Solving Linear and Integer Programs

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Outline

- **Linear Programming:**
  - Introduction to basic LP, including duality
  - Primal and dual simplex algorithms
  - Computational progress in linear programming
  - Implementing the dual simplex algorithm
- **Mixed-Integer Programming:**

Some Basic Theory

Linear Program – Definition

A **linear program** (**LP**) in **standard form** is an optimization problem of the form

$$\begin{array}{lll} \textit{Minimize} & c^T x & \\ \textit{Subject to} & Ax = b & \text{(P)} \\ & x \geq 0 & \end{array}$$

Where $c \in \boldsymbol{R}^n$, $b \in \boldsymbol{R}^m$, $A \in \boldsymbol{R}^{m \times n}$, and x is a vector of n **variables**. $c^T x$ is known as the **objective function**, $Ax=b$ as the **constraints**, and $x \geq 0$ as the **nonnegativity** conditions. b is called the **right-hand side**.

Dual Linear Program – Definition

The **dual (**or **adjoint) linear program** corresponding to (P) is the optimization problem

$$\begin{array}{lll} \textit{Maximize} & & b^T\pi \\ \textit{Subject to} & A^T\pi \leq c & \\ & \pi \ \textit{free} & \end{array} \qquad \text{(D)}$$

In this context, (P) is referred to as the **primal linear program**.

$$\left(\begin{array}{lll} & \textbf{Primal} & \\ \textit{Minimize} & & c^T x \\ \textit{Subject to} & Ax = b & \\ & x \geq 0 & \end{array} \right)$$

Weak Duality Theorem

(von Neumann 1947)

Let x be feasible for (P) and π feasible for (D). Then

Maximize $\quad b^T\pi \le c^Tx \quad$ Minimize

If $b^T\pi = c^Tx$, then x is optimal for (P) and π is optimal for (D); moreover, if either (P) or (D) is **unbounded**, then the other problem is **infeasible**.

Proof:

$$\pi^Tb \;=\; \pi^TAx \;\le\; c^Tx \quad \blacksquare$$

$Ax = b$

$\pi^TA \le c^T \;\&\; x \ge 0$

Solving Linear Programs

- **Three types of algorithms are available**
  - Primal simplex algorithms (Dantzig 1947)
  - Dual simplex algorithms (Lemke 1954)
    - Developed in context of game theory
  - Primal-dual log barrier algorithms
    - Interior-point algorithms (Karmarkar 1989)
    - Reference: Primal-Dual Interior Point Methods, S. Wright, 1997, SIAM

Primary focus: Dual simplex algorithms

Basic Solutions – Definition

Let B be an ordered set of m distinct indices $(B_1,\ldots,B_m)$ taken from $\{1,\ldots,n\}$. B is called a **basis** for (P) if A_B is nonsingular. The variables x_B are known as the **basic variables** and the variables x_N as the **non-basic** variables, where $N = \{1,\ldots,n\}\backslash B$. The corresponding **basic solution** $X \in \boldsymbol{R}^n$ is given by $X_N = 0$ and $X_B = A_B^{-1} b$. B is called (**primal**) **feasible** if $X_B \geq 0$.

Note: $AX = b \Rightarrow A_B X_B + A_N X_N = b \Rightarrow A_B X_B = b \Rightarrow X_B = A_B^{-1} b$

Primal Simplex Algorithm
(Dantzig, 1947)

Input: A feasible basis B and vectors

$$X_B = A_B^{-1}b \quad \text{and} \quad D_N = c_N - A_N^T A_B^{-T} c_B.$$

- **Step 1:** (Pricing) If $D_N \geq 0$, stop, B is optimal; else let
$$j = argmin\{D_k : k \in N\}.$$
- **Step 2:** (FTRAN) Solve $A_B y = A_j$.
- **Step 3:** (Ratio test) If $y \leq 0$, stop, (P) is unbounded; else, let
$$i = argmin\{X_{Bk}/y_k : y_k > 0\}.$$
- **Step 4:** (BTRAN) Solve $A_B^T z = e_i$.
- **Step 5:** (Update) Compute $\alpha_N = -A_N^T z$. Let $B_i = j$. Update X_B (using y) and D_N (using α_N)

Note: x_j is called the **entering** variable and x_{Bi} the **leaving** variable. The D_N values are known as **reduced costs** – like partial derivatives of the objective function relative to the nonbasic variables.

