Open Pit Mine Production Scheduling
Some IP modelling and solution techniques

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09/28/2009
Open Pit Mining Production Scheduling
   Problem description
   Ultimate pit limits
   Time-indexed MIP formulations

OPMPSP relaxations
   Standard LP relaxation
   Full relaxation of resource constraints
   Lagrangean relaxation of resource constraints

Precedence Constrained Knapsack Substructures
   Precedence Constrained Knapsack Problem
   PCKs in open pit mining
   Valid inequalities for the Precedence Constrained Knapsack Polytope
Outline

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Open Pit Mining Production Scheduling

**Input**

- block model with precedences
- deterministic block content
- ore prices and production costs
- equipment capacities

Problem

Find an order of excavation with

- max. net present value
satisfying

- precedence constraints,
- resource constraints.

Simultaneously: Determine

- opt. processing decisions for each block.
Open Pit Mining Production Scheduling

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- possibly aggregation of blocks

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Getting started: computing ultimate pit limits

Given: Profit/negative cost $w_i$ of mining each block $i \in \mathcal{N} = \{1, \ldots, N\}$.

Seek: Precedence-feasible set of blocks $X \subseteq \mathcal{N}$ with max. $\sum_{i \in X} w_i$.

(If $i \in X$ than all predecessors of $i$ are in $X$.)
Getting started: computing ultimate pit limits

Given: Profit/negative cost \( w_i \) of mining each block \( i \in \mathcal{N} = \{1, \ldots, N\} \).

Seek: Precedence-feasible set of blocks \( X \subseteq \mathcal{N} \) with max. \( \sum_{i \in X} w_i \).

(If \( i \in X \) than all predecessors of \( i \) are in \( X \).)

Observation: This is a max-weight closure problem in the precedence graph

\( (\mathcal{N}, S = \{(i, p) \in \mathcal{N} \times \mathcal{N} \mid p \text{ predecessor of } i\}) \)

with node weights \( w_i \).
Getting started: computing ultimate pit limits

Given: Profit/negative cost $w_i$ of mining each block $i \in \mathcal{N} = \{1, \ldots, N\}$.

Seek: Precedence-feasible set of blocks $X \subseteq \mathcal{N}$ with max. $\sum_{i \in X} w_i$.

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$$(\mathcal{N}, \mathcal{S} = \{(i, p) \in \mathcal{N} \times \mathcal{N} \mid p \text{ predecessor of } i\})$$

with node weights $w_i$.

▷ Solve e.g. by min-cut algorithm in a slightly extended graph:
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A new MIP formulation (Boland-Dumitrescu-Froyland-G’08)

Variables

\[ y_{k,t} = \text{fraction of aggregate } k \text{ mined during period } t \]

\[ z_{i,t} = \text{fraction of block } i \text{ processed during period } t \]
A new MIP formulation (Boland-Dumitrescu-Froyland-G’08)

Variables

\[ x_{k,t} = \begin{cases} 
1 & \text{if aggregate } k \text{ may be mined during periods } t, \ldots, T \\
0 & \text{otherwise} 
\end{cases} \]

\[ y_{k,t} = \text{fraction of aggregate } k \text{ mined during period } t \]

\[ z_{i,t} = \text{fraction of block } i \text{ processed during period } t \]
A new MIP formulation (Boland-Dumitrescu-Froyland-G’08)

Objective function

\[
NetPresentValue(y, z) = \sum_{t=1}^{T} \left( \frac{1}{1 + q} \right)^{t-1} \left( \sum_{aggs} -m_k y_{k,t} + \sum_{blocks} p_i z_{i,t} \right)
\]
A new MIP formulation (Boland-Dumitrescu-Froyland-G’08)

\[ \text{max } \text{NetPresentValue} (y, z) \]

s.t. \((x, y)\) precedence feasible

\[
x_{k,t} \in \{0, 1\} \quad \text{for } k = 1, \ldots, K, \ t = 1, \ldots, T
\]
\[
0 \leq y_{k,t} \leq 1 \quad \text{for } k = 1, \ldots, K, \ t = 1, \ldots, T
\]
\[
0 \leq z_{i,t} \leq y_{k,t} \quad \text{for } k = 1, \ldots, K, \ i \in B_k, \ t = 1, \ldots, T
\]
A new MIP formulation (Boland-Dumitrescu-Froyland-G’08)

