European Graduate Program
Berlin - Zürich

CGC Combinatorics

02M1 Lecture

The Travelling Salesman Problem and some Applications

Martin Grötschel

Block Course at TU Berlin

"Combinatorial Optimization at Work"

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Martin Grötschel
- Institut für Mathematik, Technische Universität Berlin (TUB)
- DFG-Forschungszentrum "Mathematik für Schlüsseltechnologien" (MATHEON)
- Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)

groetschel@zib.de http://www.zib.de/groetschel

Contents

1. Introduction
2. The TSP and some of its history
3. The TSP and some of its variants
4. Some applications
5. Heuristics
6. How combinatorial optimizers do it

Contents

Combinatorial optimization

Given a finite set E and a subset $\mathcal{I}$ of the power set of E (the set of feasible solutions). Given, moreover, a value (cost, length,…) c(e) for all elements e of E. Find, among all sets in $\mathcal{I}$, a set I such that its total value c(I) (= sum of the values of all elements in I) is as small (or as large) as possible.

The parameters of a combinatorial optimization problem are: (E, $\mathcal{I}$, c).

$$\min\left\{c(\mathrm{I})=\sum_{e\in\mathrm{I}}c(e)\mid \mathrm{I}\in\mathcal{I}\right\},\ \textit{where}\ \mathcal{I}\subseteq 2^{E}\ \textit{and}\ E\ \textit{finite}$$

Important issues:

- How is $\mathcal{I}$ given?
- What is the encoding length of an instance?
- How do we measure running time?

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Encoding and Running Times

Important issues:

- How is I given?
- What is the encoding length of an instance?
- How do we measure running time?

Special „simple“ combinatorial optimization problems

Finding a

- minimum spanning tree in a graph
- shortest path in a directed graph
- maximum matching in a graph
- a minimum capacity cut separating two given nodes of a graph or digraph
- cost-minimal flow through a network with capacities and costs on all edges
- …

These problems are solvable in polynomial time.

Is the number of feasible solutions relevant?

Special „hard“ combinatorial optimization problems

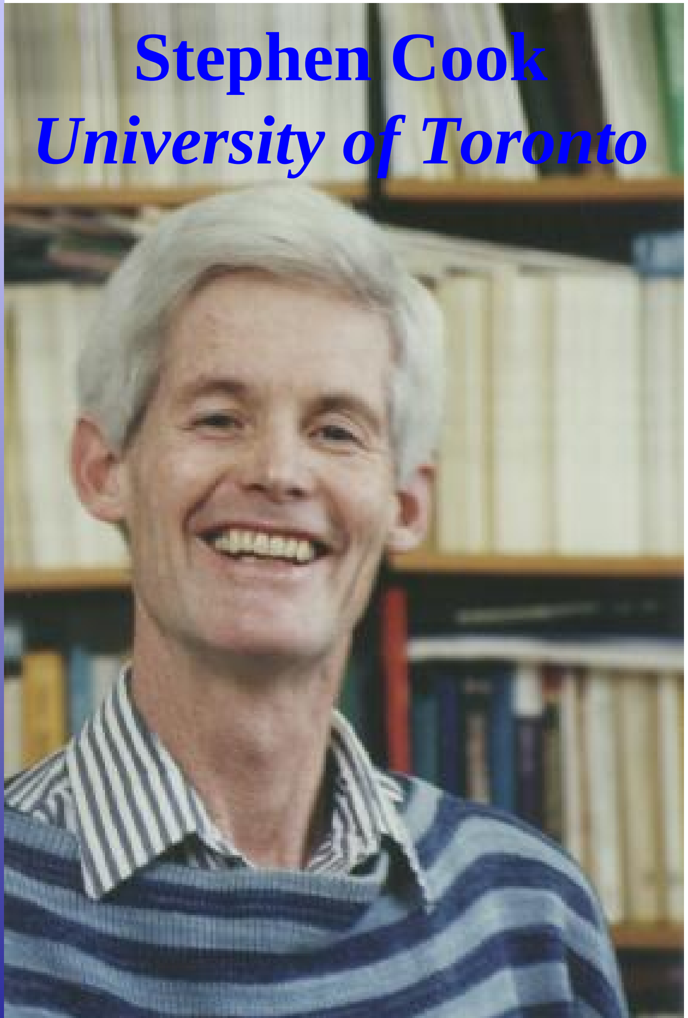
- travelling salesman problem (the prototype problem)
- location und routing
- set-packing, partitioning, -covering
- max-cut
- linear ordering
- scheduling (with a few exceptions)
- node and edge colouring
- …

These problems are NP-hard
(in the sense of complexity theory).