Primal Simplex Example

The Simplex Algorithm

Consider the following simple LP:

$$
\begin{aligned}
\text{Maximize} \quad & 3x_1 + 2x_2 + 2x_3 \\
\text{Subject to} \quad & x_1 + + x_3 \leq 8 \\
& x_1 + x_2 \leq 7 \\
& x_1 + 2x_2 \leq 12 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

The Primal Simplex Algorithm

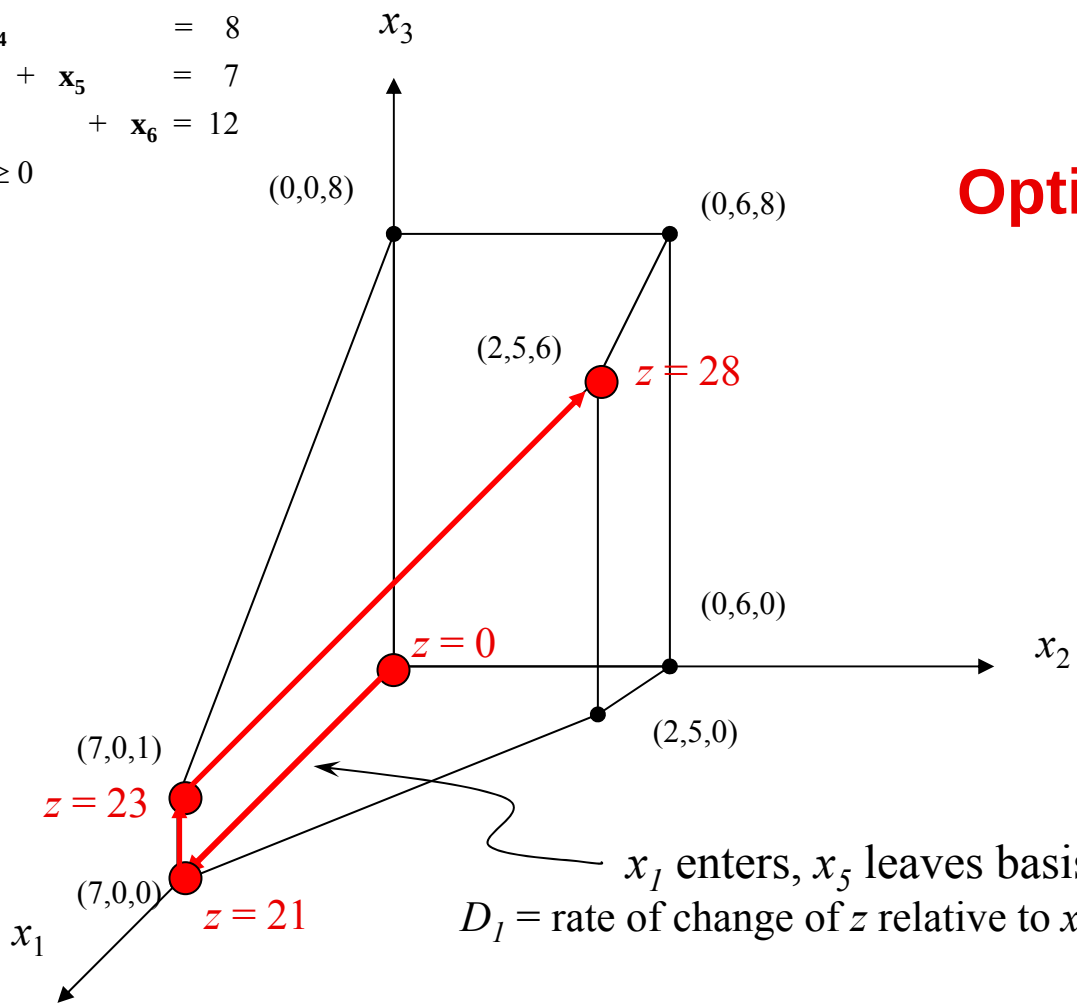
Maximize $z = 3x_1 + 2x_2 + 2x_3$

Add slacks: Initial basis B = (4,5,6)

Maximize $3x_1 + 2x_2 + 2x_3 + 0\mathbf{x_4} + 0\mathbf{x_5} + 0\mathbf{x_6}$

Subject to

$$x_1 + x_3 + \mathbf{x_4} = 8$$
$$x_1 + x_2 + \mathbf{x_5} = 7$$
$$x_1 + 2x_2 + \mathbf{x_6} = 12$$
$$x_1, x_2, x_3, \mathbf{x_4}, \mathbf{x_5}, \mathbf{x_6} \geq 0$$



