Basics of polyhedral theory, flows and networks

CO@W Berlin

Martin Grötschel

21.09.2009

14:00 – 15:30

Martin Grötschel

- Institut für Mathematik, Technische Universität Berlin (TUB)
- DFG-Forschungszentrum "Mathematik für Schlüsseltechnologien" (MATHEON)
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Linear Programming

$$\max\ c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$subject\ \ to$$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$.$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\max\ c^T x$$

$$Ax = b$$

$$x \geq 0$$

linear program
in standard form

Linear Programming

$$\max c^T x \quad Ax = b \quad x \geq 0$$

linear program in standard form

$$\max c^T x \quad Ax \leq b \quad -Ax \leq -b \quad -x \leq 0$$

$$\max c^T x \quad Ax \leq b$$

linear program in "polyhedral form"

$$\max c^T x^+ - c^T x^- \quad Ax^+ + Ax^- + Is = b \quad x^+, x^-, s \geq 0 \quad (x = x^+ - x^-)$$

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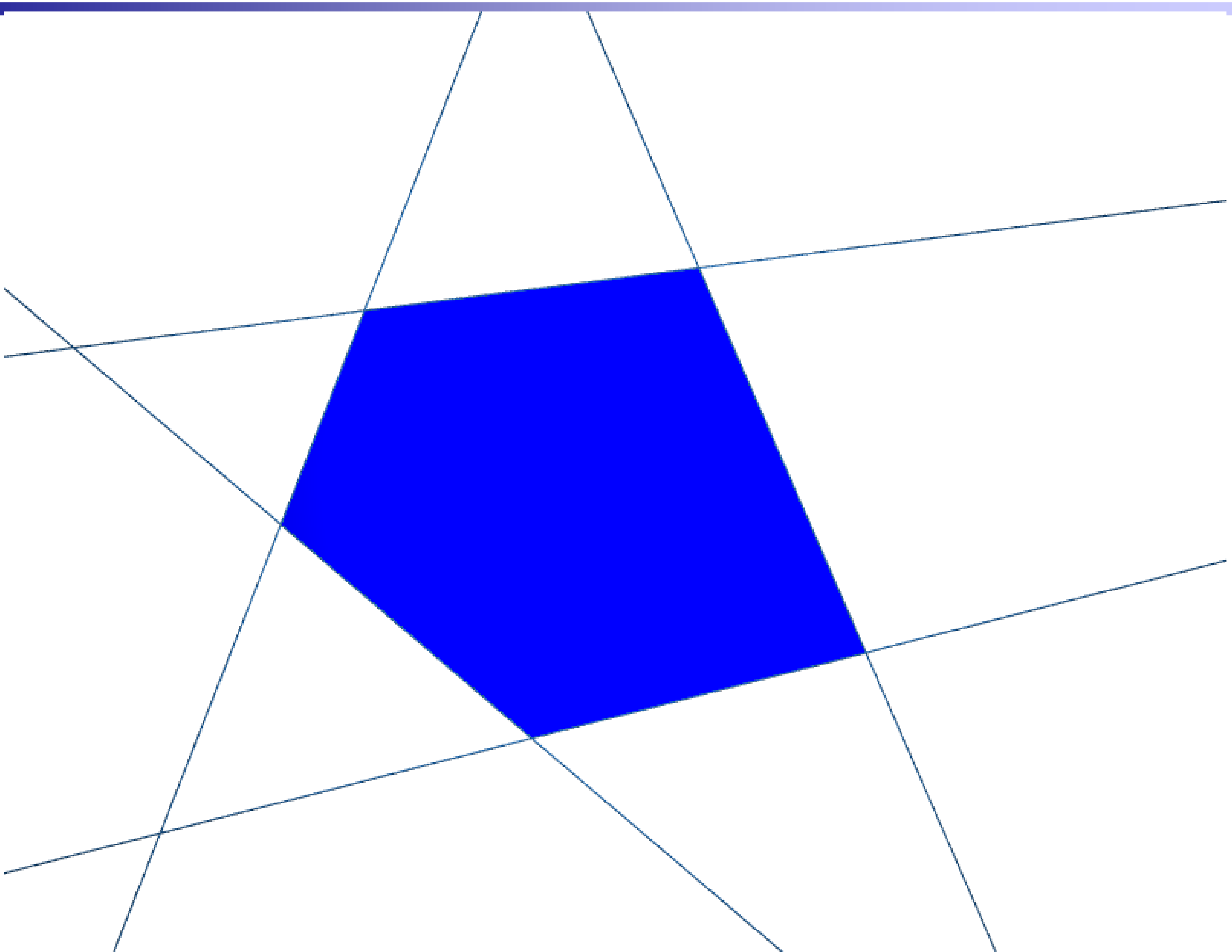
A Polytope in the Plane



A Polytope in 3-dimensional space

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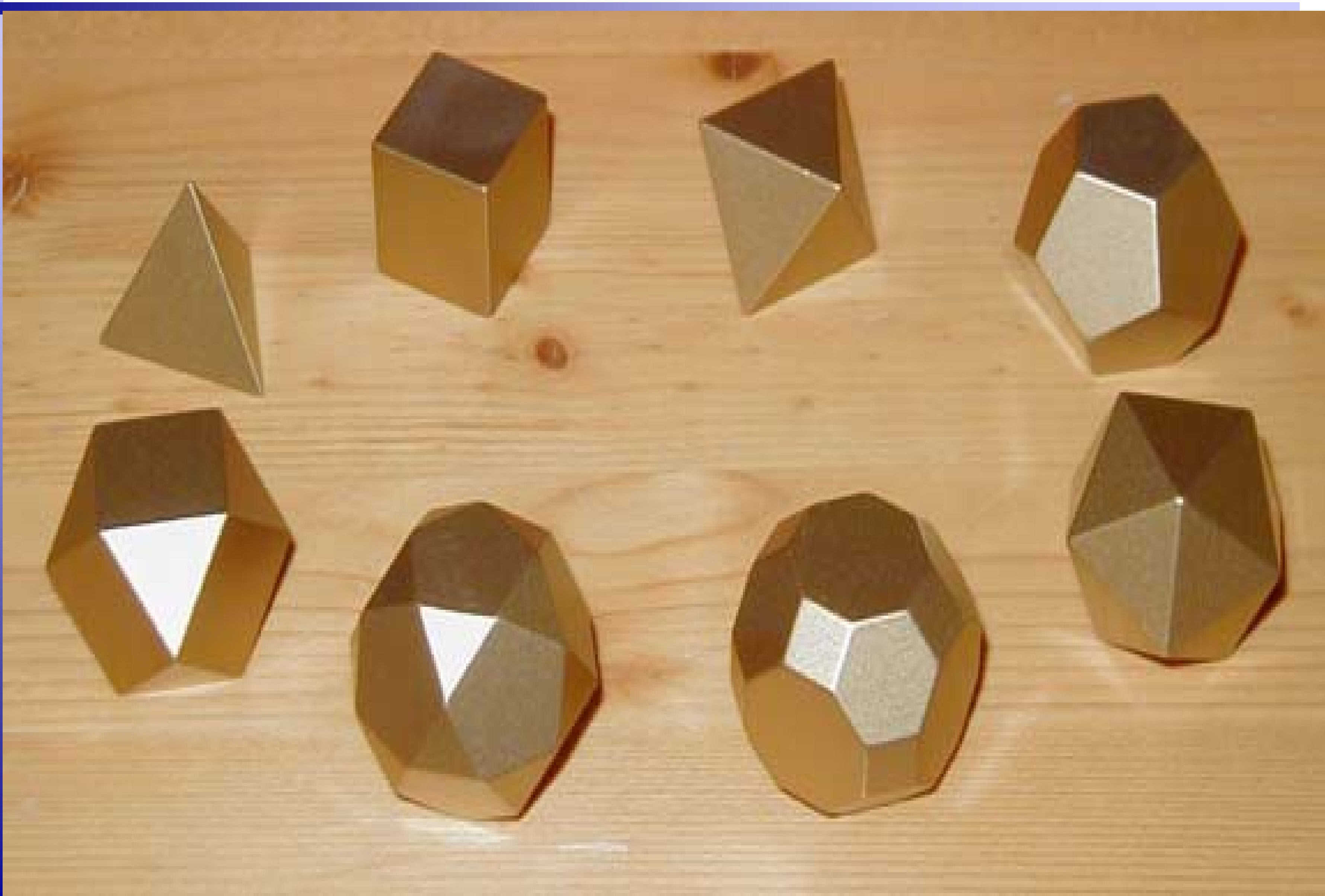
„beautiful“ polyehedra



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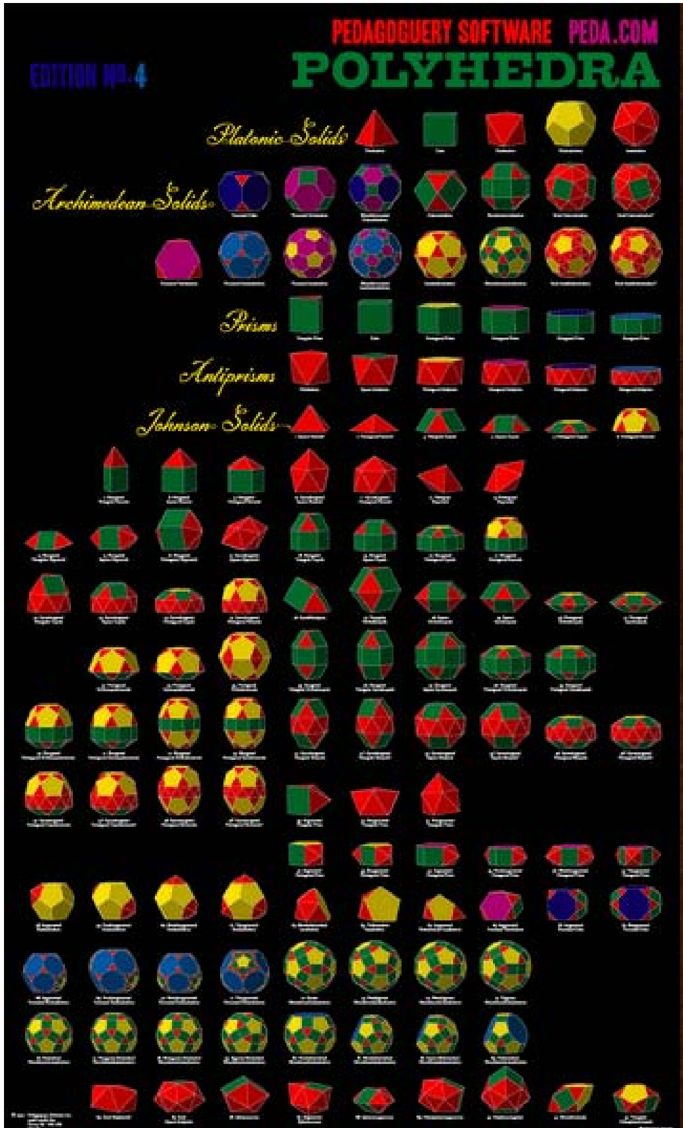
Polytopes in nature

- see examples
- diamond

Polyhedra-Poster

http://www.peda.com/posters/Welcome.html



We currently offer one poster for secure online purchasing: our *Polyhedra* poster, which displays all convex polyhedra with regular polygonal faces (a finite sampling of prisms and anti-prisms are included).

It measures 22" x 37" and is printed on glosssy paper. A protective coating was applied during printing.

The poster is shown on the left; to see a close-up of a portion of the poster, move your mouse over the image.

This is the fourth edition of the poster. Other versions of the poster are shown in our Posters Archive.

$14 FOR 1 POSTER
$28 FOR 4 POSTERS
FREE SHIPPING

Poster which displays all convex polyhedra with regular polygonal faces

http://www.eg-models.de/

EG-Models

EG-Models - a new archive of electronic geometry models
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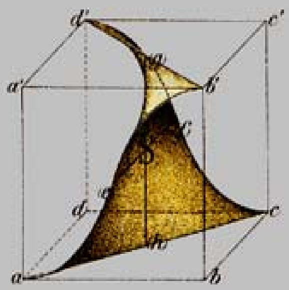


H.A. Schwarz Ges.Math.Abh
Springer Berlin 1890

Note: Some browser versions do not display Java applets. Please, press the 'No Applet' button in the navigation bar to avoid using Java.

