Polyhedral Computations
An Introduction to Porta and polymake

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00A1, 00A2: Lecture and Exercises

Block Course: "Combinatorial Optimization at Work"

Zuse Institute Berlin

This is an integrated lecture and exercise.

Goals:

- Repetition of polyhedral theory
- Introduction to polyhedral combinatorics
- Usage of software
  - `Porta`
  - `polymake`
  - `JavaView`
  - (`Zimpl`)
- Preparation for forthcoming lectures

Credits for the software we use:

- `Porta`
  Thomas Christof, Andreas Löbel (ZIB)
  originally developed at U Heidelberg, maintained at ZIB
- `polymake`
  Ewgenij Gawrilow (TU Berlin) and
  Michael Joswig (TU Darmstadt)
  developed at TU Berlin
- `JavaView`
  Konrad Polthier, Klaus Hildebrandt, Eike Preuss, and
  Ulrich Reitebuch
  originally developed at TU Berlin – now ZIB
- `Zimpl`
  Thorsten Koch (ZIB)

Outline

- Polyhedron:
  Intersection of finitely many halfspaces:

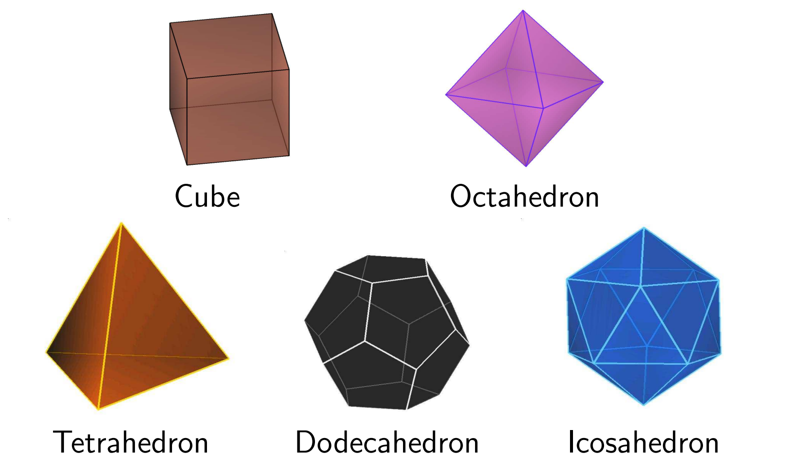
  $$P = \{x \in \mathbb{R}^d \ : \ Ax \leq b\}, \text{ where } A \in \mathbb{R}^{m \times d},\ b \in \mathbb{R}^m.$$

  Halfspaces:

  $$\{x \in \mathbb{R}^d \ : \ (a^i)^\mathsf{T} x \leq b_i\}, \quad a^i = i\text{th row of } A.$$

- Polytope:
  bounded polyhedron

Examples: Platonic Solids



Some easy examples:

- $\mathbb{R}^d$
- $\varnothing$
- $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \;:\; x \geq 0\}$
- For $v \in \mathbb{R}^d$:
  $\{v\} = \{x \in \mathbb{R}^d \;:\; x_i \leq v_i, \; -x_i \leq -v_i, \; i = 1, \ldots, d\}$

Further Examples

Some easy examples:

- $\mathbb{R}^d$
- $\varnothing$
- $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d \;:\; x \geq 0\}$
- For $v \in \mathbb{R}^d$:
  $\{v\} = \{x \in \mathbb{R}^d \;:\; x_i \leq v_i,\; -x_i \leq -v_i,\; i = 1, \ldots, d\}$

Affine space:
$S \subseteq \mathbb{R}^d$ is affine if there exists $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$ such that

$$S = \{x \in \mathbb{R}^d \;:\; Ax = b\},$$

i.e., S is the solution of a linear equation system.

As polyhedron: $S = \{x \in \mathbb{R}^d \;:\; Ax \leq b, -Ax \leq -b\}$.

P polyhedron in $\mathbb{R}^d$.

- Valid Inequalities:
  inequality $c^\mathsf{T} x \leq c_0$, $c \in \mathbb{R}^d$, $c_0 \in \mathbb{R}$, is valid if

$$P \subseteq \{x \in \mathbb{R}^d \ : \ c^\mathsf{T} x \leq c_0\}.$$

- Face of polyhedron P:

$$F = \{x \in P \ : \ c^\mathsf{T} x = c_0\},$$

  for valid inequality $c^\mathsf{T} x \leq c_0$.

P polyhedron in $\mathbb{R}^d$.

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- Face of polyhedron P:

$$F = \{x \in P \ : \ c^\mathsf{T} x = c_0\},$$

  for valid inequality $c^\mathsf{T} x \le c_0$.
- Examples:
  - P (take $0^\mathsf{T} x \le 0$)
  - $\varnothing$ (take $0^\mathsf{T} x \le -1$)

  "trivial faces".