Optimal

x_1 enters, x_5 leaves basis

D_1 = rate of change of z relative to x_1 = 21/7=3

Dual Simple Algorithm – Setup

Simplex algorithms apply to problems with constraints in equality form. We convert (D) to this form by adding the dual **slacks** d:

$$\begin{aligned} &\textit{Maximize} && b^T\pi \\ &\textit{Subject to} && A^T\pi + d = c \\ & && \pi \text{ free}, d \geq 0 \end{aligned} \qquad \boxed{\Leftrightarrow A^T\pi \leq c}$$

Dual Simple Algorithm – Setup

$$\begin{aligned} &\textit{Maximize} && b^T\pi \\ &\textit{Subject to} && A^T\pi + d = c \\ & && \pi \textit{ free}, d \geq 0 \end{aligned}$$

$$\begin{pmatrix} A_B{}^T & I_B & 0 \\ A_N{}^T & 0 & I_N \end{pmatrix} \begin{pmatrix} \pi \\ d_B \\ d_N \end{pmatrix} = \begin{pmatrix} c_B \\ \\ c_N \end{pmatrix}$$

Given a basis B, the corresponding **dual basic variables** are π and d_N. d_B are the **nonbasic variables**. The corresponding **dual basic solution** Π, D is determined as follows:

$$D_B = 0 \Rightarrow \Pi = A_B{}^{-T} c_B \Rightarrow D_N = c_N - A_N{}^T \Pi$$

B is **dual feasible** if $D_N \geq 0$.

Dual Simple Algorithm – Setup

$$\begin{aligned} &\textit{Maximize} && b^T\pi \\ &\textit{Subject to} && A^T\pi + d = c \\ & && \pi \textit{ free},\ d \geq 0 \end{aligned}$$

$$\begin{pmatrix} A_B{}^T & I_B & 0 \\ A_N{}^T & 0 & I_N \end{pmatrix} \begin{pmatrix} \pi \\ d_B \\ d_N \end{pmatrix} = \begin{pmatrix} c_B \\ \\ c_N \end{pmatrix}$$

Observation: We may assume that every dual basis has the above form.

Proof: Assuming that the primal has a basis is equivalent to assuming that *rank(A)=m* (# of rows), and this implies that all π variables can be assumed to be basic. ■

This observation establishes a 1-1 correspondence between primal and dual bases.

An Important Fact

If X and Π, D are corresponding primal and dual basic solutions determined by a basis B, then

$$\Pi^T b = c^T X.$$

Hence, by weak duality, if B is both primal and dual feasible, then X is optimal for (P) and Π is optimal for (D).

Proof:

$$\begin{aligned} c^T X &= c_B{}^T X_B && \text{(since } X_N = 0\text{)} \\ &= \Pi^T A_B X_B && \text{(since } \Pi = A_B{}^{-T} c_B\text{)} \\ &= \Pi^T b && \text{(since } A_B X_B = b\text{)} \end{aligned}$$

■

Dual Simplex Algorithm
(Lemke, 1954)

Input: A dual feasible basis B and vectors

$$X_B = A_B^{-1} b \quad \text{and} \quad D_N = c_N - A_N^T B^{-T} c_B.$$

- **Step 1:** (Pricing) If $X_B \geq 0$, stop, B is optimal; else let

$$i = argmin\{X_{Bk} : k \in \{1,\ldots,m\}\}.$$

- **Step 2:** (BTRAN) Solve $B^T z = e_i$. Compute $\alpha_N = -A_N^T z$.
- **Step 3:** (Ratio test) If $\alpha_N \leq 0$, stop, (D) is unbounded; else, let

$$j = argmin\{D_k/\alpha_k : \alpha_k > 0\}.$$

- **Step 4:** (FTRAN) Solve $A_B y = A_j$.
- **Step 5:** (Update) Set $B_i = j$. Update X_B (using y) and D_N (using α_N)

Note: d_{Bi} is the **entering** variable and d_j is the **leaving** variable. (Expressed in terms of the primal: x_{Bi} is the leaving variable and x_j is the entering variable)

Simplex Algorithms

Input: A primal feasible basis B and vectors

$X_B = A_B^{-1} b$ & $D_N = c_N - A_N^T A_B^{-T} c_B$.

