Presolve and Cutting Planes

Presolve and Cutting Planes

“Tighter” formulations

- Original MIP formulation can almost always be improved
  - **Fewer constraints and variables**
    - Less data to process
  - **Smaller difference between space of feasible continuous and feasible integer solutions**
- Two techniques:
  - **Presolve and cutting planes**

Model Reformulation

"Tighten" formulation

- **Similar steps in both cases:**
  - **Add/replace constraints in model to tighten formulation**
    - Same integer solutions
    - Fewer continuous solutions
- **Important difference:**
  - **Presolve is applied to the original model to create a new model**
  - **Cutting planes are added to an existing model (typically the presolved model) to cut off a relaxation solution**
  - **More presolve almost always helps. Too many cutting planes can hurt.**

Rounding, Lifting, and Disjunction

- Three powerful, widely used concepts in presolve and cutting planes:
- Rounding
  - Integer multiples of integer variables take integer values
- Lifting
  - Fixing a binary variable at a bound may cause a constraint to go slack
- Disjunction
  - Binary variable must take one of two values

Rounding

Simplest Form of Rounding

Rounding in presolve

- **A fractional bound on an integer variable can be truncated:**
  - **$x \leq 1.5$ implies $x \leq 1$**
- **Effects can become non-trivial when combined with bound strengthening:**
  - **$x + 2y + 4z = 4$, all variables binary**
  - **Bound strengthening and rounding together yield:**
    - $4z \geq 4 - \sup(x+2y)$; $z \geq 1/4$; $z >= 1$
    - $x=0$, $y=0$, $z=1$

GCD Reduction

More rounding in presolve

- Given a constraint involving all integer variables with integer coefficients
  - $\sum a_j x_j \leq b$
- Divide through by GCD of coefficients (g)
  - $\sum (a_j/g) x_j \leq \lfloor b/g \rfloor$
- LHS is integral, so RHS can be truncated
- Example:
  - $3x + 6y + 9z \leq 11$

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Lifting

Coefficient Reduction

Lifting in presolve

- Given a constraint involving some binary x_k:
  - $\sum a_j x_j \geq b$
- Will fixing $x_k=1$ cause constraint to go slack?
  - $a_k + \inf (\sum_{j!=k} a_j x_j) > b$?
  - $s = a_k + \inf (\sum_{j!=k} a_j x_j) - b > 0$
- If so, we can subtract the following from LHS:
  - $s x_k$
- Example:
  - $2x + y \geq 1$ becomes
  - $x + y \geq 1$

Cover (Knapsack) Cuts

- **0-1 Knapsack**

  $K = \{x \in \{0,1\}^N : \sum_{j \in N} a_j x_j \leq b\}$, **with** $a_j > 0$ **and** $b > 0$

- **The set** $C \subseteq N$ **is called a *cover* if**

  $\sum_{j \in C} a_j x_j > b$

- **The cover inequality**

  $\sum_{j \in C} x_j \leq |C| - 1$

  is valid for K

Cover+Lifting: 0-1 Knapsack

- Consider

$5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 + 8x_6 \leq 17$

- Cover inequality (in fact, this cover is minimal)