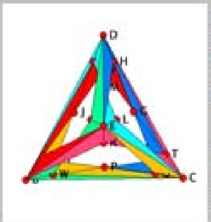
Anschauliche Geometrie - A tribute to Hilbert, Cohn-Vossen, Klein and all other geometers.

Electronic Geometry Models

This archive is open for any geometer to publish new geometric models, or to browse this site for material to be used in education and research. These geometry models cover a broad range of mathematical topics from geometry, topology, and to some extent from numerics.

Click "Models" to see the full list of published models. See here for details on the submission and review process.

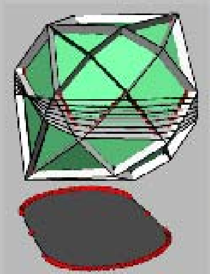
Selection of recently published models



Model 2008.11.001 by Frank H. Lutz and Günter M. Ziegler *A Small Polyhedral Z-Acyclic 2-Complex in R4.*
Section: *Polytopal Complexes*

We present a 4-dimensional polyhedral realization of a 2-dimensional Z-acyclic but non-contractible simplicial complex with 23 vertices.

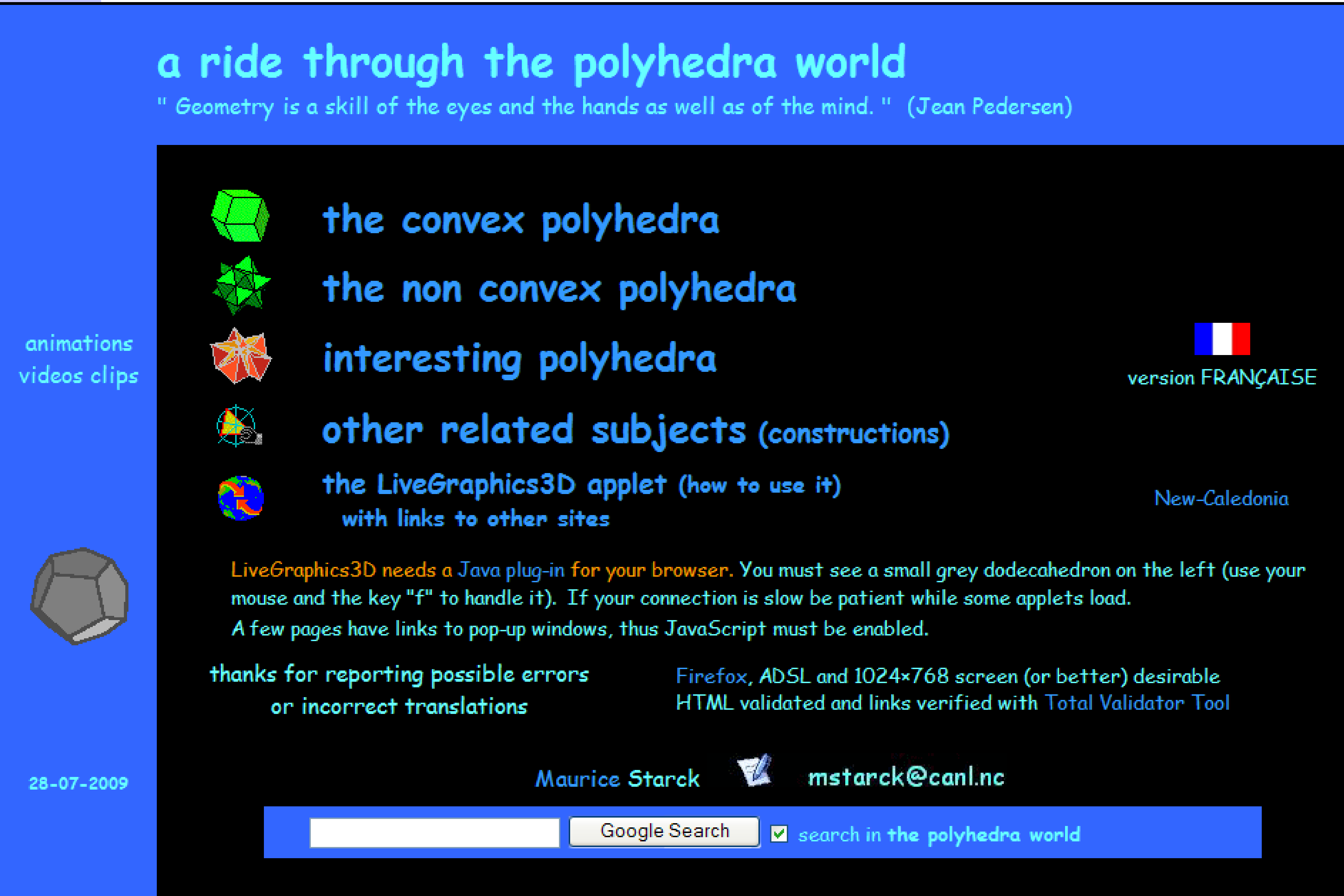
Our example answers a query by Lutz Hille (Hamburg), who in November 2006 had asked us for examples of Z-acyclic but non-contractible complexes realized in low dimensions. His question was motivated by toric geometry.



Model 2008.10.002 by Thilo Rörig, Nikolaus Witte, and Günter M. Ziegler *Zonotopes With Large 2D-Cuts.*
Section: *Polytopes*

For fixed $d \geq 2$ there are d-dimensional zonotopes with n zones for which a 2-dimensional central section has $\Omega(n^{d-1})$ vertices. The result is asymptotically optimal for all fixed $d \geq 2$.

http://www.ac-noumea.nc/maths/amc/polyhedr/index_.htm



a ride through the polyhedra world

" Geometry is a skill of the eyes and the hands as well as of the mind. " (Jean Pedersen)

animations
videos clips

the convex polyhedra

the non convex polyhedra

interesting polyhedra

other related subjects (constructions)

the LiveGraphics3D applet (how to use it)
with links to other sites

version FRANÇAISE

New-Caledonia

LiveGraphics3D needs a Java plug-in for your browser. You must see a small grey dodecahedron on the left (use your mouse and the key "f" to handle it). If your connection is slow be patient while some applets load.
A few pages have links to pop-up windows, thus JavaScript must be enabled.

thanks for reporting possible errors
or incorrect translations

Firefox, ADSL and 1024×768 screen (or better) desirable
HTML validated and links verified with Total Validator Tool

28-07-2009

Maurice Starck mstarck@canl.nc

Google Search search in the polyhedra world

Plato's five regular polyhedra

http://www.ac-noumea.nc/maths/amc/polyhedr/convex1_.htm

CO@W

convex polyhedra 1 - Microsoft Internet Explorer

Datei Bearbeiten Ansicht Favoriten Extras ?

Zurück Suchen Favoriten Medien

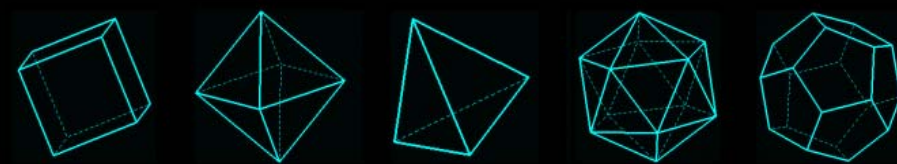
Adresse http://www.ac-noumea.nc/maths/amc/polyhedr/convex1_.htm Wechseln zu Links

Plato's five regular polyhedra

The regular polyhedra are, in the space, the analogues of the regular polygons in the plane ; their faces are regular and identical polygons, and their vertices, regular and identical, are regularly distributed on a sphere. Their analogues in dimension four are the regular polytopes.
As we do for the polygons, we recognize a convex polyhedron by the very fact that all its diagonals (segments which join two vertices not joined by an edge) are inside the polyhedron.
Whereas there exist an infinity of regular convex polygons, the regular convex polyhedra are only five.

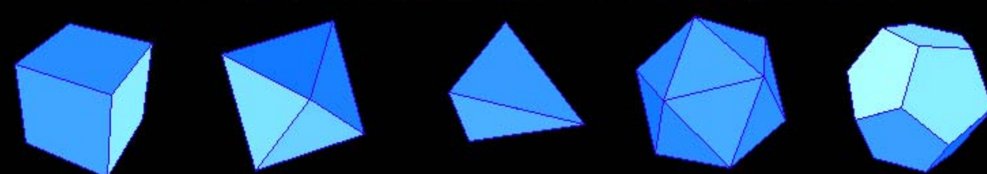
The angle of a regular polygon with n sides is 180°(n-2)/n : 60° (triangle), 90° (square), 108° (pentagon), 120° (hexagon)...

proof : On a vertex of a regular polyhedron the sum of the face's angles (there are at least three) must be smaller than 360°.
Since 6x60° = 4x90° = 3x120° = 360° < 4x108°, there are only five possibilities: 3, 4, or 5 triangles, 3 squares or 3 pentagons.



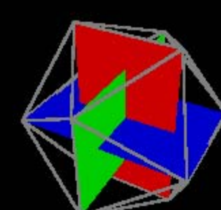
name	cube	octahedron	tetrahedron	icosahedron	dodecahedron
faces	6 squares	8 equil.triangles	4 equil.triangles	20 equil.triangles	12 regul.pentagons
vertices	8	6	4	12	20
edges	12	12	6	30	30
faces angle	90°	109°28'	70°32'	138°11'	116°34'

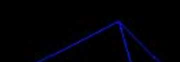
The LiveGraphics3D applet by Martin Kraus (University of Stuttgart) allows you to move these polyhedra with your mouse.



The regular octahedron's edges are the sides of three squares with the same centre and orthogonal by pairs.

The regular icosahedron's vertices are the vertices of three golden rectangles (sides in golden ratio 1.618...) with the same centre and orthogonal by pairs.





Four vertices of a cube are the vertices of a regular tetrahedron ; so we can make a regular tetrahedron by cutting four "corners" of a cube.

LiveGraphics3D 1.54: Please drag to rotate. Internet

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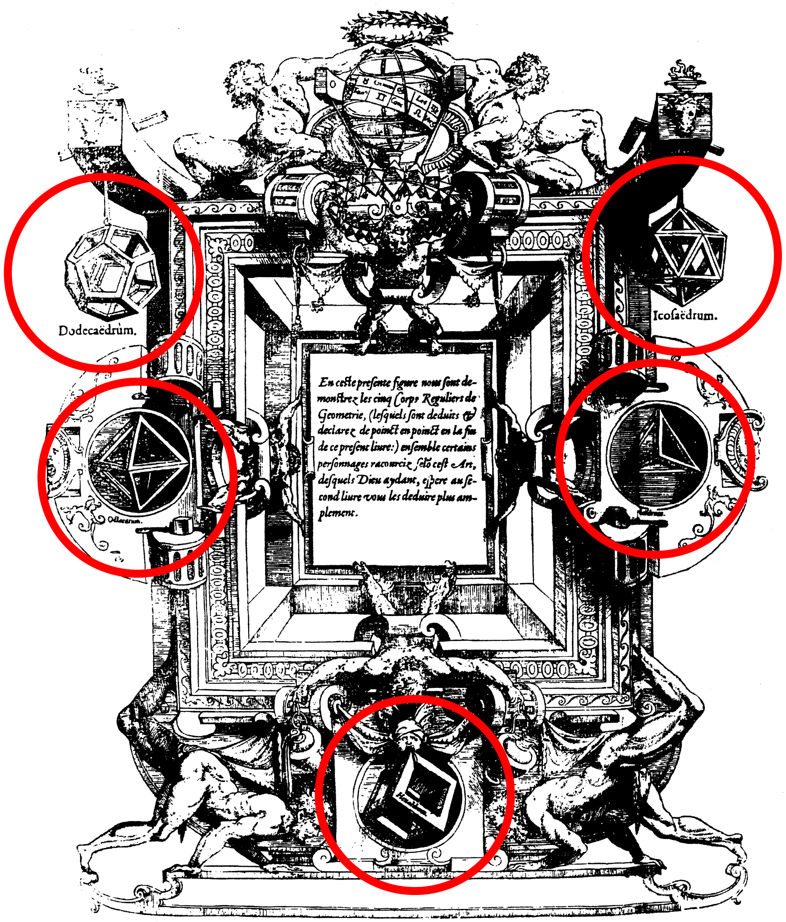
Polyhedra have fascinated people during all periods of our history



From *Livre de Perspective* by Jean Cousin, 1568.