- **Step 1:** (Pricing) If $D_N \geq 0$, stop, B is optimal; else, let

  $j = argmin\{D_k : k \in N\}$.
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Input: A dual feasible basis B and vectors

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Correctness: Dual Simplex Algorithm

- **Termination criteria**
  - Optimality **(DONE – by "An Important Fact" !!!)**
  - Unboundedness
- **Other issues**
  - Finding starting dual feasible basis, or showing that no feasible solution exists
  - Input conditions are preserved (i.e., that B is still a feasible basis)
  - Finiteness

Summary:
What we have done and what we have to do

- **Done**
  - Defined primal and dual linear programs
  - Proved the weak duality theorem
  - Introduced the concept of a basis
  - Stated primal and dual simplex algorithms
- **To do (for dual simplex algorithm)**
  - Show correctness
  - Describe key implementation ideas
  - Motivation

Dual Unboundedness
(⇒ primal infeasible)

- We carry out a key calculation
- As noted earlier, in an iteration of the dual

d_{Bi} enters basis
d_j leaves basis

in

Maximize $b^T\pi$
Subject to $A^T\pi + d = c$
π *free,* $d \geq 0$

- **The idea:** Currently $\boldsymbol{d_{Bi} = 0}$, and $\boldsymbol{X_{Bi} < 0}$ has motivated us to increase $\boldsymbol{d_{Bi}}$ to $\boldsymbol{\theta > 0}$, leaving the other components of $\boldsymbol{d_B}$ at $\boldsymbol{0}$ (the object being to increase the objective). Letting $\underline{d}, \underline{\pi}$ be the corresponding dual solution as a function of θ, we obtain

$$\underline{\boldsymbol{d}}_{\boldsymbol{B}} = \boldsymbol{\theta e_i} \qquad \underline{\boldsymbol{\pi}} = \boldsymbol{\Pi - \theta z} \qquad \underline{\boldsymbol{d}}_{\boldsymbol{N}} = \boldsymbol{D_N - \theta \alpha_N}$$

where $\boldsymbol{\alpha_N}$ and $\boldsymbol{z}$ are as computed in the algorithm.

(Dual Unboundedness – cont.)

- Letting $\underline{d}, \underline{\pi}$ be the corresponding dual solution as a function of θ. Using α_N and z from dual algorithm,

$$\underline{d}_B = \theta e_i \quad \underline{d}_N = D_N - \theta \alpha_N \quad \underline{\pi} = \pi - \theta z.$$

- Using $\theta > 0$ and $X_{Bi} < 0$ yields

$$\begin{aligned} new\_objective = \underline{\pi}^T b &= (\pi - \theta z)^T b \\ &= \pi^T b - \theta X_{Bi} \\ &= old\_objective - \theta X_{Bi} > old\_objective \end{aligned}$$

- **Conclusion 1:** If $\alpha_N \leq 0$, then $\underline{d}_N \geq 0 \ \forall \ \theta > 0 \Rightarrow$ (D) is unbounded.

- **Conclusion 2:** If α_N **not**≤ 0, then

$$\underline{d}_N \geq 0 \Rightarrow \theta \leq D_j / \alpha_j \ \forall \ \alpha_j > 0$$

$$\Rightarrow \theta_{max} = min\{D_j / \alpha_j : \alpha_j > 0\}$$

(Dual Unboundedness – cont.)

- **Finiteness:** If $\boldsymbol{D_B > 0}$ for all dual feasible bases B, then the dual simplex algorithm is finite: The dual objective strictly increases at each iteration $\Rightarrow$ no basis repeats, and there are a finite number of bases.
- There are various approaches to guaranteeing finiteness in general:
  - **Bland's Rules:** Purely combinatorial, bad in practice.
  - **CPLEX:** A perturbation is introduced to guarantee $\boldsymbol{D_B > 0}$.

Computational History of Linear Programming

“A certain wide class of practical problems appears to be just beyond the range of modern computing machinery. These problems occur in everyday life; they run the gamut from some very simple situations that confront an individual to those connected with the national economy as a whole. Typically, these problems involve a complex of different activities in which one wishes to know which activities to emphasize in order to carry out desired objectives under known limitations.”