$x_1 + x_2 + x_3 + x_4 \leq 3$

- Lifting x_5 first, then x_6

$x_1 + x_2 + x_3 + x_4 + \pi_5 x_5 \leq 3$

$\pi_5 = 3 - \max \{x_1 + x_2 + x_3 + x_4 : x_5 = 1\} = 1$

Similarly, $\pi_6 = 1$, so the lifted cover is

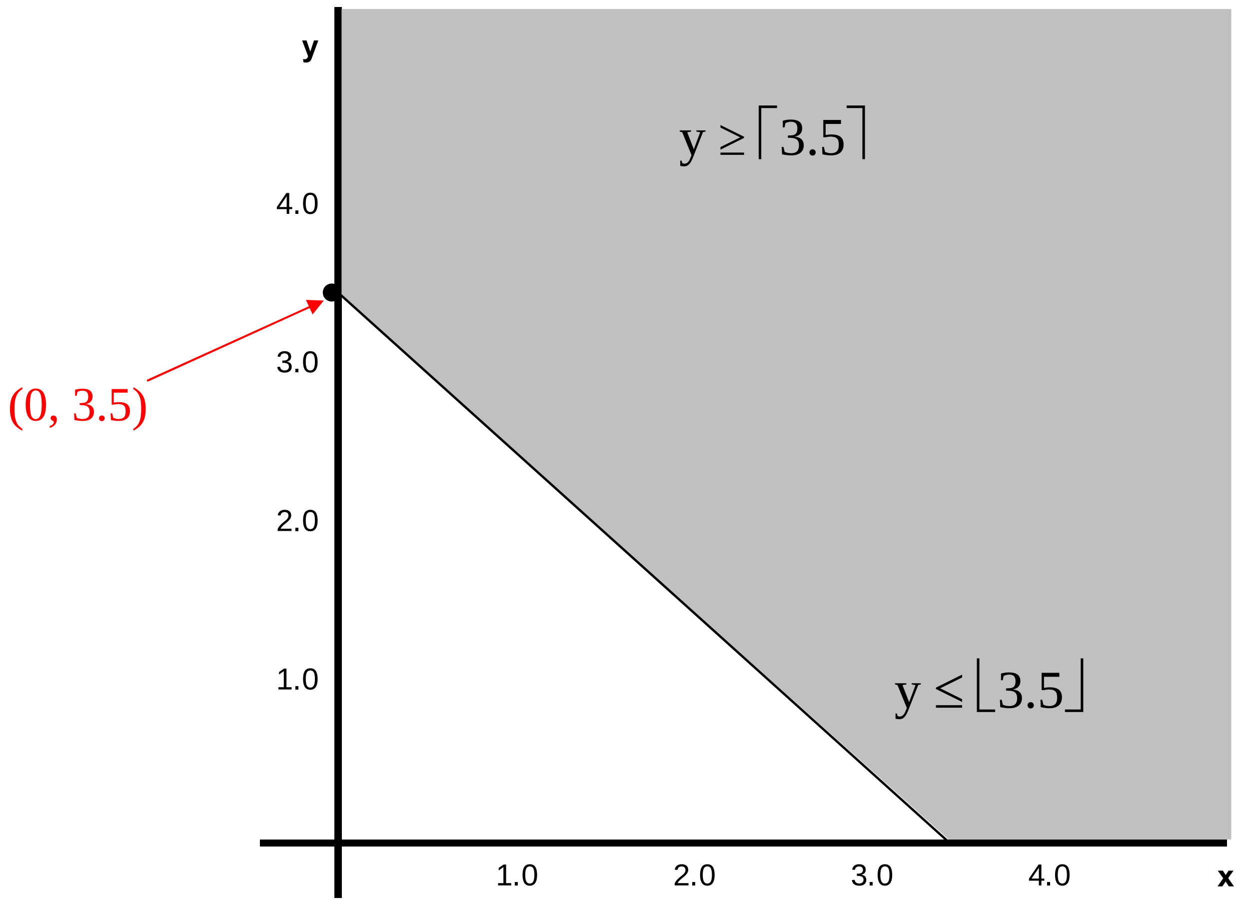
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$