- book illustrations
- magic objects
- pieces of art
- objects of symmetry
- models of the universe

Definitions

Linear programming lives (for our purposes) in the n-dimensional real (in practice: rational) vector space.

- convex polyhedral cone: conic combination (i. e., nonnegative linear combination or conical hull) of finitely many points
  K = cone(E), E a finite set in P^n.
- polytope: convex hull of finitely many points:
  P = conv(V), V a finite set in P^n.
- polyhedron: intersection of finitely many halfspaces

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$$

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Important theorems of polyhedral theory (LP-view)

When is a polyhedron nonempty?

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Important theorems of polyhedral theory (LP-view)

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

Theorem of the alternative

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Important theorems of polyhedral theory (LP-view)

Which forms of representation do polyhedra have?

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Important theorems of polyhedral theory (LP-view)

Which forms of representation do polyhedra have?

Minkowski (1896), Weyl (1935), Steinitz (1916) Motzkin (1936)

Theorem: For a subset P of $\mathbf{R}^n$ the following are equivalent:

(1) P is a polyhedron.

(2) P is the intersection of finitely many halfspaces, i.e., there exist a matrix A und ein vector b with

$$P = \{x \in \mathbf{R}^n \mid Ax \le b\}. \quad \text{(exterior representation)}$$

(3) P is the sum of a convex polytope and a finitely generated (polyhedral) cone, i.e., there exist finite sets V and E with

$$P = \mathrm{conv}(V) + \mathrm{cone}(E). \quad \text{(interior representation)}$$

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Representations of polyhedra

Carathéodory's Theorem (1911), 1873 Berlin – 1950 München

Let $x \in P = \mathrm{conv}(V) + \mathrm{cone}(E)$, there exist

$$v_0, \ldots, v_s \in V,\ \lambda_0, \ldots, \lambda_s \in \mathbf{R}_+,\ \sum_{i=0}^{s} \lambda_i = 1$$

and $e_{s+1}, \ldots, e_t \in E,\ \mu_{s+1}, \ldots, \mu_t \in \mathbf{R}_+$ with $t \leq n$ such that