Complexity Theory

- Complexity theory came formally into being in the years 1965 – 1972 with the work of Cobham (1965), Edmonds(1965), Cook (1971), Karp(1972) and many others
- Of course, there were many forerunners (Gauss has written about the number of elementary steps in a computation, von Neumann, Gödel, Turing, Post,...).
- But modern complexity theory is a the result of the combined research efforts of many, in particular, of many computer scientists and mathematical programmers trying to understand the structures underlying computational processes.

Complexity Theory



1965 Polynomial time
Class P
Nondeterministic polynomial time
Class NP
Edmonds, Cobham

1971 Cook "The Complexity of Theorem Proving Procedures"
introduced the theory of
NP completeness

Hierarchies of complexity classes...

The most important open problem:

$P = NP$?

Clay Mathematics Institute

dedicated to increasing and disseminating mathematical knowledge

Millennium Prize Problems

Announcement

Rules for the CMI Millennium Prize Problems

Publication Guidelines

Historical Context

Press Statement

Press Reaction

P versus NP

The Hodge Conjecture

The Poincaré Conjecture

The Riemann Hypothesis

Yang-Mills Existence and Mass Gap

Navier-Stokes Existence and Smoothness

The Birch and Swinnerton-Dyer Conjecture

Prize:

1 million $
for a solution

Announced 16:00, on Wednesday, May 24, 2000
Collège de France

The first NP-complete Problem

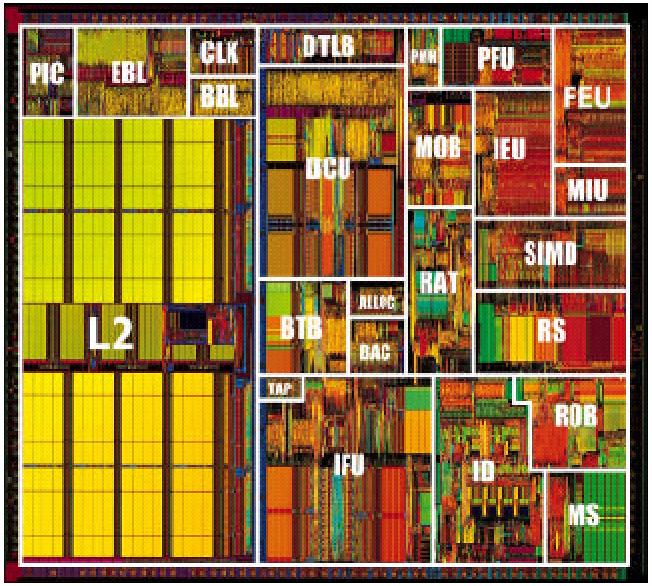
Satisfiability: Is there a truth assignmeent to the following formula:

$$(\neg x_1 \vee x_2) \wedge (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2)$$

Truly important Application:

Verification of computer chips and "systems on chips"

A design is correct iff a certain SAT formula associated with the chip has no truth assignment.



The travelling salesman problem

Given n „cities" and „distances" between them. Find a tour (roundtrip) through all cities visiting every city exactly once such that the sum of all distances travelled is as small as possible. (TSP)

The TSP is called symmetric (STSP) if, for every pair of cities i and j, the distance from i to j is the same as the one from j to i, otherwise the problem is called aysmmetric (ATSP).

The travelling salesman problem

Two mathematical formulations of the TSP

1. *Version*:

Let $K_n = (V, E)$ *be the complete graph (or digraph) with n nodes and let* c_e *be the length of* $e \in E$. *Let H be the set of all hamiltonian cycles (tours) in* K_n. *Find*

$$\min\{c(T) \mid T \in H\}.$$

2. *Version*:

Find a cyclic permutation π *of* $\{1, \dots, n\}$ *such that*

$$\sum_{i=1}^{n} c_{i\pi(i)}$$

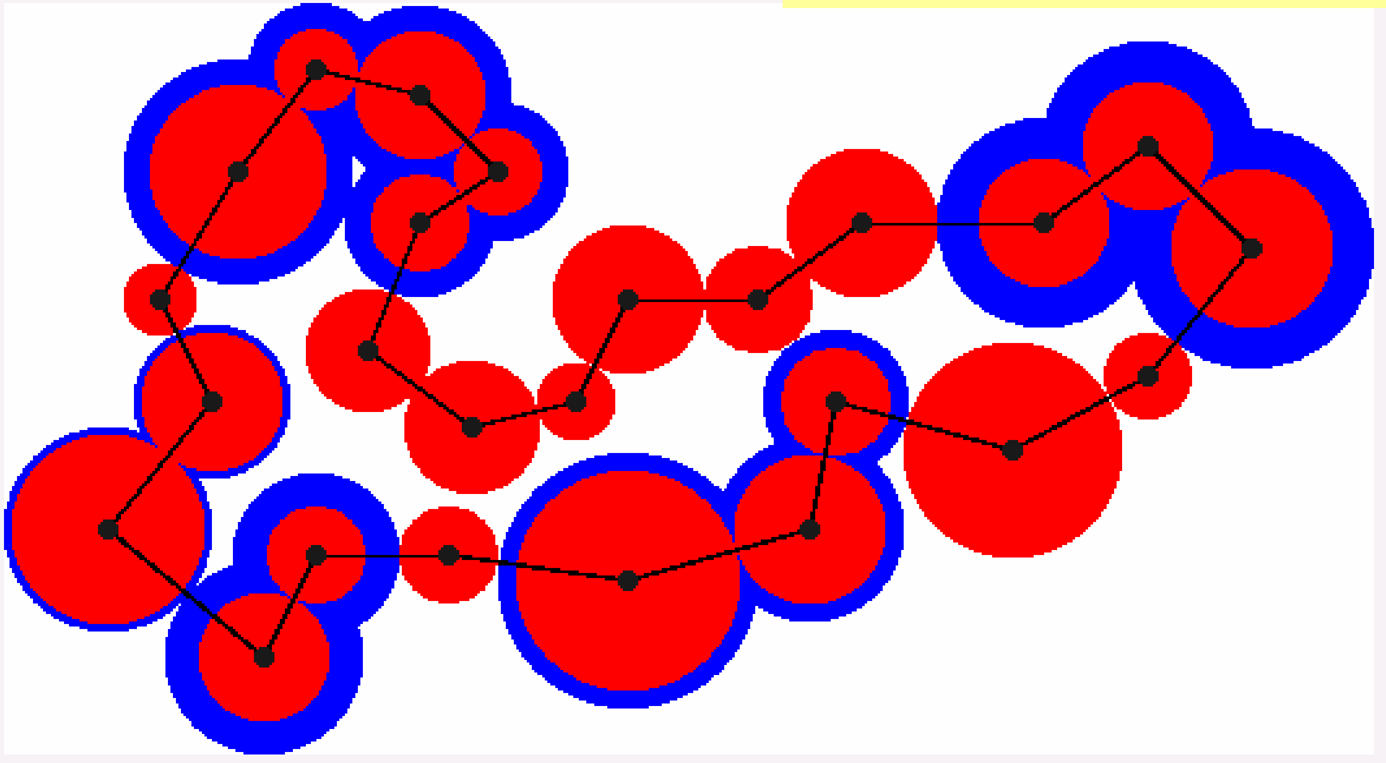
is as small as possible.

- Does that help solve the TSP?

The Problem

Given a collection of cities and the cost of travel between each pair of them, the **traveling salesman problem**, or **TSP** for short, is to find the cheapest way of visiting all of the cities and returning to your starting point. In the case we study, the travel costs are symmetric in the sense that traveling from city X to city Y costs just as much as traveling from Y to X.

http://www.tsp.gatech.edu/



The simplicity of the statement of the problem is deceptive -- the TSP is one of the most intensely studied problems in computational mathematics and yet no effective solution method is known for the general case. Indeed, the resolution of the TSP would settle the P versus NP problem and fetch a $1,000,000 prize from the Clay Mathematics Institute.

Contents

History of the TSP

Mathematical problems related to the traveling salesman problem were treated in the 1800s by the Irish mathematician Sir William Rowan Hamilton and by the British mathematician Thomas Penyngton Kirkman. The picture below is a photograph of Hamilton's Icosian Game that requires players to complete tours through the 20 points using only the specified connections. A nice discussion of the early work of Hamilton and Kirkman can be found in the book *Graph Theory 1736-1936* by N. L. Biggs, E. K. LLoyd, and R. J. Wilson, Clarendon Press, Oxford, 1976.

Usually quoted as the forerunner of the TSP →

Usually quoted as the origin of the TSP ↓

The general form of the TSP appears to be have been first studied by mathematicians starting in the 1930s by Karl Menger in Vienna and Harvard. The problem was later promoted by Hassler Whitney and Merrill Flood at Princeton. A detailed treatment of the connection between Menger and Whitney, and the growth of the TSP as a topic of study can be found in Alexander Schrijver's paper ``On the history of combinatorial optimization (till 1960)''.

about 100 years earlier

Der

Handlungsreisende

wie er sein soll

und was er zu thun hat, um Aufträge zu erhalten und eines glücklichen Erfolgs in seinen Geschäften gewiß zu sein.

Von

einem alten Commis - Voyageur.

Ilmenau 1832,

Druck und Verlag von B. Fr. Voigt.