Dimension

- Affine hull:

$$\text{aff}(S) = \cap\{S' \ : \ S' \supseteq S, \ S' \text{ affine space}\}.$$

- Dimension of $S \subseteq \mathbb{R}^d$:

$$\dim S = \dim \text{aff}(S).$$

Let P be a polyhedron in $\mathbb{R}^d$.
Face F is a

- vertex if $\dim F = 0$
- edge if $\dim F = 1$
- ridge if $\dim F = \dim(P) - 2$
- facet if $\dim F = \dim(P) - 1$

Theorem

Every polytope $P \subset \mathbb{R}^d$ is the convex hull of finitely many vectors $v_1, \ldots, v_k \in \mathbb{R}^d$, i.e., $P = \mathrm{conv}\{v_1, \ldots, v_k\}$.

Here

$$\begin{aligned}\mathrm{conv}\{v_1, \ldots, v_k\} = \{x \in \mathbb{R}^d \ : \ x = &\lambda_1 v_1 + \cdots + \lambda_k v_k, \\ &\lambda_1 + \cdots + \lambda_k = 1, \\ &\lambda_1, \ldots, \lambda_k \geq 0\}.\end{aligned}$$

Theorem

Every polyhedron $P \subseteq \mathbb{R}^d$ can be written as

$$P = \mathrm{conv}\{v_1, \dots, v_k\} + \mathrm{cone}\{r_1, \dots, r_\ell\}$$

with points $v_1, \dots, v_k \in \mathbb{R}^d$ and rays $r_1, \dots, r_\ell \in \mathbb{R}^d$.

Here

$$\mathrm{cone}\{r_1, \dots, r_\ell\} = \{x \in \mathbb{R}^d \ : \ x = \mu_1 r_1 + \cdots + \mu_\ell r_\ell, \\ \mu_1, \dots, \mu_\ell \geq 0\}.$$

With `polymake` ▷ . . .

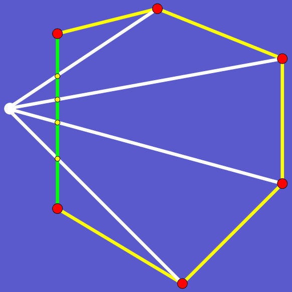
Exercise 1: Illustration of Definitions

- Start JavaView.
- Delete geometry: Inspector→Display, "Clean all".
- Load file cube3.jvx: File→Open→Browse Disk.
- Keys:
  - o: rotate
  - t: translate
  - s: scale
  - Shift-z: z-buffer on/off
- Open Inspector→Geometry→Material.
- Turn off facets ("Element").
- Change size of vertices and edges.
- Turn on facets; turn on Transparency, use slider below.
- Delete geometry. Try tetrahedron.jvx, octahedron.jvx, dodecahedron.jvx, icosahedron.jvx.

Schlegel Diagrams

One visualization technique are Schlegel diagrams:
Projection on a facet reduces dimension by 1.

Example:



Same idea for 3- and 4-dimensional polytopes.

Exercise 2: Schlegel Diagrams

- Start `JavaView`.
- Delete geometry: `Inspector`→`Display`, "`Clean all`".
- Load file `cube3.jvx`: `File`→`Open`→`Browse Disk`.
- Open `Inspector`→`Geometry`→`Material`.
- Turn off facets ("`Element`").
- Change size of vertices and edges.
- Rotate until you "see everything through a facet".

Exercise 2: Schlegel Diagrams

- Start `JavaView`.
- Delete geometry: `Inspector`→`Display`, "`Clean all`".
- Load file `cube3.jvx`: `File`→`Open`→`Browse Disk`.
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- Delete geometry. Load `cube4.jvx`.
- Delete geometry. Load `24-cell.jvx` or `cross4.jvx`.

Exercise 2: Schlegel Diagrams

- Start `JavaView`.
- Delete geometry: `Inspector→Display`, "`Clean all`".
- Load file `cube3.jvx`: `File→Open→Browse Disk`.
- Open `Inspector→Geometry→Material`.
- Turn off facets ("`Element`").
- Change size of vertices and edges.
- Rotate until you "see everything through a facet".

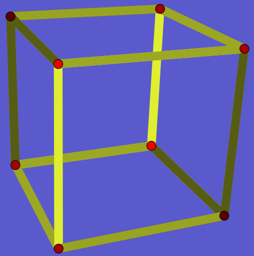
- Delete geometry. Load `cube4.jvx`.
- Delete geometry. Load `24-cell.jvx` or `cross4.jvx`.

- Load `cube4-e.jvx/cross4-e.jvx/24-cell-e.jvx`.
- `Method→Effect→Explode Group of Geometries`

d-cube:

$$C_d := \{x \in \mathbb{R}^d \ : \ -1 \leq x_i \leq 1, \ i = 1, \dots, d\}$$
$$= \mathrm{conv}\{\{-1, 1\}^d\}$$

- ▷ 1 face of dimension d
- ▷ $2 \cdot d$ facets
- ▷ $d \cdot 2^{d-1}$ edges
- ▷ 2^d vertices
- ▷ 1 face of dimension -1

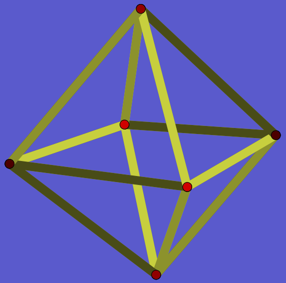


Example: d-crosspolytope

d-crosspolytope:

$$
\begin{aligned}
C_d^{\Delta} :=& \operatorname{conv}\{e_1, -e_1, \ldots, e_d, -e_d\} \\
=& \{x \in \mathbb{R}^d \;:\; a^{\mathsf{T}} x \leq 1,\; a \in \{-1, 1\}^d\}
\end{aligned}
$$

- 1 face of dimension d
- 2^d facets
- $d \cdot 2^{d-1}$ ridges
- $2 \cdot d$ vertices
- 1 face of dimension -1



How many facets can a polytope with n vertices have?