- Lifting x_6 first, then x_5, then the lifted cover is

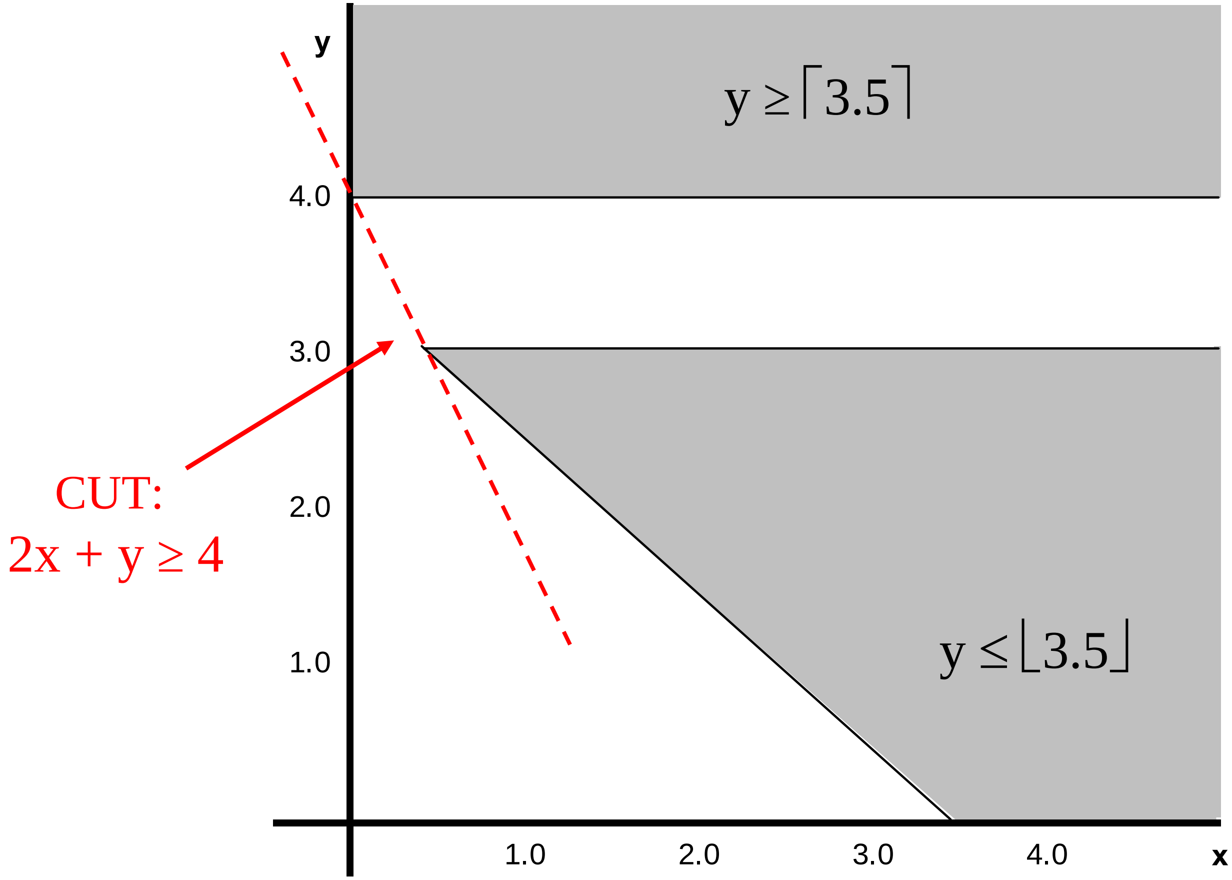
$x_1 + x_2 + x_3 + x_4 + 2x_6 \leq 3$

Disjunction

$x + y \geq 3.5, \quad x, y \geq 0, \quad y$ integral



$x + y \geq 3.5, \quad x, y \geq 0, \quad y$ integral



Gomory Mixed Cut

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- Given $y, x_j \in Z_+$, and

  $$y + \sum a_{ij}x_j = d = \lfloor d \rfloor + f, \quad f > 0$$

- Rounding: Where $a_{ij} = \lfloor a_{ij} \rfloor + f_j$, define

  $$t = y + \sum(\lfloor a_{ij} \rfloor x_j : f_j \leq f) + \sum(\lceil a_{ij} \rceil x_j : f_j > f) \in Z$$

- Then

  $$\sum(f_j\, x_j : f_j \leq f) + \sum(f_j - 1)x_j : f_j > f) = d - t$$

- Disjunction:

  $$t \leq \lfloor d \rfloor \Rightarrow \sum(f_j x_j : f_j \leq f) \geq f$$

  $$t \geq \lceil d \rceil \Rightarrow \sum((1-f_j)x_j : f_j > f) \geq 1-f$$

- Combining:

  $$\sum((f_j/f)x_j : f_j \leq f) + \sum([(1-f_j)/(1-f)]x_j : f_j > f) \geq 1$$

Computing Gomory Mixed Cuts

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1. Make a an ordered list of "sufficiently" fractional variables.
2. Take the first 100. Compute corresponding tableau rows. Reject if coeff. range too big.
3. Add to LP.
4. Repeat twice.
5. Computed only at root. Slack cuts purged at end of root computation.

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Cutting Plane Summary

Sample CPLEX Output

Default settings

```
                Nodes                                             Cuts/
         Node   Left      Objective   IInf   Best Integer       Best Node     ItCnt       Gap

            0      0      4533.5033     40                      4533.5033       125
                          8517.6222     29                     Cuts:  100       236
*          0+      0                     0      9715.0000       8517.6222       236    12.33%
                          8651.9219     10      9715.0000       Cuts:  51       266    10.94%
*          0+      0                     0      8701.0000       8651.9219       266     0.56%
                          8662.8458      4      8701.0000        Cuts:  7       273     0.44%
                          8665.4678      7      8701.0000      Covers:  2       276     0.41%
                          8667.9363      7      8701.0000      Covers:  1       278     0.38%
*           4      3                     0      8691.0000       8688.0000       282     0.03%

GUB cover cuts applied:  23
Clique cuts applied:  10
Cover cuts applied:  31
Implied bound cuts applied:  1
Gomory fractional cuts applied: 30
```

Progress: MIP

Computational Results III: 106 Models

Which Single Feature Helps Most?

(CPLEX 8.0 < 1000 seconds, 5.0 unsolvable)

▪ Cuts	53.7x
▪ Presolve	10.8x
▪ CPLEX 5.0 presolve	3.1x
▪ CPLEX 5.0 var. selection	2.9x
▪ No heuristics	1.4x
▪ No node presolve	1.3x

Progress: MIP

Computational Results IV: 106 Models

Removing Single Cuts

• Gomory mixed-integer	2.52x
• Mixed-intger rounding	1.83x
• Knapsack cover	1.40x
• Flow cover	1.22x
• Implied bound	1.19x
• Path	1.04x
• Clique	1.02x
• GUB cover	1.02x
• Disjunctive	0.53x