$$x = \sum_{i=1}^{s} \lambda_i v_i + \sum_{i=s+1}^{t} \mu_i e_i$$

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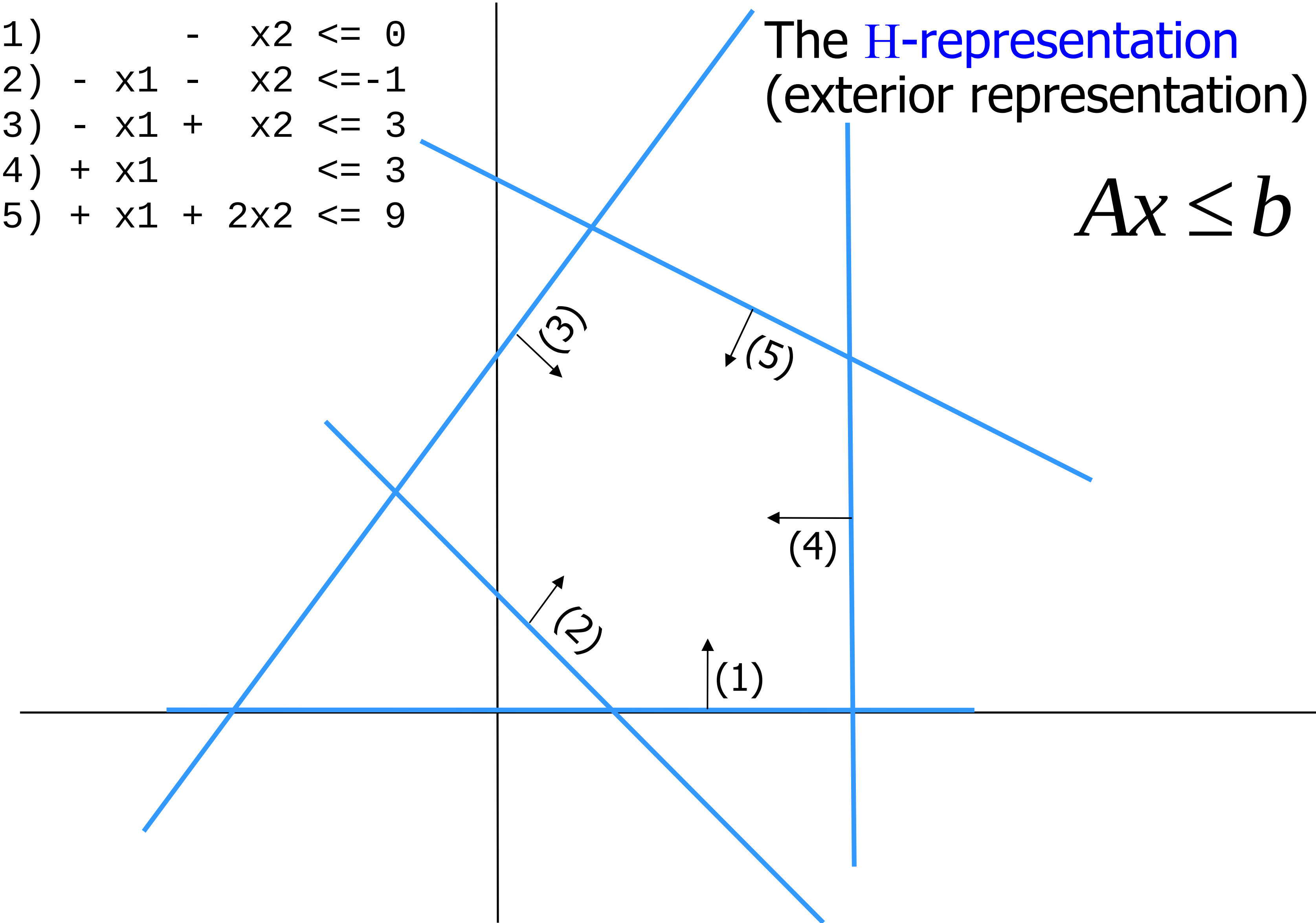
Representations of polyhedra


```
(1)          -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```

The H-representation
(exterior representation)

$$Ax \le b$$



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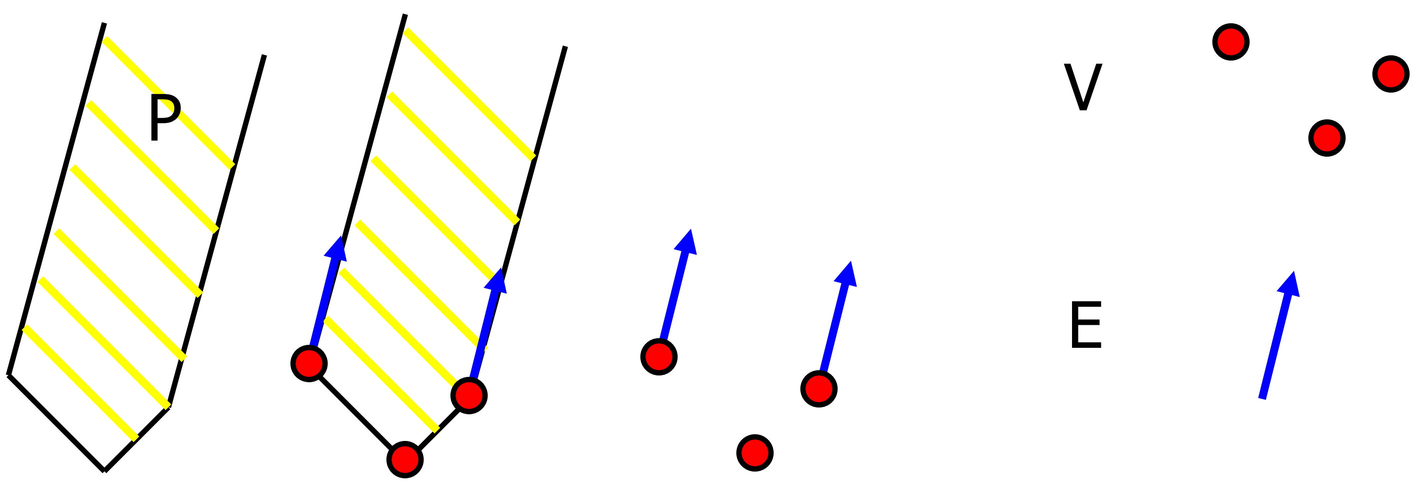
Representations of polyhedra

The ς-representation (interior representation)

$$P = \mathrm{conv}(V) + \mathrm{cone}(E).$$



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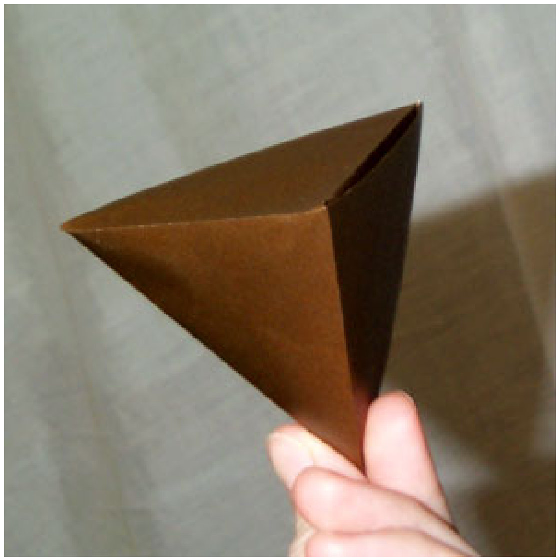
Example: the Tetrahedron



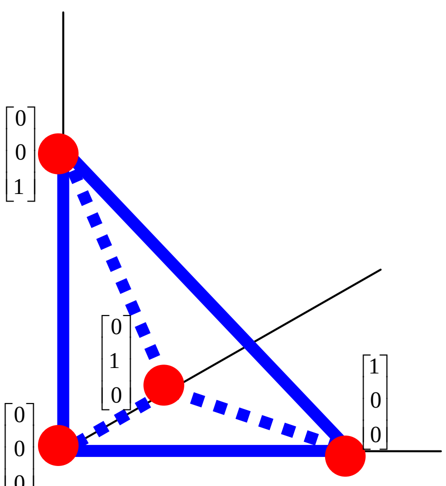
$$y \in conv\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

$$y_1 + y_2 + y_3 \leq 1$$

$$y_1 \geq 0$$

$$y_2 \geq 0$$

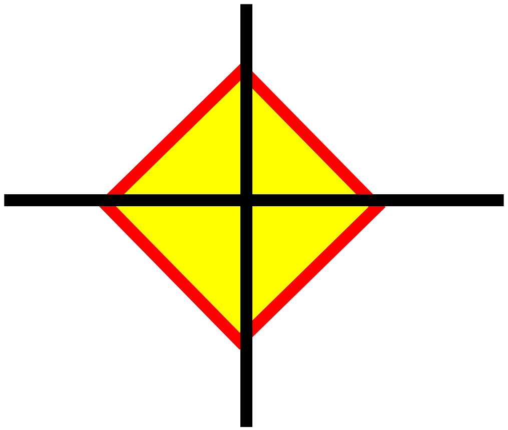
$$y_3 \geq 0$$



Example: the cross polytope

2n points

$$P = conv\{e_i, -e_i \mid i = 1, \dots, n\} \subseteq \mathbf{R}^n$$



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Example: the cross polytope

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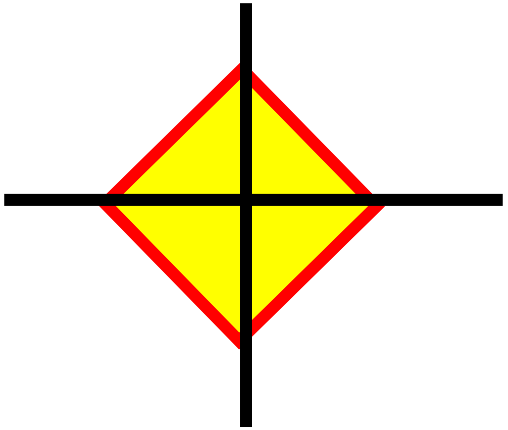

2^n inequalities

$$P = \{x \in \mathbf{R}^n \mid a^T x \leq 1 \ \forall \ a \in \{-1, 1\}^n\}$$

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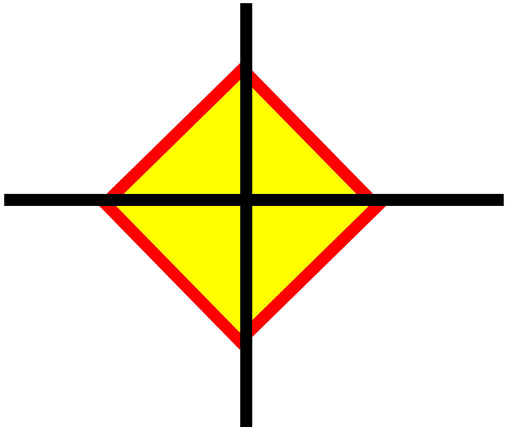
Example: the cross polytope

2n points

$$P = conv\{e_i, -e_i \mid i = 1, \ldots, n\} \subseteq \mathbf{R}^n$$

The "power" of |.|.

$$P = \left\{ x \in \mathbf{R}^n \mid \sum_{i=1}^{n} |x_i| \leq 1 \right\}$$



2^n inequalities

$$P = \left\{ x \in \mathbf{R}^n \mid a^T x \leq 1 \ \forall \ a \in \{-1, 1\}^n \right\}$$

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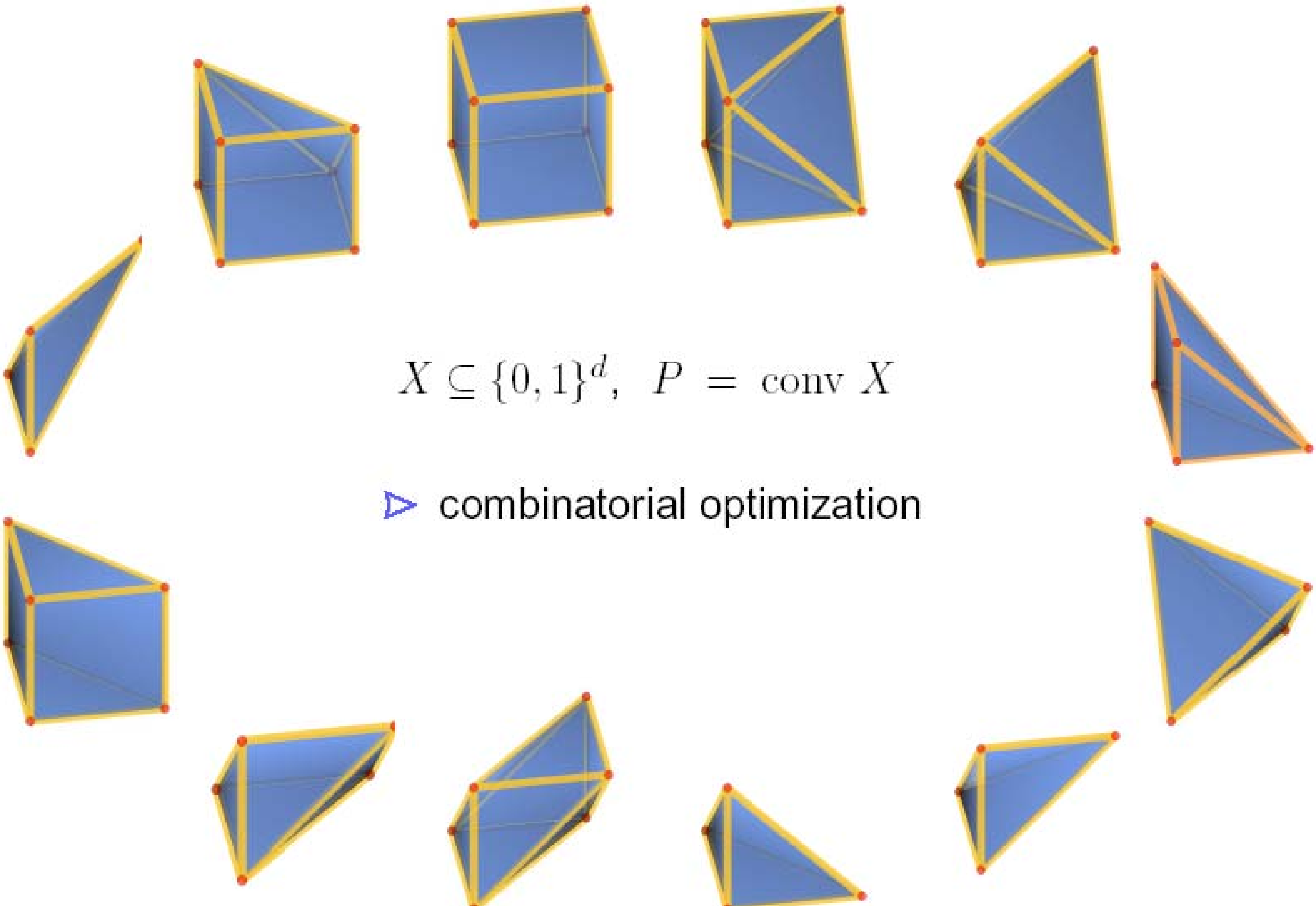
All 3-dimensional 0/1-polytopes



$$X \subseteq \{0,1\}^d, \;\; P = \operatorname{conv} X$$

▷ combinatorial optimization

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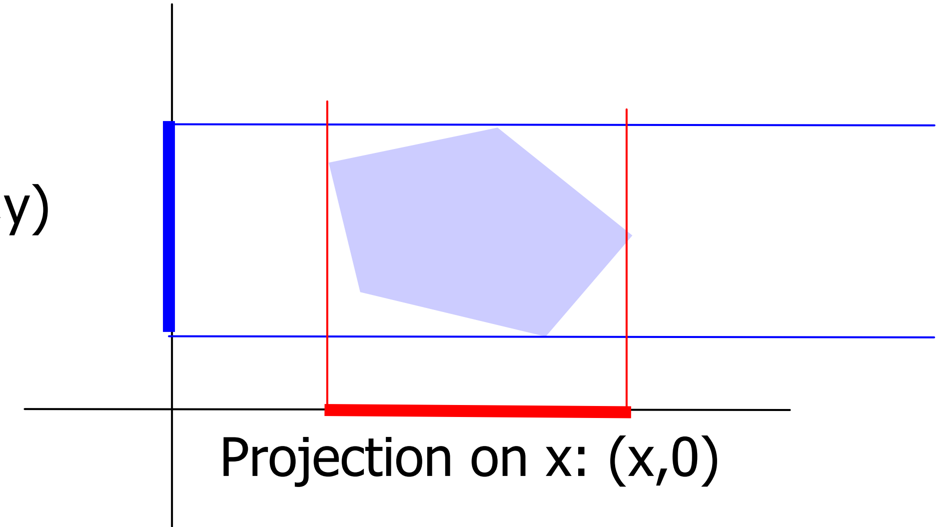
Polyedra in linear programming

- The solution sets of linear programs are polyhedra.
- If a polyhedron $P = \mathrm{conv(V)} + \mathrm{cone(E)}$ is given explicitly via finite sets V und E, linear programming is trivial.
- In linear programming, polyhedra are always given in $\mathcal{H}$-representation. Each solution method has its „standard form".

Fourier-Motzkin Elimination

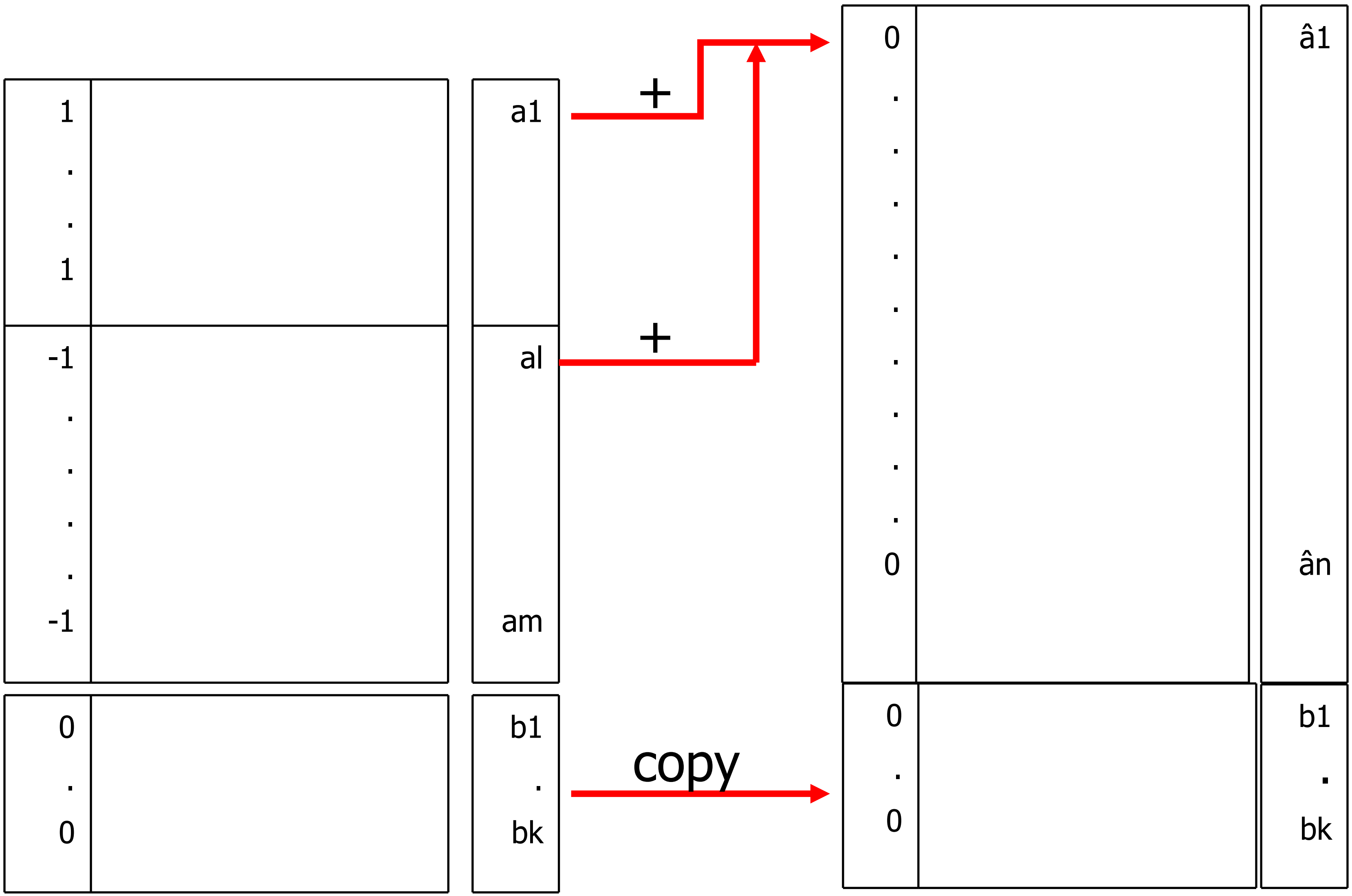
- Fourier, 1847
- Motzkin, 1938
- **Method**: successive projection of a polyhedron in n-dimensional space into a vector space of dimension n-1 by elimination of one variable.

Projection on y: (0,y)



Projection on x: (x,0)

A Fourier-Motzkin step



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Fourier-Motzkin elimination proves the Farkas Lemma

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

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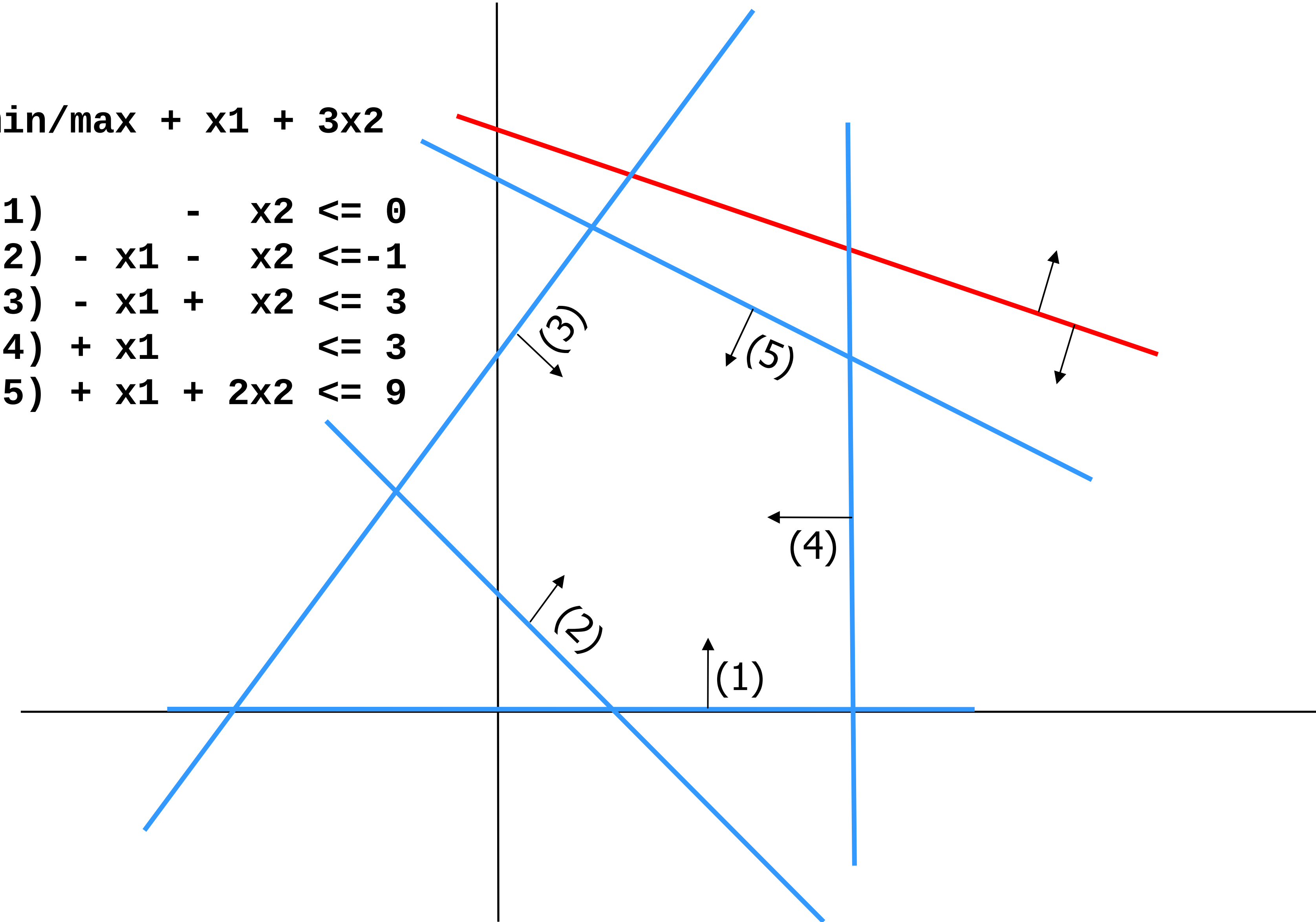
Fourier-Motzkin Elimination: an example