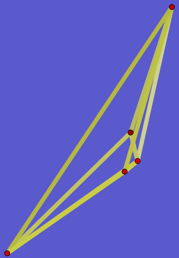
Theorem (McMullen 1970)

The maximum number of facets that a d-polytope with n vertices can have is at most

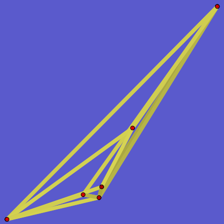
$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} 2\binom{n-d-1+i}{i} \approx c \cdot n^{\lfloor \frac{d}{2} \rfloor} \qquad (d \text{ even}).$$

The upper bound is attained by cyclic polytopes.

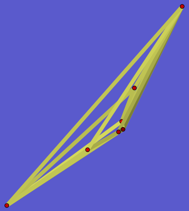
Can be explicitly constructed. Examples for $d = 3$:



$n = 5$



$n = 6$



$n = 7$

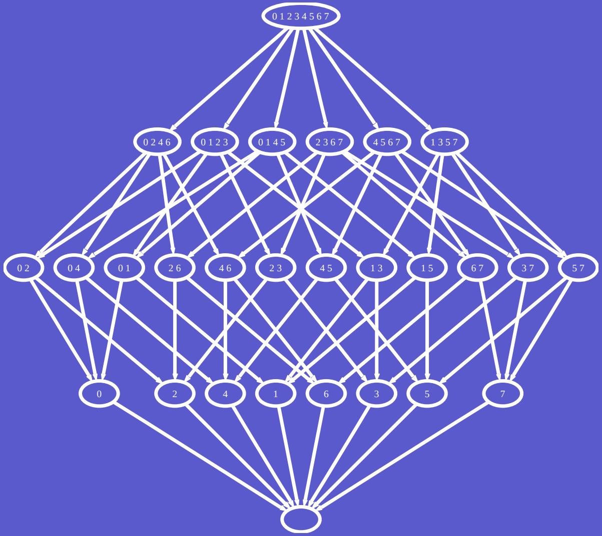
For $d \geq 4$, cyclic polytopes are neighborly, i.e., every two vertices are connected by an edge.

Hasse diagram of polytope P:

- directed graph
- nodes $=$ faces of P
- arc (G, F) if $F \subseteq G$ and $\dim(F) = \dim(G) - 1$
- has $\dim(P) + 1$ many layers

Example of Hasse Diagram

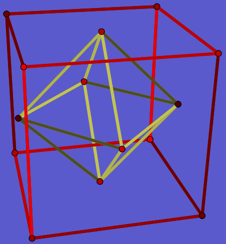
Hasse diagram of the 3-cube:

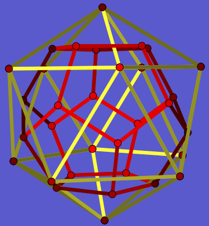


Definition

Polytopes P and Q are dual polytopes,
if the Hasse diagram of P is isomorphic to
the Hasse diagram of Q turned upside down.

$$\begin{array}{rcl} k\text{-faces} & \leftrightarrow & (d-k-1)\text{-faces} \\ \text{vertices} & \leftrightarrow & \text{facets} \\ \text{edges} & \leftrightarrow & \text{ridges} \end{array}$$

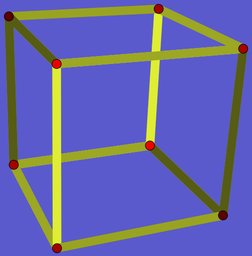


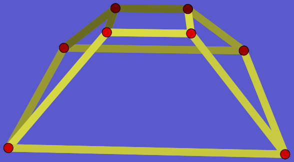


Definition

Two polytopes are combinatorially equivalent if their Hasse diagrams are isomorphic.

Isomorphism: re-order nodes in each layer such that the resulting graphs are the same.





1. A polytope P is a 0/1-polytope if it vertices have coordinates 0 and 1 only.

   Find all 3-dimensional 0/1-polytopes that are not combinatorially equivalent.
2. *Voluntary:*
   Find all 3-dimensional polytopes with 4, 5, and 6 vertices that are not combinatorially equivalent.
3. Let P be a 3-polytope and v be a vertex of P.
   Show that the number of facets equals the number of edges containing v.
   Show or provide a counterexample for the claim in higher dimensions.