max \ NetPresentValue (y, z)

s.t. \ (x, y) \ precedence \ feasible

\[
\sum_{k=1}^{K} \bar{R}_k y_{k,t} \leq M_t \quad \text{for} \ t = 1, \ldots, T
\]

\[
\sum_{i=1}^{N} R_i z_{i,t} \leq P_t \quad \text{for} \ t = 1, \ldots, T
\]

\[
x_{k,t} \in \{0, 1\} \quad \text{for} \ k = 1, \ldots, K, \ t = 1, \ldots, T
\]

\[
0 \leq y_{k,t} \leq 1 \quad \text{for} \ k = 1, \ldots, K, \ t = 1, \ldots, T
\]

\[
0 \leq z_{i,t} \leq y_{k,t} \quad \text{for} \ k = 1, \ldots, K, \ i \in B_k, \ t = 1, \ldots, T
\]

▶ integrated optimisation of processing decisions
For this talk: no aggregates, no fractional mining

\[
\begin{align*}
\text{max } & \quad \text{NetPresentValue} (x, z) \\
\text{s.t. } & \quad x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, t = 2, \ldots, T \\
& \quad x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, t = 1, \ldots, T \\
& \quad x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T \\
& \quad x_{i,0} = 0 \quad \text{for } i \in \mathcal{N} \\
& \quad 0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T
\end{align*}
\]
For this talk: no aggregates, no fractional mining

\[
\begin{align*}
\text{max} & \quad \text{NetPresentValue} (x, z) \\
\text{s.t.} & \quad x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T \\
& \quad x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, \ t = 1, \ldots, T \\
& \quad \sum_i R_i (x_{i,t} - x_{i,t-1}) \leq M_t \quad \text{for } t = 1, \ldots, T \\
& \quad \sum_i R_i z_{i,t} \leq P_t \quad \text{for } t = 1, \ldots, T \\
& \quad x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T \\
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\end{align*}
\]
For this talk: no aggregates, no fractional mining

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\begin{align*}
\text{max} \quad & \text{NetPresentValue} (x, z) \\
\text{s.t.} \quad & x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T \\
& x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, \ t = 1, \ldots, T \\
& \sum_i R_i(x_{i,t} - x_{i,t-1}) \leq M_t \quad \text{for } t = 1, \ldots, T \\
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\end{align*}
\]

\(\blacktriangleright\) NP-hard by reduction from Precedence-Constrained Knapsack
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Standard LP relaxation

\[
\text{max } \text{NetPresentValue}(x, z)
\]

s.t.
\[
\begin{align*}
& x_{i,t-1} \leq x_{i,t} \\
& x_{i,t} \leq x_{p,t} \\
\sum_{i} R_i(x_{i,t} - x_{i,t-1}) & \leq M_t & \text{for } i \in \mathcal{N}, t = 2, \ldots, T \\
\sum_{i} R_i z_{i,t} & \leq P_t & \text{for } t = 1, \ldots, T
\end{align*}
\]

\[
\begin{align*}
& 0 \leq x_{i,t} \leq 1 & \text{for } i \in \mathcal{N}, t = 1, \ldots, T \\
& x_{i,0} = 0 & \text{for } i \in \mathcal{N} \\
& 0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} & \text{for } i \in \mathcal{N}, t = 1, \ldots, T
\end{align*}
\]

\(\triangleleft\) typically: interior point outperforms simplex (when solving from scratch)
Standard LP relaxation

\[
\text{max } \text{NetPresentValue}(x, z)
\]

s.t.

\[
x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T
\]

\[
x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, t = 1, \ldots, T
\]

\[
\sum_i R_i(x_{i,t} - x_{i,t-1}) \leq M_t \quad \text{for } t = 1, \ldots, T
\]

\[
\sum_i R_i z_{i,t} \leq P_t \quad \text{for } t = 1, \ldots, T
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\[
0 \leq x_{i,t} \leq 1 \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T
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x_{i,0} = 0 \quad \text{for } i \in \mathcal{N}
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\[
0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T
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Typically: interior point outperforms simplex (when solving from scratch)
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Full relaxation of resource constraints

$$\text{max} \quad \text{NetPresentValue}(x, z)$$

s.t.