From the Commis-Voyageur

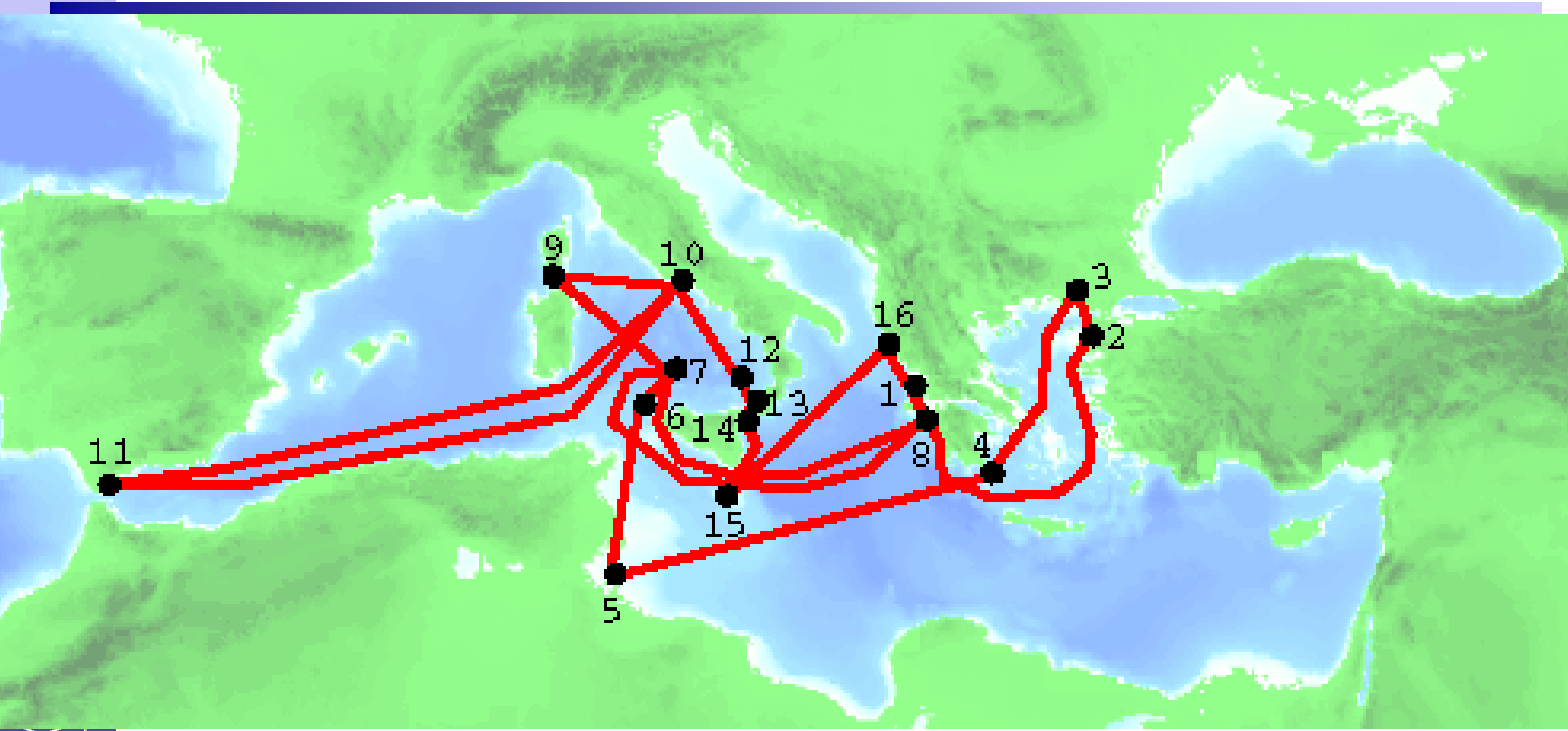
aber es kann durch eine zweckmäßige Wahl und Eintheilung der Tour, manchmal so viel Zeit gewonnen werden, daß wir es nicht glauben umgehen zu dürfen, auch hierüber einige Vorschriften zu geben.

By a proper choice and scheduling of the tour one can gain so much time that we have to make some suggestions

worauf der Reisende hauptsächlich zu sehen hat, des Hin- und Herreisens, mit mehr Oekonomie einzurichten. Die Hauptsache besteht immer darin: so viele Orte wie möglich mitzunehmen, ohne den nämlichen Ort zweimal berühren zu müssen.

The most important aspect is to cover as many locations as possible without visiting a location twice

Ulysses roundtrip (an even older TSP ?)