```
min/max + x1 + 3x2

(1)        -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1          <= 3
(5) + x1 + 2x2 <= 9
```



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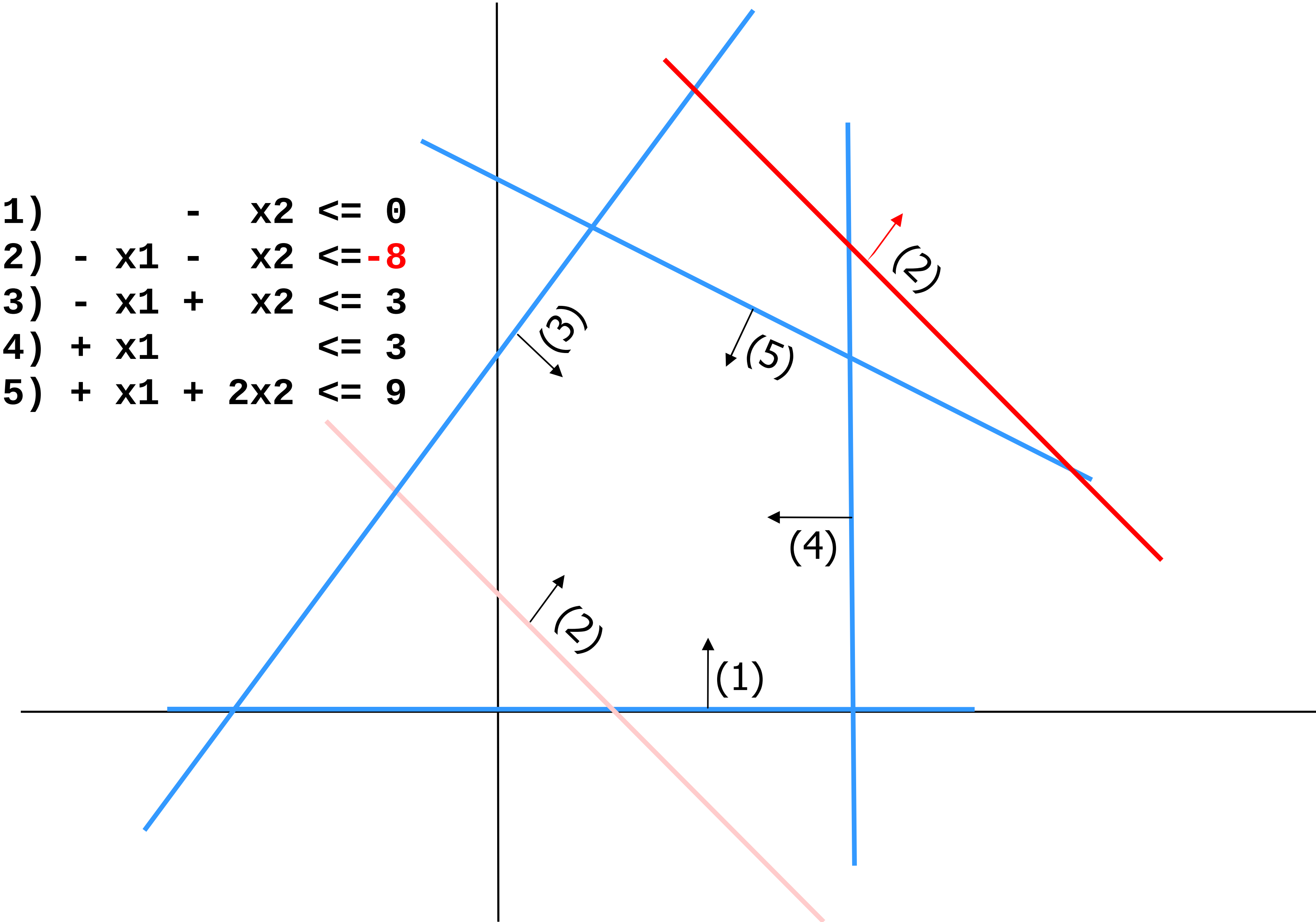
Fourier-Motzkin Elimination: an example

$$
\begin{array}{llcl}
(1) & - x_2 & \le & 0 \\
(2) & - x_1 - x_2 & \le & \color{red}{-8} \\
(3) & - x_1 + x_2 & \le & 3 \\
(4) & + x_1 & \le & 3 \\
(5) & + x_1 + 2x_2 & \le & 9
\end{array}
$$



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Fourier-Motzkin Elimination: an example, call of PORTA


```
(1)       -   x2 <= 0
(2) - x1 -   x2 <=-8
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```

```
DIM = 3

INEQUALITIES_SECTION

(1)       -   x2       <= 0
(2) - x1 -   x2       <=-8
(3) - x1 +   x2       <= 3
(4) + x1              <= 3
(5) + x1 + 2x2       <= 9


ELIMINATION_ORDER
1 0
```

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Fourier-Motzkin Elimination: an example, call of PORTA


```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2      <=  0
(2,4) (2) - x2      <= -5
(2,5) (3) + x2      <=  1
(3,4) (4) + x2      <=  6
(3,5) (5) + x2      <=  4
```

```
DIM = 3

INEQUALITIES_SECTION

(1)        -  x2      <= 0
(2) - x1 -  x2      <=-8
(3) - x1 +  x2      <= 3
(4) + x1            <= 3
(5) + x1 + 2x2      <= 9

ELIMINATION_ORDER
1 0
```

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Fourier-Motzkin Elimination: an example, call of PORTA


```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2      <=  0
(2,4) (2) - x2      <=  -5
(2,5) (3) + x2      <=  1
(3,4) (4) + x2      <=  6
(3,5) (5) + x2      <=  4

ELIMINATION_ORDER

0 1
```

```
DIM = 3

INEQUALITIES_SECTION

(2,3) 0 <= -4
```

Fourier-Motzkin elimination proves the Farkas Lemma

When is a polyhedron nonempty?

The Farkas-Lemma (1908):

A polyhedron defined by an inequality system

$$Ax \leq b$$

is empty, if and only if there is a vector y such that

$$y \geq 0,\ y^T A = 0^T,\ y^T b < 0^T$$

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Which LP solvers are used in practice?

- Fourier-Motzkin: hopeless
- Ellipsoid Method: total failure
- primal Simplex Method: good
- dual Simplex Method: better
- Barrier Method: for LPs frequently even better
- For LP relaxations of IPs: dual Simplex Method

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Fourier-Motzkin works reasonably well for polyhedral transformations:

Example: Let a polyhedron be given (as usual in combinatorial optimization implicitly) via:

$$P = \mathrm{conv}(\mathrm{V}) + \mathrm{cone}(\mathrm{E})$$

Find a non-redundant representation of P in the form:

$$P = \{x \in \mathbf{R}^d \mid Ax \leq b\}$$

Solution: Write P as follows

$$P = \{x \in \mathbf{R}^d \mid Vy + Ez - x = 0, \sum_{i=1}^{d} y_i = 1, y \geq 0, z \geq 0\}$$

and eliminate y und z.

Relations between polyhedra representations

- Given V and E, then one can compute A und b as indicated above.
- Similarly (polarity): Given A und b, one can compute V und E.
- The Transformation of a ς-representation of a polyhedron P into a H-representation and vice versa requires exponential space, and thus, also exponential running time.
- Examples: Hypercube and cross polytope.
- That is why it is OK to employ an exponential algorithm such as Fourier-Motzkin Elimination (or Double Description) for polyhedral transformations.
- Several codes for such transformations can be found in the Internet, e.g.. PORTA at ZIB and in Heidelberg.

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The Schläfli Graph S

Claw-free Graphs VI. Colouring Claw-free Graphs

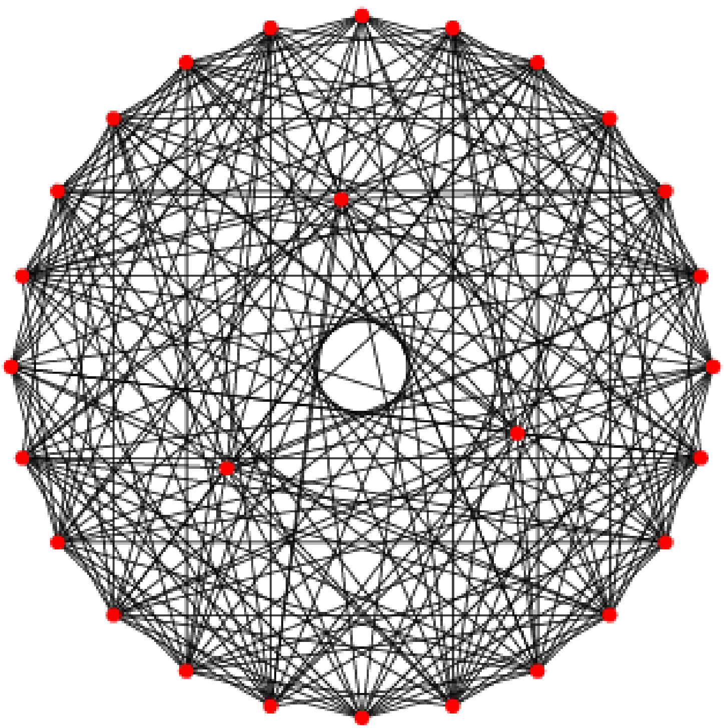
Maria Chudnovsky
Columbia University, New York NY 10027 [1]
and
Paul Seymour
Princeton University, Princeton NJ 08544 [2]

May 27, 2009

Abstract

In this paper we prove that if G is a connected claw-free graph with three pairwise non-adjacent vertices, with chromatic number χ and clique number ω, then $\chi \leq 2\omega$ and the same for the complement of G. We also prove that the choice number of G is at most 2ω, except possibly in the case when G can be obtained from a subgraph of the Schläfli graph by replicating vertices. Finally, we show that the constant 2 is best possible in all cases.