$$x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T$$

$$x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, \ t = 1, \ldots, T$$

$$x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T$$

$$x_{i,0} = 0 \quad \text{for } i \in \mathcal{N}$$

$$0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T$$
Full relaxation of resource constraints

\[
\begin{align*}
\text{max} \quad & \text{NetPresentValue}(x) \\
\text{s.t.} & \quad x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T \\
& \quad x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, t = 1, \ldots, T \\
& \quad x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T \\
& \quad x_{i,0} = 0 \quad \text{for } i \in \mathcal{N}
\end{align*}
\]

\(\triangleright\) substitute \(z\) according to objective function coefficients
Full relaxation of resource constraints

\[
\text{max} \quad \text{PresentValue}(x)
\]

s.t.

\[
x_i \leq x_p \quad \text{for } (i, p) \in S
\]

\[
x_i \in \{0, 1\} \quad \text{for } i \in N
\]

▷ substitute \( z \) according to objective function coefficients
▷ costs and profits decrease over time \( \Rightarrow \) all blocks are mined in period 1 (if at all) \( \Leftrightarrow \) equivalent to computing the **Ultimate Pit Limit**
Full relaxation of resource constraints

\[
\text{max} \quad \text{PresentValue}(x)
\]

s.t.

\[
\begin{align*}
  x_i & \leq x_p & \text{for } (i, p) \in S \\
  x_i & \in \{0, 1\} & \text{for } i \in \mathcal{N}
\end{align*}
\]

▷ substitute \( z \) according to objective function coefficients
▷ costs and profits decrease over time \( \Rightarrow \) all blocks are mined in period 1 (if at all) \( \Leftrightarrow \) equivalent to computing the Ultimate Pit Limit
▷ solve efficiently as max-weight closure problem
▷ use as preprocessing: all blocks in an optimal OPMPSP schedule must be contained in the Ultimate Pit – remove the others from the model
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Lagrangean relaxation of resource constraints

\[
\begin{align*}
\text{max} \quad \text{NetPresentValue}(x, z) \\
\text{s.t.} \quad x_{i,t-1} &\leq x_{i,t} \quad \text{for } i \in \mathcal{N}, t = 2, \ldots, T \\
&\quad x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, t = 1, \ldots, T \\
&\quad \sum_i R_i(x_{i,t} - x_{i,t-1}) \leq M_t \quad \text{for } t = 1, \ldots, T \\
&\quad \sum_i R_i z_{i,t} \leq P_t \quad \text{for } t = 1, \ldots, T \\
&\quad x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T \\
&\quad x_{i,0} = 0 \quad \text{for } i \in \mathcal{N} \\
&\quad 0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T
\end{align*}
\]
Lagrangean relaxation of resource constraints

\[
\begin{align*}
\text{max} \quad & \text{NetPresentValue}(x, z) \\
& + \sum_t \mu_t \left[ M_t - \sum_i R_i (x_{i,t} - x_{i,t-1}) \right] + \sum_t \pi_t \left[ P_t - \sum_i R_i z_{i,t} \right] \\
\text{s.t.} \quad & x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T \\
& x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, \ t = 1, \ldots, T \\
& x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T \\
& x_{i,0} = 0 \quad \text{for } i \in \mathcal{N} \\
& 0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T
\end{align*}
\]
Lagrangean relaxation of resource constraints

$$\text{max } \text{NetPresentValue} (x, z)$$

$$+ \sum_t \mu_t [M_t - \sum_i R_i (x_{i,t} - x_{i,t-1})] + \sum_t \pi_t [P_t - \sum_i R_i z_{i,t}]$$

s.t.

$$x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, t = 2, \ldots, T$$

$$x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in S, t = 1, \ldots, T$$

$$x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T$$

$$x_{i,0} = 0 \quad \text{for } i \in \mathcal{N}$$

$$0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T$$

- feas. region is integral $\Rightarrow$ opt. multipliers as in LP, best dual bound equal to LP bound (Geoffrion 1974)
Lagrangean relaxation of resource constraints

\[
\max \quad \text{NetPresentValue}(x, z) \\
+ \sum_t \mu_t \left[ M_t - \sum_i R_i(x_{i,t} - x_{i,t-1}) \right] + \sum_t \pi_t \left[ P_t - \sum_i R_i z_{i,t} \right]
\]

s.t.
\[
\begin{align*}
x_{i,t-1} & \leq x_{i,t} & \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T \\
x_{i,t} & \leq x_{p,t} & \text{for } (i, p) \in \mathcal{S}, t = 1, \ldots, T \\
x_{i,t} & \in \{0, 1\} & \text{for } i \in \mathcal{N}, t = 1, \ldots, T \\
x_{i,0} & = 0 & \text{for } i \in \mathcal{N} \\
0 & \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} & \text{for } i \in \mathcal{N}, t = 1, \ldots, T
\end{align*}
\]