Ulysses

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	509	501	312	1019	736	656	60	1039	726	2314	479	448	479	619	150
2		126	474	1526	1226	1133	532	1449	1122	2789	958	941	978	1127	542
3			541	1516	1184	1084	536	1371	1045	2728	913	904	946	1115	499
4				1157	980	919	271	1333	1029	2553	751	704	720	783	455
5					478	583	996	858	855	1504	677	651	600	401	1033
6						115	740	470	379	1581	271	289	261	308	687
7							667	455	288	1661	177	216	207	343	592
8								1066	759	2320	493	454	479	598	206
9									328	1387	591	650	656	776	933
10										1697	333	400	427	622	610
11											1838	1868	1841	1789	2248
12												68	105	336	417
13													52	287	406
14														237	449
15															636

Table 2. The distance table for Ulysses 2000.

The distance table

1	Ithaca	38.24N	20.42E
2	Troy	39.57N	26.15E
3	Maronia	40.56N	25.32E
4	Malea	36.26N	23.12E
5	Djerba	33.48N	10.54E
6	Favignana	37.56N	12.19E
7	Ustica	38.42N	13.11E
8	Zakinthos	37.52N	20.44E
9	Bonifaccio	41.23N	9.10E
10	Circeo	41.17N	13.05E
11	Gibraltar	36.08N	5.21W
12	Stromboli	38.47N	15.13E
13	Messina	38.15N	15.35E
14	Taormina	37.51N	15.17E
15	Birzebbuga	35.49N	14.32E
16	Corfu	39.36N	19.56E

Table 1. Polar coordinates of the 16 locations in the Mediterranean.

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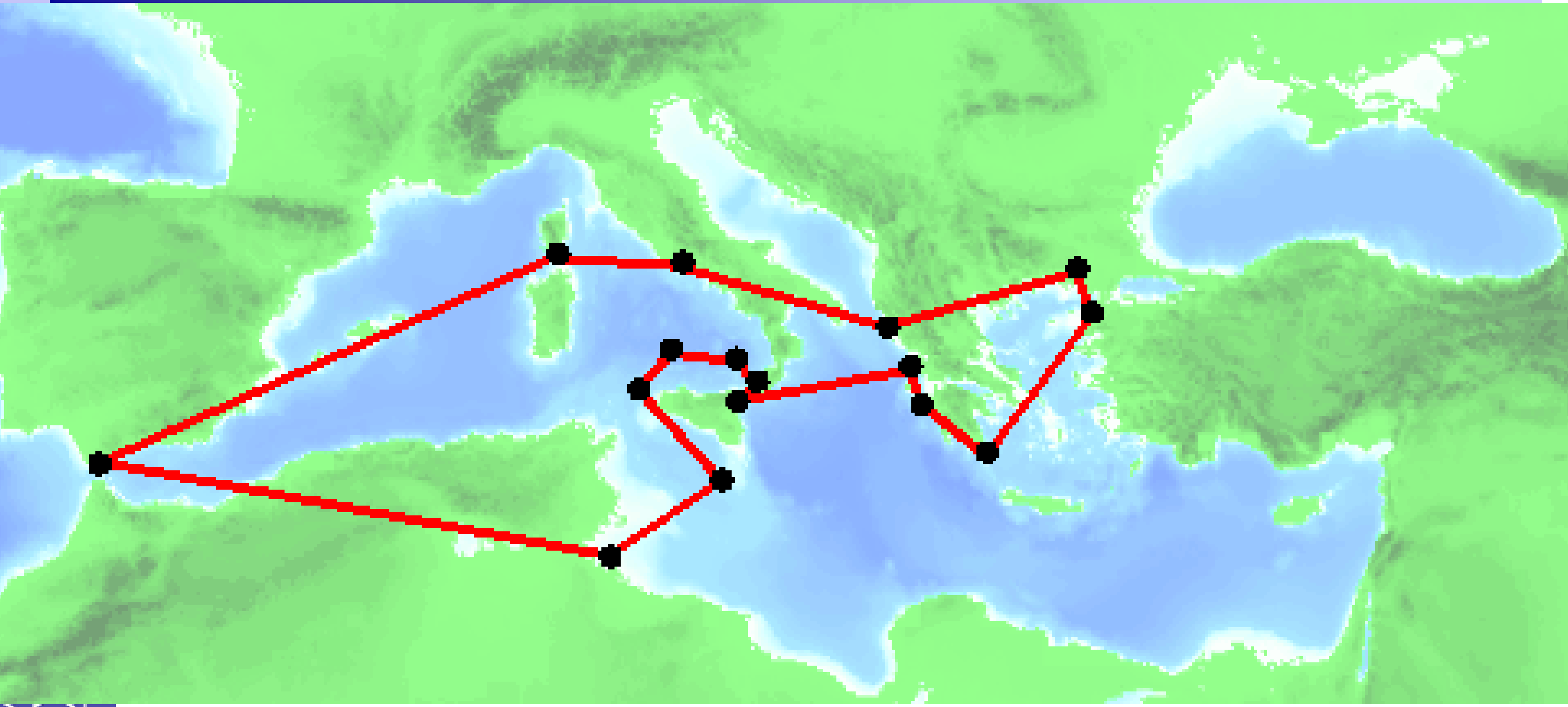
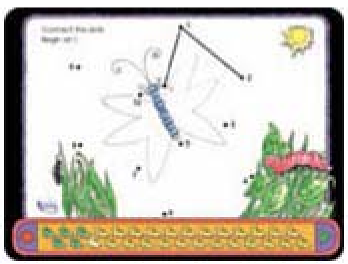
Ulysses roundtrip