Outline

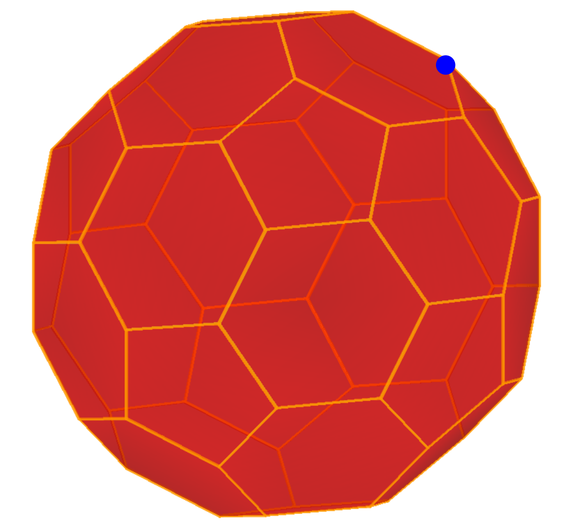
Linear programming problem:

- Given: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$
- Find: $x^\star \in P := \{x \ : \ Ax \leq b\}$ maximizing $c^\mathsf{T} x^\star$

Useful facts:

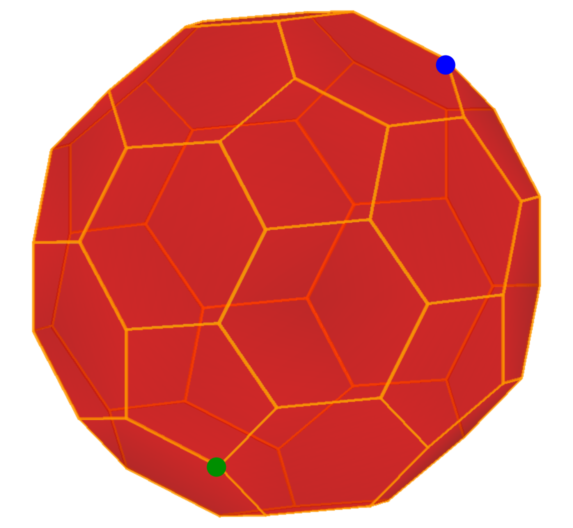
- Optimum is attained at vertex of P.
- Suffices to go from vertex to vertex $\rightsquigarrow$ Simplex-Algorithm.

Geometric View on the Simplex Algorithm



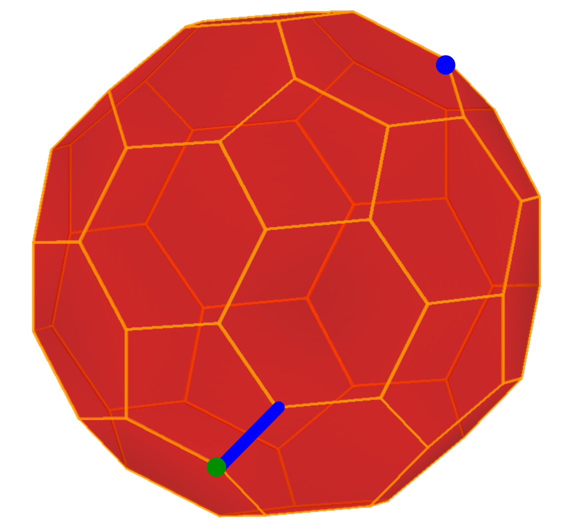
Objective function oriented to right top