▷ feas. region is integral ⇒ opt. multipliers as in LP, best dual bound equal to LP bound (Geoffrion 1974)
▷ $z$-variables can be substituted according to obj. coefficient
▷ transformation to unconstrained project scheduling
Lagrangean relaxation of resource constraints

\[
\begin{align*}
\text{max} & \quad \text{NetPresentValue}(x, z) \\
& + \sum_t \mu_t [M_t - \sum_i R_i(x_i,t - x_i,t-1)] + \sum_t \pi_t [P_t - \sum_i R_iz_i,t] \\
\text{s.t.} & \quad x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, t = 2, \ldots, T \\
& \quad x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, t = 1, \ldots, T \\
& \quad x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T \\
& \quad x_{i,0} = 0 \quad \text{for } i \in \mathcal{N} \\
& \quad 0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, t = 1, \ldots, T
\end{align*}
\]

▷ feas. region is integral ⇒ opt. multipliers as in LP, best dual bound equal to LP bound (Geoffrion 1974)
▷ \(z\)-variables can be substituted according to obj. coefficient
▷ transformation to unconstrained project scheduling
▷ efficient solution by min-cut computations (Möhring et al. 2003)
Lagrangean dual (with Helmberg’s ConicBundle 0.2d) vs. LP-relaxation (with CPLEX 11.0 Barrier, no crossover)

| Problem instance | LP-relaxation | | Lagrangean dual |
|------------------|---------------|-----------|
| ma-115-8513      | 11.2          | 7.230097   | 1.8      | 7.230097   | 0.158787 | 1.000000 |
| ma-296-8513      | 27.2          | 7.276624   | 2.6      | 7.276631   | 0.094993 | 1.000001 |
| ma-1038-8513     | 259.5         | 7.306851   | 9.5      | 7.306854   | 0.036415 | 1.000001 |
| ma-115-8513      | 11.2          | 7.230097   | 1.8      | 7.230097   | 0.158787 | 1.000000 |
| ma-296-8513      | 27.2          | 7.276624   | 2.6      | 7.276631   | 0.094993 | 1.000001 |
| ma-1038-8513     | 259.5         | 7.306851   | 9.5      | 7.306854   | 0.036415 | 1.000001 |


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Precedence Constrained Knapsack Problem

Given:

- Items $i \in \mathcal{N}$ with sizes $a_i$ and values $c_i$
- Knapsack capacity $b$
- Precedence order $S \subseteq \mathcal{N} \times \mathcal{N}$ (transitively reduced)

Feasible Solution:

- set $X \subseteq \mathcal{N}$ with $\sum_{i \in X} a_i \leq b$ and
- $X$ is closed under $S$ (If $i \in X$ and $(i, j) \in S$, then $j \in X$.)

Goal: maximize $\sum_{i \in X} c_i$

Important substructure in many applications:

- Resource constrained scheduling
- Multi-level facility location, access & distribution network design
- Routing problems: confluent flows, Internet routing, . . .
- . . .
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PCKs in open pit mining

\[
\begin{align*}
\text{max} & \quad \text{NetPresentValue}(x, z) \\
\text{s.t.} & \quad x_{i,t-1} \leq x_{i,t} \quad \text{for } i \in \mathcal{N}, \ t = 2, \ldots, T \\
& \quad x_{i,t} \leq x_{p,t} \quad \text{for } (i, p) \in \mathcal{S}, \ t = 1, \ldots, T \\
& \quad \sum_i R_i (x_{i,t} - x_{i,t-1}) \leq M_t \quad \text{for } t = 1, \ldots, T \quad \leftarrow \text{add up} \\
& \quad \sum_i R_i z_{i,t} \leq P_t \quad \text{for } t = 1, \ldots, T \\
& \quad x_{i,t} \in \{0, 1\} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T \\
& \quad x_{i,0} = 0 \quad \text{for } i \in \mathcal{N} \\
& \quad 0 \leq z_{i,t} \leq x_{i,t} - x_{i,t-1} \quad \text{for } i \in \mathcal{N}, \ t = 1, \ldots, T
\end{align*}
\]
For each time period, the OPMPSP formulation contains one precedence constrained knapsack.
Open Pit Mining Production Scheduling