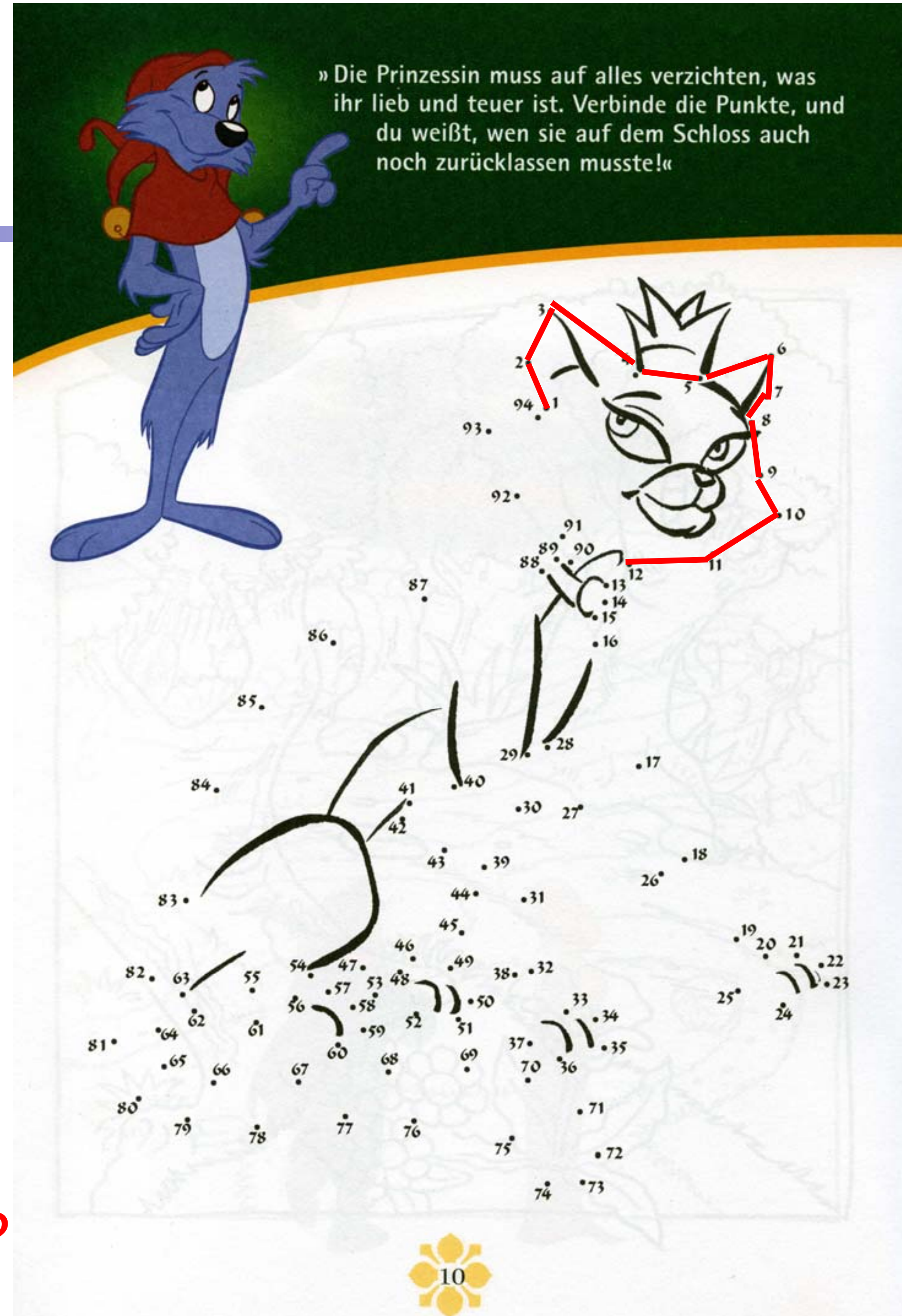
optimal „Ulysses tour“

Malen nach Zahlen
TSP in art ?

Number Sequence



Drawing By Numbers



- When was this invented?

