quad & \sum_{s \in \mathcal{S}} x_s \\ \text{subject to} \quad & \sum_{s \in \mathcal{S}} (\lambda_s)_i \, x_s \geq 1 && \forall i \in \mathcal{I} \\ & x_s \geq 0 && \forall s \in \mathcal{S} \end{aligned}$$

Dual (D)

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{I}} y_i \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i \leq 1 && \forall s \in \mathcal{S} \\ & y_i \geq 0 && \forall i \in \mathcal{I} \end{aligned}$$

Strong Duality

If there exists an optimal solution x^* for (P), then there exists an optimal solution y^* for (D) and their objective values are equal.

Column Generation

s_i	$\tilde{c}$ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						0.5
2	2		1					1					1	1				0.5
3	3			1					1						1			0.5
2	4				1					1			1			1		0.5
3	5					1					1						1	0.5
1	6						1					1		1	1	1	1	0.5
	x^*								0.5		0.5		1		0.5		0.5	3

Strong Duality

If there exists an optimal solution x^* for (P), then there exists an optimal solution y^* for (D) and their objective values are equal.

Example

$$\mathbb{1}^T x^* = (y^*)^T \mathbb{1}$$

Column Generation

	$\tilde{c}$																	
s_i	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						
2	2		1					1					1	1				
3	3			1					1						1			
2	4				1					1			1			1		
3	5					1					1						1	
1	6						1					1		1	1	1	1	
	x^*																	

- Standard technique to solve LPs with huge number of columns
- Work only with a subset of $\mathcal{S}'$ from the set of all feasible columns $\mathcal{S}$
- Add only potentially improving columns using reduced cost criteria

$$\tilde{c}_s = c_s - \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i^* < 0$$

Column Generation

	$\tilde{c}$																	
s_i	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						
2	2		1					1					1	1				
3	3			1					1						1			
2	4				1					1			1			1		
3	5					1					1						1	
1	6						1					1		1	1	1	1	
	x^*																	

- ▷ Start with a subset $\mathcal{S}'$ of columns containing a feasible solution

Column Generation

s_i	$\tilde{c}$ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						1
2	2		1					1					1	1				1
3	3			1					1						1			1
2	4				1					1			1			1		1
3	5					1					1						1	1
1	6						1					1		1	1	1	1	1
	x^*	1	1	1	1	1	1											6

Restricted Dual (RD)

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{I}} y_i \\ \text{subject to} \quad & \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i \leq 1 & \forall s \in \mathcal{S}' \\ & y_i \geq 0 & \forall i \in \mathcal{I} \end{aligned}$$

- Use optimal (restricted) dual solution y^* to generate new columns

$$\exists s \in \mathcal{S} \setminus \mathcal{S}' : \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i^* > 1$$

Column Generation

	$\tilde{c}$							-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	
s_i	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						1
2	2		1					1					1	1				1
3	3			1					1						1			1
2	4				1					1			1			1		1
3	5					1					1						1	1
1	6						1					1		1	1	1	1	1
	x^*																	

Restricted Dual (RD)

$$\max \sum_{i \in \mathcal{I}} y_i$$

$$\text{subject to} \quad \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i \leq 1 \qquad \forall s \in \mathcal{S}'$$

$$y_i \geq 0 \qquad \forall i \in \mathcal{I}$$

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Column Generation

	$\tilde{c}$																	
s_i	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						0.5
2	2		1					1					1	1				1
3	3			1					1						1			0.5
2	4				1					1			1			1		1
3	5					1					1						1	0.5
1	6						1					1		1	1	1	1	0.5
	x^*		1		1				1								1	4

Restricted Dual (RD)

$$\max \sum_{i \in \mathcal{I}} y_i$$

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$$y_i \geq 0 \qquad \forall i \in \mathcal{I}$$

- Use optimal (restricted) dual solution y^* to generate new columns

$$\exists s \in \mathcal{S} \setminus \mathcal{S}' : \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i^* > 1$$

Column Generation

s_i	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	y^*
	$\tilde{c}$	0.5		0.5		0.5	0.5	-0.5		0.5	-0.5		-1	-0.5		-0.5		
1	1	1						1	1	1	1	1						0.5
2	2		1					1					1	1				1
3	3			1					1						1			0.5
2	4				1					1			1			1		1
3	5					1					1						1	0.5
1	6						1					1		1	1	1	1	0.5
	x^*																	

Restricted Dual (RD)

$$\max \sum_{i \in \mathcal{I}} y_i$$

$$\text{subject to} \quad \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i \leq 1 \qquad \forall s \in \mathcal{S}'$$

$$y_i \geq 0 \qquad \forall i \in \mathcal{I}$$

- Use optimal (restricted) dual solution y^* to generate new columns

$$\exists s \in \mathcal{S} \setminus \mathcal{S}' : \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i^* > 1$$

Column Generation

s_i	$\tilde{c}$ / i	0.5 1	0.5 2	0.5 3	0.5 4	0.5 5	0.5 6	7	8	9	10	11	12	13	14	15	16	y^*
1	1	1						1	1	1	1	1						0.5
2	2		1					1					1	1				0.5
3	3			1					1						1			0.5
2	4				1					1			1			1		0.5
3	5					1					1						1	0.5
1	6						1					1		1	1	1	1	0.5
	x^*							1					1				1	3

Restricted Dual (RD)

$$\max \sum_{i \in \mathcal{I}} y_i$$

$$\text{subject to} \quad \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i \leq 1 \qquad \forall s \in \mathcal{S}'$$

$$y_i \geq 0 \qquad \forall i \in \mathcal{I}$$

- ▷ Use optimal (restricted) dual solution y^* to prove optimality

$$\forall s \in \mathcal{S} \setminus \mathcal{S}' : \sum_{i \in \mathcal{I}} (\lambda_s)_i \, y_i^* \leq 1$$

Exercise 2

1. Formulate the pricing problem for the binpacking problem
2. Use SCIP to solve the binpacking problem using column generation
3. Run automated tests to compare with the previous results

Pricing Problem

Restricted Dual (RD)

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{I}} y_i & \\ \text{subject to} \quad & \sum_{i\in\mathcal{I}} (\lambda_s)_i \, y_i \le 1 & \forall s \in \mathcal{S}' \\ & y_i \ge 0 & \forall i \in \mathcal{I} \end{aligned}$$

$$\begin{aligned} \exists s \in \mathcal{S} : \sum_{i\in\mathcal{I}} (\lambda_s)_i \, y_i^* > 1 \Leftrightarrow \min \quad & 1 - \sum_{i\in\mathcal{I}} (\lambda_s)_i y_i^* & \\ \text{subject to} \quad & \sum_{i\in\mathcal{I}} (\lambda_s)_i s_i \le \kappa & \\ & (\lambda_s)_i \in \{0,1\} & \forall i \in \mathcal{I} \end{aligned}$$