The Schläfli Graph S



Clique and stability number

Maximal cliques in S have size 6.
Maximal stable sets in S have size 3.
S has chromatic number 9 and there are two essentially different ways to color S with 9 colors. The complementary graph has chromatic number 6.

The Schläfli graph is a strongly regular graph on 27 nodes which is the graph complement of the generalized quadrangle $GQ(2, 4)$. It is the unique strongly regular graph with parameters $(27, 16, 10, 8)$ (Godsil and Royle 2001, p. 259).

http://mathworld.wolfram.com/SchlaefliGraph.html

The Polytope of stable sets of the Schläfli Graph

input file Schlaefli.poi
dimension : 27
number of cone-points : 0
number of conv-points : 208

The incidence vectors of the stable sets of the Schläfli graph

sum of inequalities over all iterations : 527962
maximal number of inequalities : 14230

transformation to integer values
sorting system

number of equations : 0
number of inequalities : 4086

The Polytope of stable sets of the Schläfli Graph

FOURIER - MOTZKIN - ELIMINATION:

iter- ation	upper bound # ineq	# ineq	max bit- length	long arith metic	non- zeros in %	mem used in kB	time used in sec
180	29	29	1	n	0.04	522	1.00
179	30	29	1	n	0.04	522	1.00
10	8748283	13408	3	n	0.93	6376	349.00
9	13879262	12662	3	n	0.93	6376	368.00
8	12576986	11877	3	n	0.93	6376	385.00
7	11816187	11556	3	n	0.93	6376	404.00
6	11337192	10431	3	n	0.93	6376	417.00
5	9642291	9295	3	n	0.93	6376	429.00
4	10238785	5848	3	n	0.92	6376	441.00
3	3700762	4967	3	n	0.92	6376	445.00
2	2924601	4087	2	n	0.92	6376	448.00
1	8073	4086	2	n	0.92	6376	448.00

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The Polytope of stable sets of the Schläfli Graph

INEQUALITIES_SECTION

(1) - x1 <= 0

(4086) +2x1+2x2+2x3+ x4+ x5+ x6 + x10+ x11+ x12+ x13+ x14+ x15
+x16+ x17+ x18+ x19+2x20 + x22+2x23 + x25+2x26 <= 3

8 different classes of inequalities found in total, among these, 5 classes have been unknown so far.

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Data resources at ZIB, open access

- MIPLIB
- FAPLIB
- STEINLIB

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ZIB offerings

- **PORTA -** POlyhedron Representation Transformation Algorithm
- **SoPlex -** The **S**equential **o**bject-oriented sim**plex** class library
- **Zimpl -** A mathematical modelling language
- **SCIP -** Solving constraint integer programs (IP & MIP)

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Semi-algebraic Geometry
Real-algebraic Geometry

$\mathrm{f}_i(x), g_j(x), h_k(x)$ are polynomials in d real variables

$$S_{\geq} := \{x \in \mathbf{R}^{d^{\mathbf{d}}} : \mathrm{f}_1(x) \geq 0, \ldots, \mathrm{f}_l(x) \geq 0\}$$ basic closed

$$S_{>} := \{x \in \mathbf{R}^{d^{\mathbf{d}}} : g_1(x) > 0, \ldots, g_m(x) > 0\}$$ basic open

$$S_{=} := \{x \in \mathbf{R}^{d^{\mathbf{d}}} : h_1(x) = 0, \ldots, h_n(x) = 0\}$$

$S := S_{\geq} \bigcup S_{>} \bigcup S_{=}$ is a semi-algebraic set

Theorem of Bröcker(1991) & Scheiderer(1989) basic closed case

Every basic closed semi-algebraic set of the form

$$S = \{x \in \mathbf{R}^{d\,\mathbf{d}} : \mathrm{f}_1(x) \geq 0, \ldots, \mathrm{f}_l(x) \geq 0\},$$

where $\mathrm{f}_i \in \mathbf{R}[x_1, \ldots, x_d], 1 \leq i \leq l,$ are polynomials,

can be represented by at most $d(d+1)/2$

polynomials, i.e., there exist polynomials

such that

$$\mathrm{p}_1, \ldots, \mathrm{p}_{d(d+1)/2} \in \mathbf{R}[x_1, \ldots, x_d]$$

$$S = \{x \in \mathbf{R}^d : \mathrm{p}_1(x) \geq 0, \ldots, \mathrm{p}_{d(d+1)/2}(x) \geq 0\}.$$

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Theorem of Bröcker(1991) & Scheiderer(1989) basic open case

Every basic open semi-algebraic set of the form

$$S = \{x \in \mathbf{R}^{d}: f_1(x) > 0, \ldots, f_l(x) > 0\},$$

where $f_i \in \mathbf{R}[x_1, \ldots, x_d], 1 \leq i \leq l,$ are polynomials,

can be represented by at most d

polynomials, i.e., there exist polynomials

such that

$$p_1, \ldots, p_d \in \mathbf{R}[x_1, \ldots, x_d]$$

$$S = \{x \in \mathbf{R}^{d} : p_1(x) > 0, \ldots, p_d(x) > 0\}.$$

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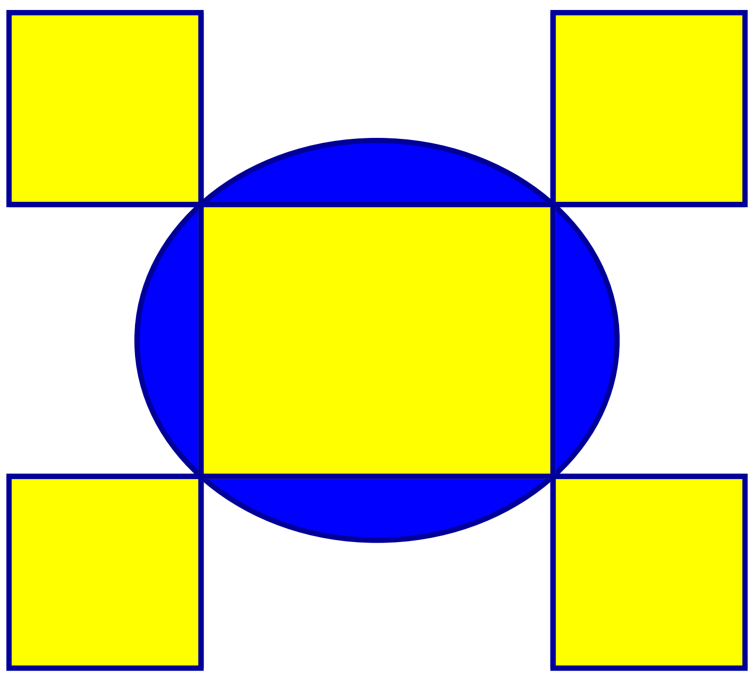
A first constructive result

Bernig [1998] proved that, for $d=2$, every convex polygon can be represented by two polynomial inequalities.

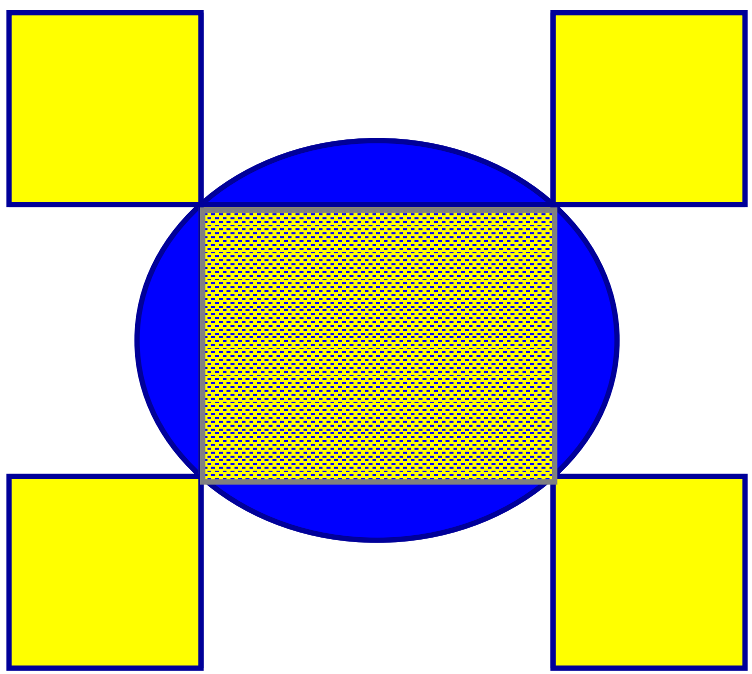


p(1)= product of all linear inequalities

p(2)= ellipse

A first Constructive Result

Bernig [1998] proved that, for $d=2$, every convex polygon can be represented by two polynomial inequalities.



p(1)= product of all
linear inequalities

p(2)= ellipse

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Our first theorem

Theorem Let $P \subset \mathbf{R}^n$ be a n-dimensional polytope given by an inequality representation. Then

$k \leq n^n$ polynomials $p_i \in \mathbf{R}[x_1, \ldots, x_n]$

can be constructed such that

$$P = \mathrm{P}(p_1, \ldots, p_k).$$

Martin Grötschel, Martin Henk:

The Representation of Polyhedra by Polynomial Inequalities

Discrete & Computational Geometry, 29:4 (2003) 485-504

Our main theorem

Theorem Let $P \subset \mathbf{R}^n$ be a n-dimensional polytope given by an inequality representation. Then

2n polynomials $\mathrm{p}_i \in \mathbf{R}[x_1,...,x_n]$

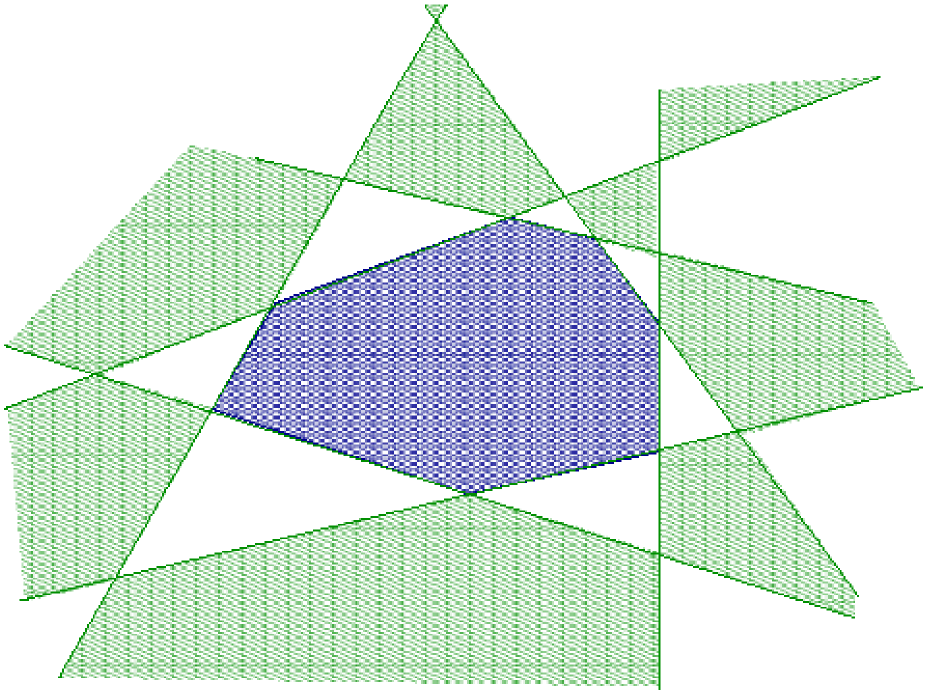
can be constructed such that

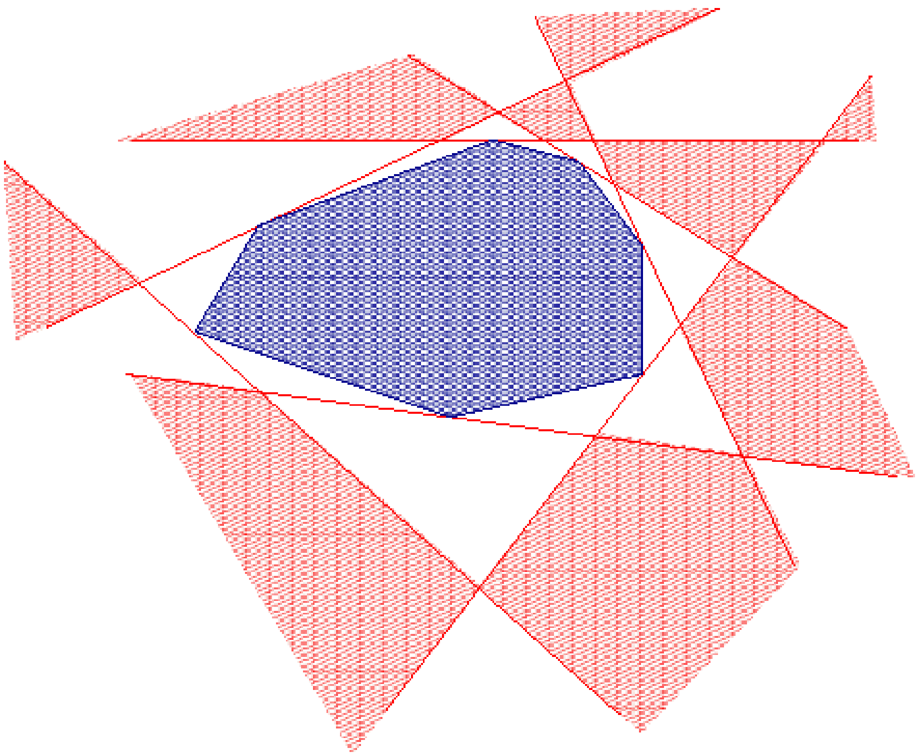
$$P = \mathrm{P}\,(\mathrm{p}_1,...,\mathrm{p}_{2n}).$$