- Problem description
- Ultimate pit limits
- Time-indexed MIP formulations

OPMPSP relaxations

- Standard LP relaxation
- Full relaxation of resource constraints
- Lagrangean relaxation of resource constraints

Precedence Constrained Knapsack Substructures

- Precedence Constrained Knapsack Problem
- PCKs in open pit mining

Valid inequalities for the Precedence Constrained Knapsack Polytope
Precedence Constrained Knapsack Polytope

\[ PCKP := \text{conv}\{ x \in \{0, 1\}^N \mid \sum_{i \in N} a_i x_i \leq b, \quad (i) \}
\]
\[ x_i \leq x_j \quad \text{for} \quad (i, j) \in S \quad (ii) \]

Polyhedral structure of \( PCKP \) well studied:

- Dimension, Conditions for (i) and (ii) to be facet-defining
- Knapsack-based inequalities \([\text{Boyd’93, Park-Park’97}]\)
  - Induced cover inequalities
  - (Induced) k-cover inequalities
  - (Induced) (1,k)-configuration inequalities
- Sequential lifting results

But:

- Nothing implemented in MIP solvers
Variable fixing

Notation

- $\text{Pred}(i) := \{ j : j \text{ predecessor of } i \} \cup \{ i \}$
  (Items that must be included with $i$.)
- $A(i) := \sum_{j \in \text{Pred}(i)} a_j$
  (Total size induced by $i$.)
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Observation: If $A(i) > b$, then $x_i = 0$ for all $x \in PCKP$. 
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Variable elimination scheme

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Remarks

- Can compute all $A(i)$ in $O(S)$.
- Fixing is very efficient in practice (reduces LP gaps substantially).
- CPLEX does not find these fixings even with most aggressive probing.
Pairwise conflicts

Apply same idea to pairs of items:

\[ A(i, j) := \sum_{k \in \text{Pred}(i) \cup \text{Pred}(j)} a_k \]

(Total size of \(i, j\) and predecessors.)
Pairwise conflicts

Apply same idea to pairs of items:

\[ A(i, j) := \sum_{k \in \text{Pred}(i) \cup \text{Pred}(j)} a_k \]

(Total size of \( i \), \( j \) and predecessors.)

Observation: If \( A(i, j) > b \), then \( x_i + x_j \leq 1 \) for all \( x \in PCKP \).
Apply same idea to pairs of items:

\[ A(i, j) := \sum_{k \in \text{Pred}(i) \cup \text{Pred}(j)} a_k \]

(Total size of \(i, j\) and predecessors.)

Observation: If \(A(i, j) > b\), then \(x_i + x_j \leq 1\) for all \(x \in PCKP\).

- \(\{i, j\}\) with \(A(i, j) > b\): Induced cover of size 2
- Induced cover: \(X \subset N\) with \(A(X) > b\)
- [Boyd’93, Park-Park’97] Conditions when induced cover inequalities are facet-defining (for low-dimensional faces of PCKP)
Clique inequalities

Idea

- Consider conflict graph $CG = (\mathcal{N}, E)$ with
  
  
  $E := \{ij \mid A(i, j) > b\}$

Precedence order

Conflict graph $CG$
Clique inequalities

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Precedence order

Conflict graph $CG$
Idea

Consider conflict graph $CG = (\mathcal{N}, E)$ with

$$E := \{ij \mid A(i, j) > b\}$$

Observation: Any valid inequality for $\text{STAB}(CG)$ is valid for $\text{PCKP}$.

Corollary: Let $C$ be a clique in $CG$. Then (1) is valid for $\text{PCKP}$.

$$\sum_{i \in C} x_i \leq 1$$
Clique inequalities

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  \[ E := \{ij \mid A(i, j) > b\} \]

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Conflict graph $CG$
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Precedence order

Conflict graph $CG$

Observation: Any valid inequality for $STAB(CG)$ is valid for $PCKP$.