George B. Dantzig, 1948

Application of LP & MIP - I

- **Transportation-airlines**
  - Fleet assignment
  - Crew scheduling
  - Ground personnel scheduling
  - Yield management
  - Fuel allocation
  - Passenger mix
  - Booking control
  - Maintenance scheduling
  - Load balancing/freight packing
  - Airport traffic planning
  - Gate scheduling/assignment
  - Upset recover and management
- **Transportation-other**
  - Vehicle routing
  - Freight vehicle scheduling and assignment
  - Depot/warehouse location
  - Freight vehicle packing
  - Public transportation system operation
  - Rental car fleet management
- **Process industries**
  - Plant production scheduling and logistics
  - Capacity expansion planning
  - Pipeline transportation planning
  - Gasoline and chemical blending

Application of LP & MIP - II

- **Financial**
  - Portfolio selection and optimization
  - Cash management
  - Synthetic option development
  - Lease analysis
  - Capital budgeting and rationing
  - Bank financial planning
  - Accounting allocations
  - Securities industry surveillance
  - Audit staff planning
  - Assets/liabilities management
  - Unit costing
  - Financial valuation
  - Bank shift scheduling
  - Consumer credit delinquency management
  - Check clearing systems
  - Municipal bond bidding
  - Stock exchange operations
  - Debt financing
- **Manufacturing**
  - Product mix planning
  - Blending
  - Manufacturing scheduling
  - Inventory management
  - Job scheduling
  - Personnel scheduling
  - Maintenance scheduling and planning
  - Steel production scheduling
- **Coal Industry**
  - Coal sourcing/transportation logistics
  - Coal blending
  - Mining operations management
- **Forestry**
  - Forest land management
  - Forest valuation models
  - Planting and harvesting models