Hartwig Bosse, Martin Grötschel, Martin Henk:
Polynomial inequalities representing polyhedra
Mathematical Programming 103 (2005)35-44

http://www.springerlink.com/index/10.1007/s10107-004-0563-2

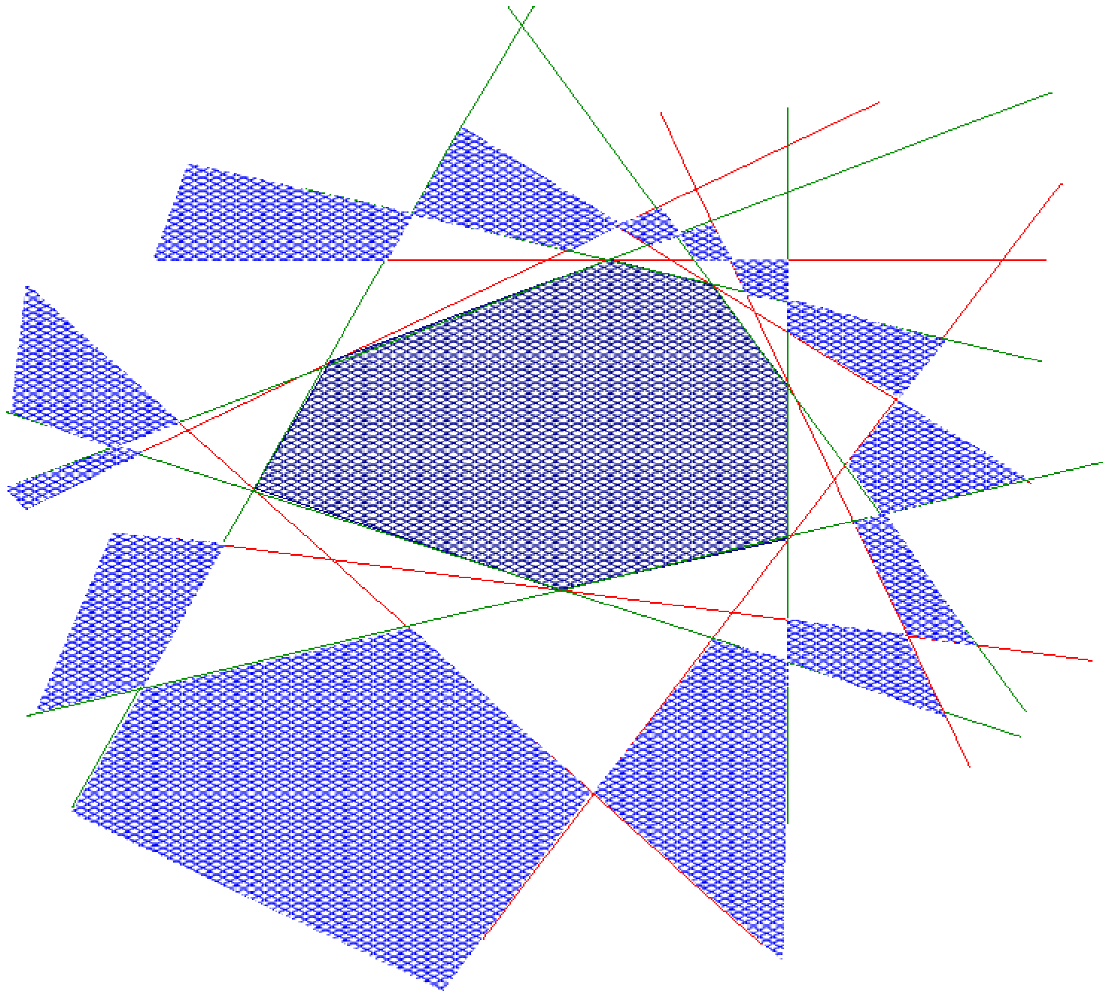
The construction in the 2-dimensional case





$\{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0\}$ $\{x \in \mathbb{R}^d : \mathfrak{p}_0(x) \geq 0\}$

The construction in the 2-dimensional case



$$\{x \in \mathbb{R}^d : \mathfrak{p}_1(x) \geq 0 \text{ and } \mathfrak{p}_0(x) \geq 0\}$$

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Recent “Semi-algebraic Progress”

three-dimensional polyhedra can be described by three polynomial inequalities

jointly with Gennadiy Averkov

Discrete Comput. Geom.,**42**(2), 2009, 166-186; arXiv:0807.2137

representing simple d-dimensional polytopes by d polynomials

jointly with Gennadiy Averkov

to appear in Math. Prog. (A); arXiv:0709.2099v1

http://fma2.math.unimagdeburg.de/~henk/preprints/henk&polynomdarstellungen%20von%20polyedern.pdf

Bröcker

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Faces etc.

- Important concept: dimension
- face
- vertex
- edge
- (neighbourly polytopes)
- ridge = subfacet
- facet

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Linear Programming: The DualityTheorem

The most important and influential theorem in optimization.

$$\min\left\{wx \mid Ax \geq b\right\} = \max\left\{yb \mid y \geq 0, yA = w\right\}$$

A good research idea is to try to mimic this result:

$$\min\left\{\textit{something}\right\} = \max\left\{\textit{something}\right\}$$

A relation of this type is called min-max result.

Max-flow min-cut theorem
(Ford & Fulkerson, 1956)

Let $D = (V, A)$ be a directed graph, let $r, s \in V$ and let $c: A \rightarrow \mathbb{R}_+$ be a capacity function. Then the maximum value of an r-s -flow subject to the capacity c is equal to the minimum capacity of an r-s -cut.

If all capacities are integer, there exists an integer optimum flow.

Here an r-s-flow is a vector $x: A \rightarrow \mathbb{R}$ such that

(1) (i) $x(a) \geq 0 \quad \forall a \in A$

(ii) $x\left(\delta^-(v)\right) = x\left(\delta^+(v)\right) \quad \forall v \in V,\ r \neq v \neq s$

The value of the flow is the net amount of flow leaving r, i.e., is

(2) $x\left(\delta^+(r)\right) - x\left(\partial^-(r)\right)$

(which is equal to the net amount of flow entering s). The flow x is *subject to* c if $x(a) \leq c(a)$ for all a in A.

Ford-Fulkerson animation

- http://www.cse.yorku.ca/~aaw/Wang/MaxFlowStart.htm

Flow Algorithms

- The Ford-Fulkerson Algorithm
  The grandfather of augmenting paths algorithms
- The Dinic-Malhorta-Kumar-Maheshwari Algorithm
- Preflow (Push-Relabel) Algorithms

Complexity survey

from Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, 2003 Springer

10.8b. Complexity survey for the maximum flow problem

Complexity survey ($*$ indicates an asymptotically best bound in the table):

$O(n^2mC)$	Dantzig [1951a] simplex method
$O(nmC)$	Ford and Fulkerson [1955,1957b] augmenting path
$O(nm^2)$	Dinits [1970], Edmonds and Karp [1972] shortest augmenting path
$O(n^2m \log nC)$	Edmonds and Karp [1972] fattest augmenting path
$O(n^2m)$	Dinits [1970] shortest augmenting path, layered network
$O(m^2 \log C)$	Edmonds and Karp [1970,1972] capacity-scaling
$O(nm \log C)$	Dinits [1973a], Gabow [1983b,1985b] capacity-scaling
$O(n^3)$	Karzanov [1974] (preflow push); cf. Malhotra, Kumar, and Maheshwari [1978], Tarjan [1984]
$O(n^2\sqrt{m})$	Cherkasskiĭ [1977a] blocking preflow with long pushes
$O(nm \log^2 n)$	Shiloach [1978], Galil and Naamad [1979,1980]
$O(n^{5/3}m^{2/3})$	Galil [1978,1980a]

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Complexity survey

from Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, 2003 Springer

continued

	$O(nm \log n)$	Sleator [1980], Sleator and Tarjan [1981,1983a] dynamic trees
*	$O(nm \log(n^2/m))$	Goldberg and Tarjan [1986,1988a] push-relabel+dynamic trees
	$O(nm + n^2 \log C)$	Ahuja and Orlin [1989] push-relabel + excess scaling
	$O(nm + n^2 \sqrt{\log C})$	Ahuja, Orlin, and Tarjan [1989] Ahuja-Orlin improved
*	$O(nm \log((n/m)\sqrt{\log C} + 2))$	Ahuja, Orlin, and Tarjan [1989] Ahuja-Orlin improved + dynamic trees
*	$O(n^3/\log n)$	Cheriyan, Hagerup, and Mehlhorn [1990,1996]
	$O(n(m + n^{5/3} \log n))$	Alon [1990] (derandomization of Cheriyan and Hagerup [1989,1995])
	$O(nm + n^{2+\varepsilon})$	(for each $\varepsilon > 0$) King, Rao, and Tarjan [1992]
*	$O(nm \log_{m/n} n + n^2 \log^{2+\varepsilon} n)$	(for each $\varepsilon > 0$) Phillips and Westbrook [1993,1998]
*	$O(nm \log_{\frac{m}{n \log n}} n)$	King, Rao, and Tarjan [1994]
*	$O(m^{3/2} \log(n^2/m) \log C)$	Goldberg and Rao [1997a,1998]
*	$O(n^{2/3} m \log(n^2/m) \log C)$	Goldberg and Rao [1997a,1998]

Here $C := \|c\|_\infty$ for integer capacity function c. For a complexity survey for unit capacities, see Section 9.6a.

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Complexity survey

from Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, 2003 Springer

Research problem: Is there an $O(nm)$-time maximum flow algorithm?

For the special case of *planar* undirected graphs:

	$O(n^2 \log n)$	Itai and Shiloach [1979]
	$O(n \log^2 n)$	Reif [1983] (minimum cut), Hassin and Johnson [1985] (maximum flow)
	$O(n \log n \log^* n)$	Frederickson [1983b]
*	$O(n \log n)$	Frederickson [1987b]

For *directed* planar graphs:

	$O(n^{3/2} \log n)$	Johnson and Venkatesan [1982]
	$O(n^{4/3} \log^2 n \log C)$	Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]
*	$O(n \log n)$	Weihe [1994b,1997b]

Min-cost flow

Let $D = (V, A)$ be a directed graph, let $r, s \in V$, let $c: A \to \mathbb{R}_+$ be a capacity function, $w: A \to \mathbb{R}$ a cost function, and f a flow value. Find a flow x of value f subject to c with minimum value $w^T x$.

$$\min \sum_{a \in A} w(a)x(a)$$

$$0 \leq x(a) \leq c(a) \quad \forall a \in A$$

$$x\left(\delta^+(v)\right) - x\left(\partial^-(v)\right) = 0 \quad \forall r \neq v \neq s$$

$$x\left(\delta^+(r)\right) - x\left(\partial^-(r)\right) = f$$

There is a similarly large number of algorithms with varying complexity, see Schrijver (2003).