Corollary: Let $C$ be a clique in $CG$. Then (1) is valid for $PCKP$.

\[ \sum_{i \in C} x_i \leq 1 \quad (1) \]
Observation: Let $C$ be a clique in $CG$ with $P(C) = \bigcap_{i \in C} \text{Pred}(i) \neq \emptyset$. Then, for any $i \in P(C)$, $\sum_{j \in C} x_j \leq x_i (2)$ is valid for PCKP.
Observation: Let $C$ be a clique in $CG$ with $P(C) = \bigcap_{i \in C} \text{Pred}(i) \neq \emptyset$. Then, for any $i \in P(C)$,

$$\sum_{j \in C} x_j \leq x_i$$

is valid for $PCKP$. 
Observation: Let $C$ be a clique in $CG$ with $P(C) = \bigcap_{i \in C} \text{Pred}(i) \neq \emptyset$. Then, for any $i \in P(C)$,

$$\sum_{j \in C} x_j \leq x_i$$

is valid for $PCKP$. 

\[(2)\]
Observation: Let \( C \) be a clique in \( CG \) with 
\[ P(C) = \bigcap_{i \in C} \text{Pred}(i) \neq \emptyset. \] Then, for any \( i \in P(C) \), 
\[ \sum_{j \in C} x_j \leq x_i \quad (2) \]
is valid for \( PCKP \).
Observation: Let $C$ be a clique in $CG$ with $P(C) = \bigcap_{i \in C} \text{Pred}(i) \neq \emptyset$. Then, for any $i \in P(C)$,

$$\sum_{j \in C} x_j + (1 - x_i) \leq 1$$

is valid for $PCKP$. 
**Observation:** Let $C$ be a clique in $CG$ with $P(C') = \bigcap_{i \in C} \text{Pred}(i) \neq \emptyset$. Then, for any $i \in P(C)$,

$$\sum_{j \in C} x_j + x'_i \leq 1$$

is valid for $PCKP$. 

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**Precedence order**

**Extended conflict graph** $CG'$
Separation of clique-based inequalities

Given: \( x \in [0, 1]^N \) fractional

Seek:

(1) clique \( C \) in \( CG \) such that \( \sum_{j \in C} x_j > 1 \), or

(2) clique \( C \) in \( CG \) and \( i \in P(C) \) such that \( \sum_{j \in C} x_j > x_i \)
Separation of clique-based inequalities

Given: $x \in [0, 1]^N$ fractional

Seek:

(1) clique $C$ in $CG$ such that $\sum_{j \in C} x_j > 1$, or

(2) clique $C$ in $CG$ and $i \in P(C)$ such that $\sum_{j \in C} x_j > x_i$

 seperation of (1) and (2) is equivalent to finding a maximum-weight clique in $CG'$ with node weights $w_k := x_k$ and $w_{k'} := 1 - x_k$ for all $k \in N$

$w(C^*) \leq 1$: all inequalities (1) and (2) satisfied

$w(C^*) > 1$ and $C^* \subseteq N$: inequality (1) violated

$w(C^*) > 1$ and $C^* = C \cup \{i'\}$: inequality (2) violated
Separation of clique-based inequalities

Given: \( x \in [0, 1]^N \) fractional
Seek:

(1) clique \( C \) in \( CG \) such that \( \sum_{j \in C} x_j > 1 \), or

(2) clique \( C \) in \( CG \) and \( i \in P(C) \) such that \( \sum_{j \in C} x_j > x_i \)

- Separation of (1) and (2) is equivalent to finding a maximum-weight clique in \( CG' \) with node weights \( w_k := x_k \) and \( w_k' := 1 - x_k \) for all \( k \in N \)
  - \( w(C^*) \leq 1 \): all inequalities (1) and (2) satisfied
  - \( w(C^*) > 1 \) and \( C^* \subseteq N \): inequality (1) violated
  - \( w(C^*) > 1 \) and \( C^* = C \cup \{i'\} \): inequality (2) violated

- NP-hard, but: Efficient MWC implementations exist in any state-of-the-art MIP solver.
- Ideally: extend the conflict graph of your MIP solver.

More details on facets and computational results: see e.g. Boland-Froyland-Fricke-Sotirov’06, Bley-Boland-Fricke-Froyland’09.
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Some IP modelling and solution techniques

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09/28/2009