Geometric View on the Simplex Algorithm



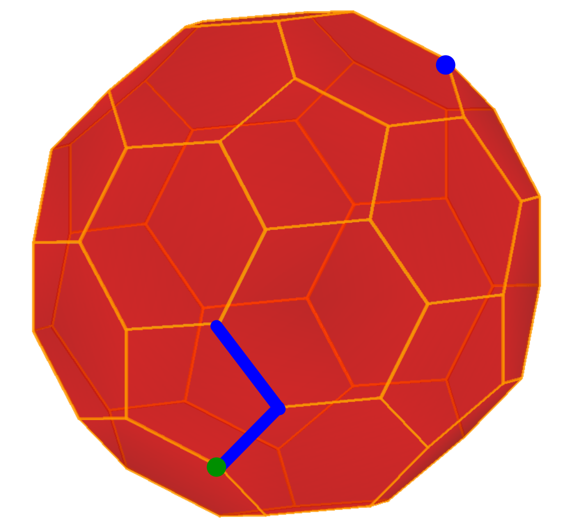
Start at vertex

Geometric View on the Simplex Algorithm



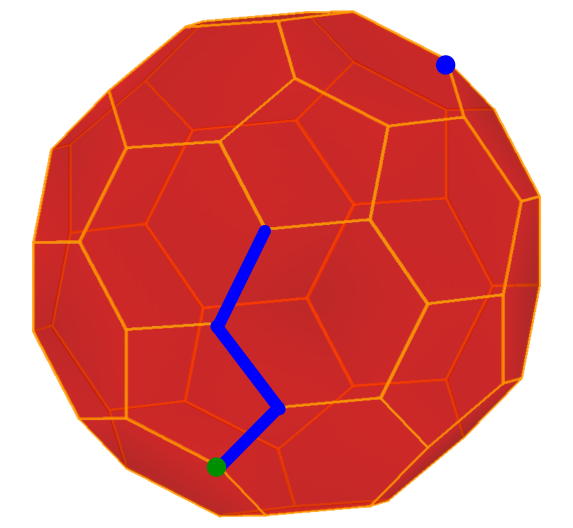
Walk along improving edge

Geometric View on the Simplex Algorithm



Keep walking as long as we can improve

Geometric View on the Simplex Algorithm



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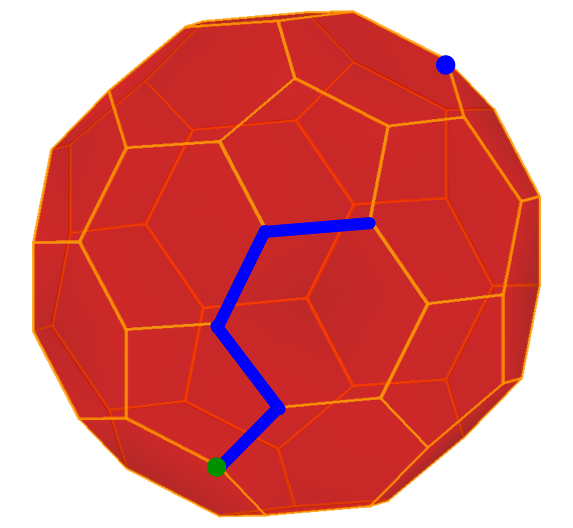

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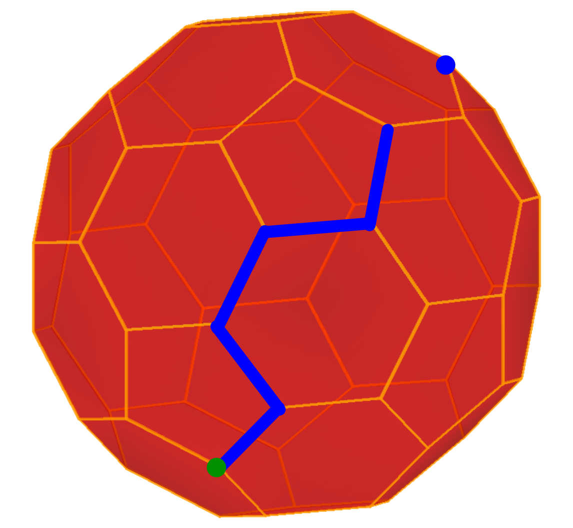
Geometric View on the Simplex Algorithm



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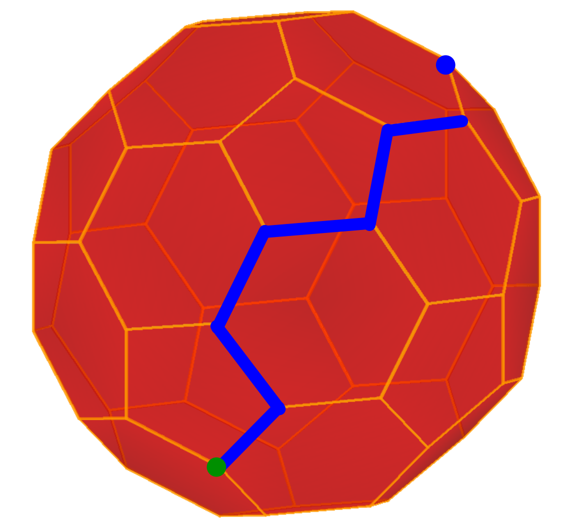
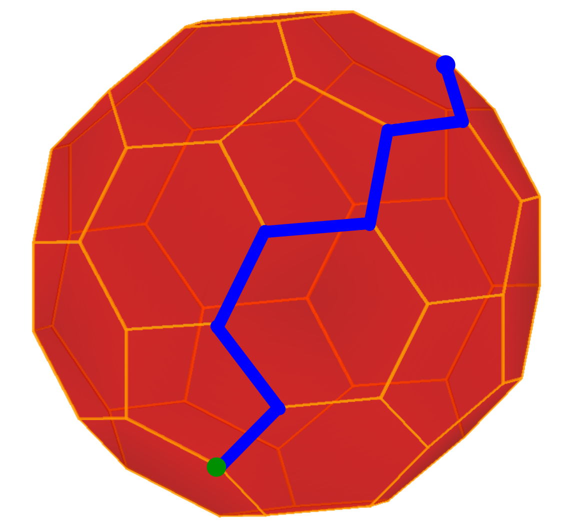
Geometric View on the Simplex Algorithm



Keep walking as long as we can improve

Geometric View on the Simplex Algorithm



Keep walking as long as we can improve

More examples:

- soccerball.jvx
- Schrijver_KleeMinty.3.jvx

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- `soccerball.jvx`
- `Schrijver_KleeMinty.3.jvx`

For most pivot rules:
Exist examples in which simplex algorithm behaves bad,
i.e., takes very (exponentially) long path.