Min-Max Results

CO@W

König´s Matching Theorem (1931) (Frobenius, 1912)

The maximum size of a matching in a bipartite graph is equal to the minimum number of vertices covering all edges, i. e.,

$$\nu\left(G\right)=\tau\left(G\right)$$

for bipartite graphs G.

Tutte-Berge Formula (Tutte(1947), Berge(1958))

$$\max\left\{|M|: M \subseteq E \text{ matching}\right\} = \min_{W\subseteq V} \frac{1}{2}(|V|+|W|-O(G-W))$$

where G=(V,E) is an arbitrary graph.

Total unimodularity

A matrix A is called *totally unimodular* if each square submatrix of A has determinant 0, +1 or –1. In particular, each entry of A is 0, +1 or –1.

The interest of totally unimodular matrices for optimization was discovered by the following theorem of Hoffman and Kruskal (1956):

If A is totally unimodular and b and w are integer vectors, then both sides of the LP-duality equation

$$\max\{wx \mid Ax \le b\} = \min\{yb \mid y \ge 0, yA = w\}$$

have integer optimum solutions.

Total unimodularity

CO@W

There have been many characterizations of totally unimodular matrices:

Ghouila-Houri (1962)

Camion (1965)

Padberg (1976)

Truemper(1977)

....

Full understanding was achieved by establishing a link to regular matroids, Seymour (1980). This connection also yields a polynomial time algorithm to recognize totally unimodular matrices.

Min-Max Results

Dilworth's theorem (1950)

The maximum size of an antichain in a partially ordered set (P, <) is equal to the minimum number of chains needed to cover P.

Fulkerson's optimum branching theorem (1974)

Let $D=(V,A)$ be a directed graph, let $r \in V$ and let $l : A \to R_+$ be a length function. Then the minimum length of an r-arborescence is equal to the maximum number t of r-cuts $C_1,\ldots, C_t$ (repetition allowed) such that no arc a is in more than $l(a)$ of the C_i.

Edmonds' disjoint branching theorem (1973)

Let $D=(V,A)$ be a directed graph, and let $r \in V$. Then the maximum number of pairwise disjoint r-arborescences is equal to the minimum size of an r-cut.

Min-Max Results

Edmonds' matroid intersection theorem (1970)

Let $M_1 = (S, \mathcal{J}_1)$ and $M_2 = (S, \mathcal{J}_2)$ be matroids, with rank functions r_1 and r_2, respectively. Then the maximum size of a set in $\mathcal{J}_1 \cap \mathcal{J}_2$ is equal to

$$\min_{S' \subseteq S} \left(r_1(S') + r_2(S \setminus S') \right).$$

Min-Max Results and Polyhedra

- Min-max results almost always provide polyhedral insight and can be employed to prove integrality of polyhedra.
- For instance, the matroid intersection theorem can be used to prove a theorem on the integrality of the intersection of two matroid polytopes.

Min-Max Results and Polyhedra

Let M=(E, $\mathcal{I}$) be a matroid with rank function r.

Define IND($\mathcal{I}$):=conv{x^I | I is an Element of $\mathcal{I}$}.

IND($\mathcal{I}$) is called matroid polytope. Let

$$P(\mathcal{I}) := \left\{ x \in \mathbf{R}^E : \sum_{e \in F} x_e \leq r(F) \ \forall \ F \subseteq E, \ x_e \geq 0 \ \forall \ e \in E \ \right\}$$

Theorem: P($\mathcal{I}$) = IND($\mathcal{I}$).

Theorem: Let M_1=(E, $\mathcal{I}_1$) and M_2=(E, $\mathcal{I}_2$) be two matroids

with rank functions r_1 and r_2, respectively. Then

IND($\mathcal{I}_1$I$\mathcal{I}_2$) = P($\mathcal{I}_1$)IP($\mathcal{I}_2$)

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Min-Max Results and Polyhedra

In other words, if $M_1=(E, \mathcal{I}_1)$ and $M_2=(E, \mathcal{I}_2)$ are two matroids on the same ground set E with rank functions r_1 and r_2, respectively, and if c_e is a weight for all elements e of E, then a set that is independent in M_1 and M_2 and has the largest possible weight can be found via the following linear program

$$\max \sum_{e \in E} c_e x_e$$

$$\sum_{e \in F} x_e \leq r_1(F) \ \forall \ F \subseteq E$$

$$\sum_{e \in F} x_e \leq r_2(F) \ \forall \ F \subseteq E$$

$$x_e \geq 0 \ \forall \ e \in E$$

An Excursion into Matroid Theory

- If time permits

Matroids and Independence Systems

Let E be a finite set, $\mathcal{I}$ a subset of the power set of E.

The pair (E,$\mathcal{I}$) is called independence system on E if the following axioms are satisfied:

(I.1) The empty set is in $\mathcal{I}$.

(I.2) If J is in $\mathcal{I}$ and I is a subset of J then I belongs to $\mathcal{I}$.

Let (E,$\mathcal{I}$) satisfy in addition:

(I.3) If I and J are in $\mathcal{I}$ and if J is larger than I then there is an element j in J, j not in I, such that the union of I and j is in $\mathcal{I}$.

Then M=(E,$\mathcal{I}$) is called a matroid.

Notation

Let $(E,\mathcal{I})$ be an independence system.

- Every set in $\mathcal{I}$ is called independent.
- Every subset of E not in $\mathcal{I}$ is called dependent.
- For every subset F of E, a basis of F is a subset of F that is independent and maximal with respect to this property.

The rank r(F) of a subset F of E is the cardinality of a largest basis of F. The lower rank $r_u(F)$ of F is the cardinality of a smallest basis of F.

The Largest Independent Set Problem

Problem:

Let (E,$\mathcal{I}$) be an independence system with weights on the elements of E. Find an independent set of largest weight.

We may assume w.l.o.g. that all weights are nonnegative (or even positive), since deleting an element with nonpositive weight from an optimum solution, will not decrease the value of the solution.

The Greedy Algorithm

Let (E,$\mathcal{I}$) be an independence system with weights c(e) on the elements of E. Find an independent set of largest weight.

The Greedy Algorithm:

1. Sort the elements of E such that $c_1 \geq c_2 \geq \ldots \geq c_n \geq 0$.
2. Let $\mathrm{I}_{\text{greedy}} := \emptyset$.
3. FOR i=1 TO n DO:

   IF $\mathrm{I}_{\text{greedy}} \cup \{i\} \in \mathcal{I}$ THEN $\mathrm{I}_{\text{greedy}} := \mathrm{I}_{\text{greedy}} \cup \{i\}$.

4. OUTPUT $\mathrm{I}_{\text{greedy}}$.

A key idea is to interprete the greedy solution as the solution of a linear program.

Polytopes and LPs

Let M=(E,$\mathcal{I}$) be an independence system

with weights c(e) on the elements of E.

$$\text{IND(M)} = conv\left\{x^{\mathrm{I}} \in \mathbf{R}^{E} \;\middle|\; \mathrm{I} \in \mathcal{I}\right\}$$

$$= conv\left\{x \in \mathbf{R}^{E} \;\middle|\; \sum_{e\in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E\right\}$$

The LP relaxation

$$\min c^T x \qquad \text{s.t.} \sum_{e\in F} x_e \le r(F)\ \forall\ F \subseteq E,$$

$$x_e \ge 0 \qquad \forall\ e \in E$$

The dual LP

$$\min \sum_{F\subseteq E} y_F r(F) \quad \text{s.t.} \sum_{F\ni e} y_F \ge c_e \qquad \forall\ e \in E,$$

$$y_F \ge 0 \qquad \forall\ F \subseteq E$$

The Dual Greedy Algorithm

Let (E,$\mathcal{I}$) be an independence system with weights c(e) for all e.

After sorting the elements of E so that

$c_1 \geq c_2 \geq \ldots \geq c_n \geq 0,\ c_{n+1} := 0$ set

$$E_i := \{1, 2, \ldots, i\},\ i=1, 2, \ldots, n \text{ and}$$

$$y_{E_i} := c_i - c_{i+1}, \qquad i=1, 2, \ldots, n.$$

Then $y_{E_i} = c_i - c_{i+1}$, i=1, 2, ..., n

is a feasible solution of the dual LP

$$\min \sum_{F \subseteq E} y_F r_u(F),$$

$$\text{s.t.} \sum_{F \ni e} y_F \geq c_e \quad \forall\ e \in E,$$

$$y_F \geq 0 \quad \forall\ F \subseteq E$$

Observationn

Let (E,$\mathcal{I}$) be an independence system with weights c(e) for all e.

After sorting the elements of E so that

$c_1 \geq c_2 \geq \ldots \geq c_n \geq 0,\ c_{n+1} := 0$

we can express every greedy and optimum solution as follows

$$\mathrm{c}(\mathrm{I}_{\mathrm{greedy}}) = \sum_{i=1}^{n} (c_i - c_{i+1}) \left| \mathrm{I}_{\mathrm{greedy}} \cap E_i \right|$$

$$\mathrm{c}(\mathrm{I}_{\mathrm{opt}}) = \sum_{i=1}^{n} (c_i - c_{i+1}) \left| \mathrm{I}_{\mathrm{opt}} \cap E_i \right|$$

Rank Quotient

Let (E,$\mathcal{I}$) be an independence system with weights c(e) for all e.

$$q := \min_{\substack{F \subseteq E \\ r(F) > 0}} \frac{r_u(F)}{r(F)}$$

The number q is between 0 and 1 and is called rank quotient of (E,$\mathcal{I}$).

Observation: q = 1 iff (E,$\mathcal{I}$) is a matroid.

The General Greedy Quality Guarantee

$$\max \sum_{e\in E} c_e x_e, \text{s.t.} \sum_{e\in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E$$

$$\ge\ \max \sum_{e\in E} c_e x_e, \text{s.t.} \sum_{e\in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E,\ x \text{ integral}$$

$$= c(\mathrm{I}_{\text{opt}}) \ge\ c(\mathrm{I}_{\text{greedy}}) = \sum_{i=1}^{n} (c_i - c_{i+1}) \left|\mathrm{I}_{\text{greedy}} \cap E_i\right| \ge \sum_{i=1}^{n} (c_i - c_{i+1}) r_u(E_i)$$

$$= \sum_{i=1}^{n} y_{E_i} r_u(E_i)$$

$$\ge\ \min \sum_{F\subseteq E} y_F r_u(F), \text{s.t.} \sum_{F\ni e} y_F \ge c_e\ \forall\ e \in E, y_F \ge 0\ \forall\ F \subseteq E$$

$$\ge\ q \min \sum_{F\subseteq E} y_F r(F), \text{s.t.} \sum_{F\ni e} y_F \ge c_e\ \forall\ e \in E, y_F \ge 0\ \forall\ F \subseteq E$$

$$=\ q \max \sum_{e\in E} c_e x_e, \text{s.t.} \sum_{e\in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E$$

$$\ge\ q \max \sum_{e\in E} c_e x_e, \text{s.t.} \sum_{e\in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E,\ x \text{ integral}$$

$$=\ q\ c(\mathrm{I}_{\text{opt}})$$ a quality guarantee

Consequences

Let M=(E,$\mathcal{I}$) be an independence system with weights c(e) on the elements of E.

$$\mathrm{IND(M)} = conv\left\{x^{\mathrm{I}} \,\middle|\, \mathrm{I} \in \mathcal{I}\right\}$$

$$\mathrm{P(M)} = \left\{x \in \mathbf{R}^E \,\middle|\, \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0\ \forall\ e \in E\right\}$$

Theorem: (a) P(M) = IND(M) if and only if M is a matroid

(b) If M is a matroid then all optimum vertex solutions of the primal LP

$$\max c^T x \qquad \text{s.t.} \sum_{e \in F} x_e \le r(F)\ \forall\ F \subseteq E,\ x_e \ge 0 \quad \forall\ e \in E$$

are integral. If the weights are integral then the dual LP

$$\min \sum_{F \subseteq E} y_F r(F) \qquad \text{s.t.} \sum_{F \ni e} y_F \ge c_e \quad \forall\ e \in E,\ y_F \ge 0 \quad \forall\ F \subseteq E$$

also has integral optimum solutions,

i.e., the system is totally dual integral.

Min-Max Results: Challenges

Despite all the beautiful min-max results mentioned before (and the not mentioned far reaching generalizations such as submodular flows or matroid matching), there is still a great challenge:

understand integral duality.

Where and when does it occur?

Why?....

Basics of polyhedral theory, flows and networks

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The End

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