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The TSP in archeology

- Flinders Petrie (1853-1942) and the Luxor graves
- In the words of James Baikie, author of the book *A Century of Excavation in the Land of the Pharaohs,* "if the name of any one man must be associated with modern excavation as that of the chief begetter of its principles and methods, it must be the name of Professor Sir W.M. Flinders Petrie. It was he…who first called the attention of modern excavators to the importance of "unconsidered trifles" as means for the construction of the past…the broken earthenware of a people may be of far greater value than its most gigantic monuments."
- Petrie began to analyze the grave goods methodically. Grave A might contain certain types of pot in common with Grave B; Grave B also contained a later style of pot, the only type to be found in Grave C. By writing cards for each grave and filing them in logical order, Petrie established a full sequence for the cemetery, concluding that the last graves were probably contemporary with the First Dynasty. The development of life along the Nile thus was revealed, from early settlers to farmers to political stratification.

The TSP in archeology: Flinders Petrie’s contribution

- Introduction of the “Hamming distance of graves”, before Richard Wesley Hamming (1915 –1998) introduced it in mathematics.

  (The Hamming distance is used in telecommunication to count the number of flipped bits in a fixed-length binary word, an estimate of error. Hamming weight analysis of bits is used in several disciplines including information theory, coding theory, and cryptography.)
- Definition of the hamiltonian path problem through “graves”.

Survey Books

Literature: more than 800 entries in Zentralblatt/Math

Zbl 0562.00014 **Lawler, E.L.(ed.)**; **Lenstra, J.K.(ed.)**; **Rinnooy Kan, A.H.G.(ed.)**; **Shmoys, D.B.(ed.)**
The traveling salesman problem. A guided tour of combinatorial optimization. Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience publication. Chichester etc.: John Wiley \& Sons. X, 465 p. (1985). *MSC 2000:* *00Bxx 90-06

Zbl 0996.00026 **Gutin, Gregory (ed.)**; **Punnen, Abraham P.(ed.)**
The traveling salesman problem and its variations. Combinatorial Optimization. 12. Dordrecht: Kluwer Academic Publishers. xviii, 830 p. (2002). *MSC 2000:* *00B15 90-06 90Cxx

The Seminal DFJ-Paper of 1954

SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

G. DANTZIG, R. FULKERSON, AND S. JOHNSON

The Rand Corporation, Santa Monica, California

(Received August 9, 1954)

It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance.

The Seminal DFJ-Paper of 1954 preprint

SOLUTION OF A LARGE SCALE TRAVELING SALESMAN PROBLEM

by

G. Dantzig, R. Fulkerson and S. Johnson

P-510

12 April 1954

G. Dantzig, R. Fulkerson, S. Johnson, *Solution of a Large Scale Traveling Salesman Problem*, Paper P-510, The RAND Corporation, Santa Monica, California, [12 April] 1954. [53, 984, 997, 999, 1003]

G. Dantzig, R. Fulkerson, S. Johnson, Solution of a large-scale traveling-salesman problem, *Journal of the Operations Research Society of America* 2 (1954) 393–410. [6, 53, 984, 995]

Some Quotes from DFJ 1954

Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n.

For undirected tours, the symbol x_{IJ} will be treated identically with x_{JI} so that we may rewrite (1) as

$$\sum_{J=1}^{n} x_{IJ} = 2. \qquad (x_{IJ} \geq 0;\ I = 1, 2, \cdots, n;\ I \neq J;\ x_{IJ} = x_{JI}) \qquad (2)$$

IP Formulation

The problem is to find the minimum of the linear form

$$D(x) = \sum_{I>J} d_{IJ}\, x_{IJ}, \qquad (3)$$

where the $x_{IJ} = 0$ or 1 and the $x_{IJ} = 1$ form a tour, and where the summation in (3) extends over all indices (I, J) such that $I > J$.

To make a linear programming problem out of this (see ref. 2) one

Polyhedral Approach

397

needs, as we have observed, a way to describe tours by more linear restraints than that given by (2). This is extremely difficult to do as illustrated by work of I. Heller[4] and H. Kuhn.[6] They point out that such relations always exist. However, there seems to be no simple way to characterize them and for moderate size n the number of such restraints appears to be astronomical. In spite of these difficulties, this paper will

An important class of conditions that tours satisfy, which excludes many non-tour cases satisfying (2), are the 'loop conditions.' These are linear inequality restraints that exclude subcycles or loops. Consider a non-tour solution to (2) which has a subtour of $n_1 < n$ cities; we note that the sum of the x_{IJ} for those links (I,J) in the subtour is n_1. Hence we can

eliminate this type of solution by imposing the condition that the sum of x_{IJ} over all links (I,J) connecting cities in the subset S of n_1 cities be less than n_1, i.e.,

$$\sum_{S} x_{IJ} \leq n_1 - 1 \quad (4)$$

where the summation extends over all (I,J) with I and J in the n_1 cities S. From (2) we note that two other conditions, each equivalent to (4), are

$$\sum_{\bar{S}} x_{IJ} \leq n - n_1 - 1, \quad (5)$$

where $\bar{S}$ means the summation extends over all (I,J) such that neither I nor J is in S, and

$$\sum_{S\bar{S}} x_{IJ} \geq 2, \quad (6)$$

where $S\bar{S}$ means that the summation extends over all (I,J) such that I is in S and J not in $\bar{S}$.