More examples:

- soccerball.jvx
- Schrijver_KleeMinty.3.jvx

For most pivot rules:
Exist examples in which simplex algorithm behaves bad,
i.e., takes very (exponentially) long path.

On the other hand:

Conjecture (Hirsch 1957)

The shortest path between any two vertices in a d-polytope with m facets is at most $m - d$.

unsolved

Recall the main theorem:

Theorem

Every polytope P can be described by an H-description $P = \{x : Ax \leq b\}$ or by a V-description $P = \text{conv}\{v_1, \ldots, v_k\}$.

Given, $v_1, \ldots, v_k \in \mathbb{R}^d$, describe $P = \text{conv}\{v_1, \ldots, v_k\}$ by inequalities:

$$\begin{aligned} x &= \lambda_1 v_1 + \ldots \lambda_k v_k \\ 1 &= \lambda_1 + \ldots \lambda_k \\ 0 &\leq \lambda_1, \ldots, \lambda_k \end{aligned}$$

(the variables are x and $\lambda_1, \ldots, \lambda_k$).
Then project out $\lambda_1, \ldots, \lambda_k$ by Fourier-Motzkin Elimination.

Polarity = exchange roles of points and inequalities.

Definition

The polar P^Δ of a polytope P is:

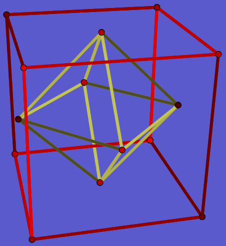
$$P^\Delta = \{c \in \mathbb{R}^d \ : \ c^\mathsf{T} x \le 1, \text{ for all } x \in P\}.$$

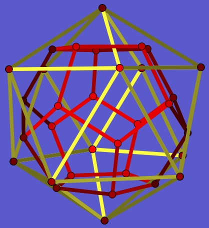
Proposition

1. *If* $P = \operatorname{conv}\{v_1, \ldots, v_m\}$, 0 *is in the interior of* P, *and* $V = [v_1, \ldots, v_m]$, *then* $P^\Delta = \{x \in \mathbb{R}^d \ : \ V^T x \le \mathbb{1}\}$.
2. *If* $P = \{x \in \mathbb{R}^d \ : \ Ax \le \mathbb{1}\}$, *then* $P^\Delta = \operatorname{conv}\{A_1, \ldots, A_m\}$, *where* A_i *is the* i*th column of* A.

Consequences from Polarity

- One can prove: $P^{\Delta\Delta} = P$, if $0 \in P$.
- If we can convert from V to H-description, we can convert in the other direction as well:
  Exchange roles of inequalities and points.
- It turns out that P^{Δ} is combinatorially equivalent to "the" dual of P.





Example: 3-dimensional cube:

```
DIM = 11
ELIMINATION_ORDER
1 2 3 4 5 6 7 8 0 0 0
INEQUALITIES_SECTION
1 x1 -1 x2 +1 x3 +1 x4 -1 x5 +1 x6 -1 x7 -1 x8 - x9 == 0
1 x1 +1 x2 -1 x3 +1 x4 -1 x5 -1 x6 +1 x7 -1 x8 - x10 == 0
1 x1 +1 x2 +1 x3 -1 x4 +1 x5 -1 x6 -1 x7 -1 x8 - x11 == 0
x1 + x2 + x3 + x4 + x5 + x6 + x7 + x8 == 1
x1 >= 0
x2 >= 0
x3 >= 0
x4 >= 0
x5 >= 0
x6 >= 0
x7 >= 0
x8 >= 0
END
```

Exercise 4: Fourier-Motzkin Elimination

1. Choose your favorite polytope.
2. Generate a file `filename.ieq` and insert data as above.
3. Call: `fmel filename.ieq`
4. Investigate output `filename.ieq.poi`.

- We know that the output can be exponentially large.
- Therefore hope for output polynomial algorithm:
  runs in polynomial time in the input and output.
  Open
- Exist examples in which Fourier-Motzkin elimination generates exponentially many inequalities in between.
- Exist many other algorithms, but almost all of them are known to have bad examples.
- It is in general hard to only generate vertices of an unbounded polyhedron (Khachiyan et. al [2005]).

Outline

Combinatorial Optimization Problems

Combinatorial Optimization problem:

Given:

- finite set S (groundset)
- set $\mathcal{F} \subseteq 2^S$ (feasible solutions)
- weight $w_s \in \mathbb{R}$ for each $s \in S$

Goal:
Find $S' \in \mathcal{F}$ with maximal weight $w(S')$.

$$w(S') := \sum_{s \in S'} w_s \quad \text{for } S' \subseteq S.$$

Example: Stable Set Problem

Given:

- graph $G = (V, E)$
- groundset $= V$
- $U \subseteq V$ is feasible if
  no two vertices $u, v \in U$ are connected by an edge.
  U is called stable or independent set.
- $w_v = 1$ for all $v \in V$.

Goal:
Find stable set U of maximal cardinality.