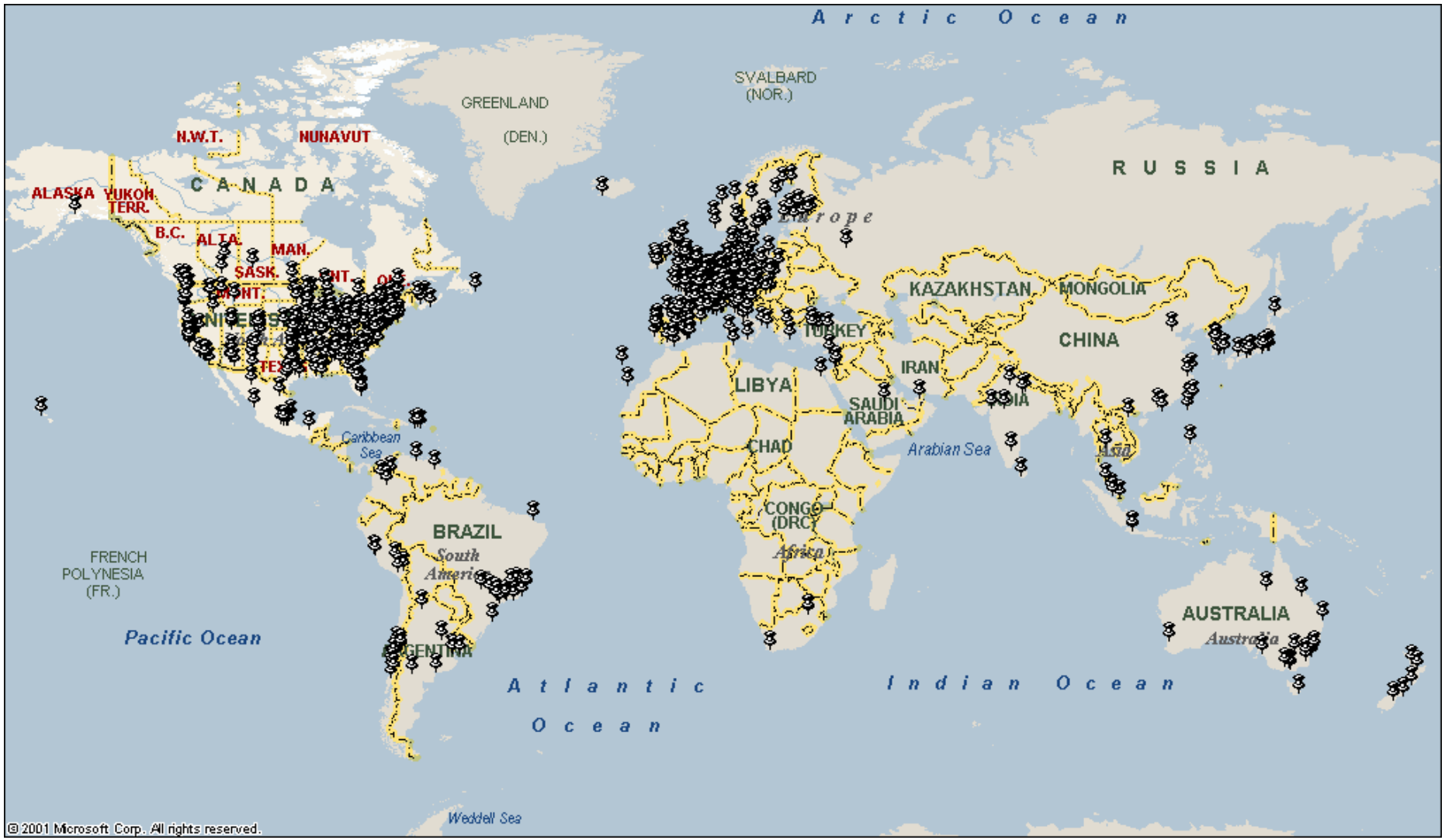
Application of LP & MIP - III

- **Agriculture**
  - Production planning
  - Farm land management
  - Agricultural pricing models
  - Crop and product mix decision models
  - Product distribution
- **Public utilities and natural resources**
  - Electric power distribution
  - Power generator scheduling
  - Power tariff rate determination
  - Natural gas distribution planning
  - Natural gas pipeline transportation
  - Water resource management
  - Alternative water supply evaluation
  - Water reservoir management
  - Public water transportation models
  - Mining excavation models
- **Oil and gas exploration and production**
  - Oil and gas production scheduling
  - Natural gas transportation scheduling
- **Communications and computing**
  - Circuit board (VLSI) layout
  - Logical circuit design
  - Magnetic field design
  - Complex computer graphics
  - Curve fitting
  - Virtual reality systems
  - Computer system capacity planning
  - Office automation
  - Multiprocessor scheduling
  - Telecommunications scheduling
  - Telephone operator scheduling
  - Telemarketing site selection

Application of LP & MIP - IV

- **Food processing**
  - Food blending
  - Recipe optimization
  - Food transportation logistics
  - Food manufacturing logistics and scheduling
- **Health care**
  - Hospital staff scheduling
  - Hospital layout
  - Health cost reimbursement
  - Ambulance scheduling
  - Radiation exposure models
- **Pulp and paper industry**
  - Inventory planning
  - Trim loss minimization
  - Waste water recycling
  - Transportation planning
- **Textile industry**
  - Pattern layout and cutting optimization
  - Production scheduling
- **Government and military**
  - Post office scheduling and planning
  - Military logistics
  - Target assignment
  - Missile detection
  - Manpower deployment
- **Miscellaneous applications**
  - Advertising mix/media scheduling
  - Pollution control models
  - Sales region definition
  - Sales force deployment

CPLEX Across the World



1194 Cities – Excluding ISV Deployments

LP History

- **George Dantzig, 1947**
  - Introduced LP and recognized it as more than a conceptual tool: Computing answers important.
  - Invented "primal" simplex algorithm.
  - First LP solved: Laderman, 9 cons., 77 vars., 120 MAN-DAYS.
- **First computer code – 1951**
- **LP used commercially – Early 60s**
- **Powerful mainframe codes introduced – Early 70s**
- **Computational progress stagnated – Mid 80s**
- **Remarkable progress last 15 years (PCs, new computer science and mathematics)**
  - We now have three algorithms: Primal & Dual Simplex, Barrier

Example: A Production Planning Model

401,640 constraints 1,584,000 variables

Solution time line (2.0 GHz P4)**:**

- 1988 (CPLEX 1.0): Houston, 13 Nov 2002

Example: A Production Planning Model

401,640 constraints 1,584,000 variables

Solution time line (2.0 GHz P4)**:**

- 1988 (CPLEX 1.0): 8.0 days (Berlin, 21 Nov)

Example: A Production Planning Model

401,640 constraints 1,584,000 variables

Solution time line (2.0 GHz P4)**:**

- 1988 (CPLEX 1.0): 15.0 days (Dagstuhl, 28 Nov)

Example: A Production Planning Model

401,640 constraints 1,584,000 variables

Solution time line (2.0 GHz P4)**:**

- 1988 (CPLEX 1.0): 19.0 days (Amsterdam, 2 Dec)