Stable Set Problem II

Integer programming formulation:

$$\begin{aligned} \max \quad & \mathbb{1}^{\mathsf{T}} x \\ \text{s.t.} \quad & x_u + x_v \leq 1 && \text{for all edges } \{u, v\} \in E \\ & x \in \{0, 1\}^n. \end{aligned}$$

Will use LP relaxation:

$$\begin{aligned} \max \quad & \mathbb{1}^{\mathsf{T}} x \\ \text{s.t.} \quad & x_u + x_v \leq 1 && \text{for all edges } \{u, v\} \in E \\ & Ax \leq b && \text{add. inequalities} \\ & x \in [0, 1]^n. \end{aligned}$$

Geometry and Combinatorial Optimization

Explain general approach to combinatorial optimization problems for the stable set problem.

Study stable set polytope:

$$\begin{aligned} P_S &:= \mathrm{conv}\{\chi(U) \;:\; U \text{ is a stable set}\} \\ &= \mathrm{conv}\{x \in \{0,1\}^n \;:\; x_u + x_v \le 1, \text{ for all } \{u,v\} \in E\}, \end{aligned}$$

where $\chi(U)$ is the incidence vector of U:

$$\chi(U)_i = \begin{cases} 1 & \text{if } i \in U \\ 0 & \text{otherwise.} \end{cases}$$

"Recipe" to solve the stable set problem:

1. Find classes of facets of polytope P_S.
2. For a class of facets:
   Construct (fast) separation algorithm:
   Given $x^\star$, does there exist facet defined by $c^\mathsf{T} x \leq d$ such that $c^\mathsf{T} x^\star > d$ (inequality is violated).
3. Start cutting plane algorithm:
   3.1 Find solution $x^\star$ of LP relaxation.
   3.2 If $x^\star$ is integer $\longrightarrow$ optimal solution found!
   3.3 Solve separation problems for classes of facets.
   3.4 Add violated inequalities to the LP relaxation.
   Goto 3.1.

Exercise 5: Finding Facets – Part I

1. Take small graph of your choice.
2. Find all stable sets.
3. Put corresponding incidence vectors into file.
   Format (filename ending with ".poi"):

```
DIM = <n>
CONV_SECTION
<x_1> <x_2> . . . <x_n>
. . .
CONE_SECTION
END
```

4. Compute facets: traf <file>.poi.
5. Analyze result. Can you explain the facets?

Exercise 5: Finding Facets – Part II

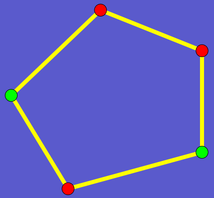
1. Write ZIMPL program, which reads any graph.
   Hint: Write graph in file: put one edge in each line
   `u v`
   Use
   `set E := read FILE as "<1n,2n>";`
   to read file. Call `zimpl -D FILE="..."`.
2. For many graphs, generate LP format output.
3. Use `lp2porta` to generate Porta output.
4. Use `vint` to generate all integer points that satisfy the given inequalities.
5. Compute facets.
6. Analyze result. Can you explain the facets?
   Can you find a separation algorithm? (May be hard!)

“Solution”: Famous Classes

1. Nonnegativity constraints: $x_u \geq 0$
2. Odd hole inequalities:

$$x(S) \leq \frac{|S|-1}{2}$$

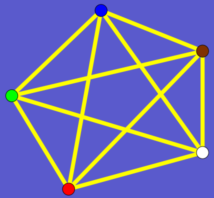
$S =$ vertices of odd cycle without chord.



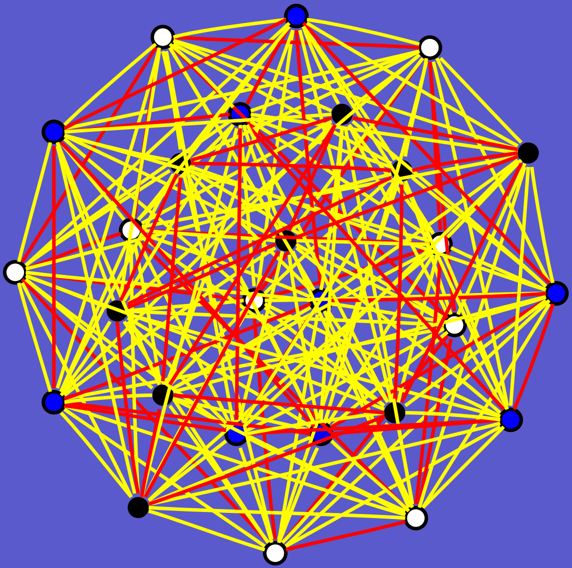
3. Clique inequalities:

$$x(S) \leq 1$$

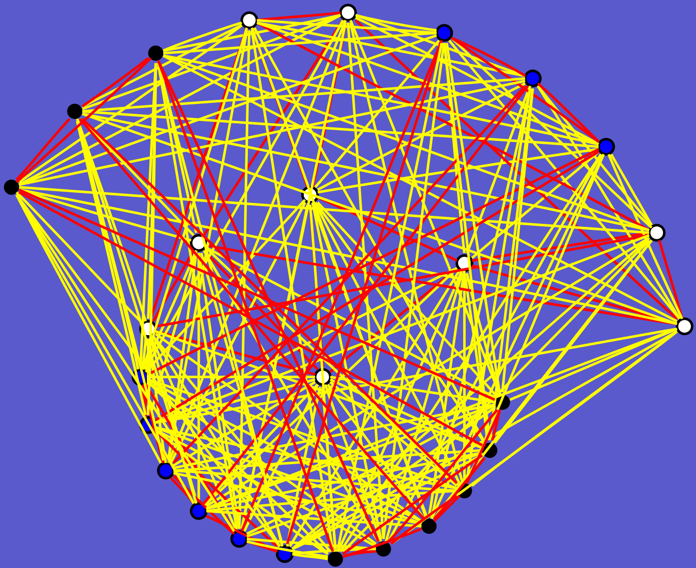
$S =$ vertices of a clique.