Example: A Production Planning Model

401,640 constraints 1,584,000 variables

Solution time line (2.0 GHz P4)**:**

- 1988 (CPLEX 1.0): 23.0 days (Houston, 6 Dec)

Example: A Production Planning Model

401,640 constraints 1,584,000 variables

Solution time line (2.0 GHz P4)**:**

- 1988 (CPLEX 1.0): 29.8 days
- 1997 (CPLEX 5.0): 1.5 hours
- 2002 (CPLEX 8.0): 86.7 seconds
- 2003 (February): 59.1 seconds

Speedup: >43500x

BIG TEST: The testing methodology

- Not possible for one test to cover 10+ years: Combined several tests.
- The biggest single test:
  - **Assembled 680 real LPs**
  - Test runs: Using a time limit **(4 days per LP) two chosen methods would be compared as follows:**
    - Run method 1: Generate 680 solve times
    - Run method 2: Generate 680 solve times
    - Compute 680 ratios and form **GEOMETRIC MEAN** (not arithmetic mean!)

(part of a) BIG Test Set

Model	Rows	Cols	NZs
M1	16223	28568	88340
M2	16768	39474	203112
M3	17681	165188	690273
M4	18262	23211	136324
M5	19103	33490	276895
M6	19374	180670	5392558
M7	19519	45832	124280
M8	19844	55528	152952
M9	19999	85191	170369
M10	21019	115761	728432
M11	22513	99785	337746
M12	22797	63995	172018
M13	23610	44063	154822
M14	23700	23005	169045
M15	23712	31680	81245
M16	24377	46592	2139096
M17	26618	38904	1067713
M18	27349	97710	288421
M19	27441	15128	96118
M20	27899	26243	261968
M21	28240	55200	161640
M22	28420	164024	505253
M23	29002	111722	2632880
M24	29017	20074	2001102
M25	29147	9984	1013168
M26	29724	98124	196524
M27	30190	57000	623730

Model	Rows	Cols	NZs
M28	31770	272372	829040
M29	33440	56624	161831
M30	34994	87510	208179
M31	35519	43582	557466
M32	35645	34675	208769
M33	36400	92878	246006
M34	38782	261079	1508199
M35	39951	125000	381259
M36	41340	64162	370839
M37	41344	163569	1928534
M38	41366	78750	2110518
M39	43387	107164	189864
M40	43687	164831	722066
M41	44150	200077	4966017
M42	44211	37199	321663
M43	47423	81915	228565
M44	48548	163200	617683
M45	54447	326504	1807146
M46	55020	117910	391081
M47	55463	191233	840986
M48	60384	100078	485414
M49	63856	144693	717229
M50	66185	157496	418321
M51	67745	111891	305125
M52	69418	612608	1722112
M53	84840	316800	1899600
M54	95011	197489	749771

Model	Rows	Cols	NZs
M55	99578	326504	2102273
M56	105127	154699	358171
M57	108393	112955	602948
M58	118158	487427	974854
M59	123964	93288	459680
M60	125211	159109	457198
M61	129181	467192	1025706
M62	155265	377918	930166
M63	175147	358239	1211488
M64	179080	707556	1570514
M65	185929	189867	2787708
M66	186441	23732	397080
M67	209760	363092	1061495
M68	269640	1205640	6481640
M69	280756	920198	5936426
M70	319256	638512	1231403
M71	344297	559428	1909649
M72	589250	1533590	5327318
M73	716772	1169910	2511088
M74	1000000	1685236	3370472
M75	1128152	1587664	4496138
M76	1285665	1313795	4998121
M77	1372333	2627821	13321433
M78	1709857	1903725	4959650
M79	5034171	7365337	25596099
M80	5822606	15228380	30960543
M81	6662791	9781747	34053329
M82	10810259	19916187	49228262

LP Progress: 1988 – Present

(Operations Research, Jan 2002, pp. 3—15)

- Algorithms (*machine independent*):

  Primal *versus* best of **Primal/Dual/Barrier** **3300x**

- Machines (workstations →PCs): **1600x**

- **NET: Algorithm × Machine** **5 300 000x**

  (2 months/5300000 ~= 1 second)

Algorithm comparison and other remarks …

(≥ 50000 rows)

- Dual simplex vs. primal: Dual 2.70x faster
- Dual simplex vs. barrier: Dual 1.06x faster