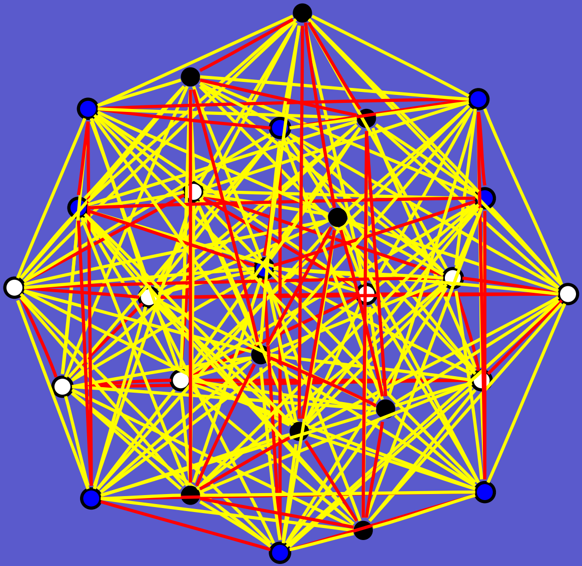
Schläfli Graph



Schläfli Graph



Schläfli Graph



Stable Set Polytope for the Schläfli Graph

- The Schläfli graph has 27 nodes and 216 edges.
- The Schläfli graph has 208 stable sets.
- P_S has 4086 facets with 7 classes.
- `Porta` generated 527,962 inequalities in total.
- Maximal number of inequalities in one iteration: 14230.

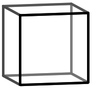
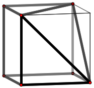
Facet Classes

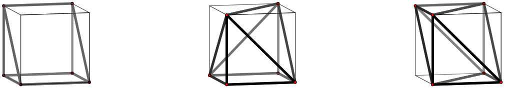
1. 27 nonnegativity constraints
2. 215 5-clique inequalities, 72 6-clique inequalities
   $x_{16} + x_{17} + x_{18} + x_{21} + x_{27} \leq 1$
3. 216 inequalities with 12 non-zero coefficients (two 2's and ten 1's) and right-hand side 2
   $x_7 + x_8 + x_9 + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + 2x_{25} + x_{26} + 2x_{27} \leq 2$
4. 1080 inequalities with 14 non-zero coefficients (all 1's) and right-hand side 2
   $x_7 + x_8 + x_9 + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{22} + x_{24} + x_{25} + x_{26} + x_{27} \leq 2$
5. 216 inequalities with 15 non-zero coefficients (all 1's) and right-hand side 2
   $x_7 + x_8 + x_9 + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{21} + x_{22} + x_{24} + x_{25} + x_{27} \leq 2$
6. 27 inequalities with 17 non-zero coefficients (sixteen 1's and one 2) and right-hand side 2
   $x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{21} + x_{24} + x_{25} + x_{26} + 2x_{27} \leq 2$
7. 2232 inequalities with 21 non-zero coefficients (sixteen 1's and five 2's) and right-hand side 3
   $2x_7 + 2x_8 + 2x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{21} + x_{22} + x_{24} + 2x_{25} + 2x_{27} \leq 3$

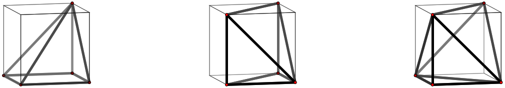
Further Reading

- Günter M. Ziegler
  Lectures on Polytopes
  Springer-Verlag, revised edition 1998.
- Branko Grünbaum
  Convex Polytopes
  Edited by V. Kaibel, V. Klee, G. M. Ziegler,
  Springer-Verlag, 2004
- Alexander Schrijver
  Theory of Linear and Integer Programming
  John Wiley & Sons, 1986
- Martin Grötschel, László Lovász, Alexander Schrijver
  Geometric Algorithms and Combinatorial Optimization
  Springer-Verlag, 2nd ed., 1993

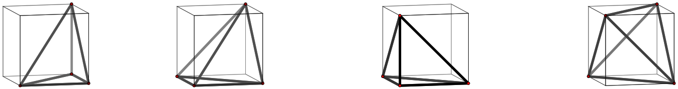
Solution Exercise 3 – Part I

$n = 8$:  $n = 7$: 

$n = 6$: 

$n = 5$: 

the first two are equivalent.



$n = 4$: all equivalent

Solution to Exercise 3 – Part II