There are, however, other more complicated types of restraints which sometimes must be added to (2) in addition to an assortment of loop conditions in order to exclude solutions involving fractional weights x_{IJ}. In the 49-city case we needed two such conditions. However, later when

Subtour Elimination Constraints in several forms

Remarks

- The preprint version is much clearer than the published paper. The editors have replaced abstract insight by a sequence of examples and thus almost destroyed the “real” contents of the paper.
- The authors outline the branch and bound technique.
- They explain the cutting plane methodology and observe clearly where the difficulties and chances of this method are.
- They mention the importance of heuristics.
- They are modest:

CONCLUDING REMARK

It is clear that we have left unanswered practically any question one might pose of a theoretical nature concerning the traveling-salesman problem; however, we hope that the feasibility of attacking problems involving a moderate number of points has been successfully demonstrated, and that perhaps some of the ideas can be used in problems of similar nature.

The Authors provide data

Table of Road Distances between Cities in Adjusted Units

City

City	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
1																																										
2	8																																									
3	39	45																																								
4	37	47	9																																							
5	50	49	21	15																																						
6	61	62	21	20	17																																					
7	58	60	16	17	18	6																																				
8	59	60	15	20	26	17	10																																			
9	62	66	20	25	31	22	15	5																																		
10	81	81	40	44	50	41	35	24	20																																	
11	103	107	62	67	72	63	57	46	41	23																																
12	108	117	66	71	77	68	61	51	46	26	11																															
13	145	149	104	108	114	106	99	88	84	63	49	40																														
14	181	185	140	144	150	142	135	124	120	99	85	76	35																													
15	187	191	146	150	156	142	137	130	125	105	90	81	41	10																												
16	161	170	120	124	130	115	110	104	105	90	72	64	34	31	27																											
17	142	146	101	104	111	97	91	85	86	75	51	59	29	53	48	21																										
18	174	178	133	138	143	129	123	117	118	107	83	84	54	46	35	26	31																									
19	185	186	142	143	140	130	126	124	128	118	93	101	72	69	58	58	43	26																								
20	164	165	120	123	124	106	106	105	110	104	86	97	71	93	82	62	42	45	22																							
21	87	139	94	96	94	80	78	77	84	77	56	64	65	90	87	58	36	68	50	30																						
22	117	122	77	80	83	68	62	60	61	50	34	42	49	82	77	60	30	62	70	49	21																					
23	114	118	73	78	84	69	63	57	59	48	28	36	43	77	72	45	27	59	69	55	27	5																				
24	85	89	44	48	53	41	34	28	29	22	23	35	69	105	102	74	56	88	99	81	59	32	29																			
25	77	80	36	40	46	34	27	19	21	14	29	40	77	114	111	84	64	96	107	87	60	40	37	8																		
26	87	89	44	46	46	30	28	29	32	27	36	47	78	116	112	84	66	98	95	75	47	36	39	12	11																	
27	91	93	48	50	48	34	32	33	36	30	34	45	77	115	110	83	63	97	91	72	44	32	36	9	15	3																
28	105	106	62	63	64	47	46	49	54	48	46	59	85	119	115	88	66	98	79	59	31	36	42	28	33	21	20															
29	111	113	69	71	66	51	53	56	61	57	59	71	96	130	126	98	75	98	85	62	38	47	53	39	42	29	30	12														
30	91	92	50	51	46	30	34	38	43	49	60	71	103	141	136	109	90	115	99	81	53	61	62	36	34	24	28	20	20													
31	83	85	42	43	38	22	26	32	36	51	63	75	106	142	140	112	93	126	108	88	60	64	66	39	36	27	31	28	28	8												
32	89	91	55	55	50	34	39	44	49	63	76	87	120	155	150	123	100	123	109	86	62	71	78	52	49	39	44	35	24	15	12											
33	95	97	64	63	56	42	49	56	60	75	86	97	126	160	155	128	104	128	113	90	67	76	82	62	59	49	53	40	29	25	23	11										
34	74	81	44	43	35	23	30	39	44	62	78	89	121	159	155	127	108	136	124	101	75	79	81	54	50	42	46	43	39	23	14	14	21									
35	67	69	42	41	31	25	32	41	46	64	83	90	130	164	160	133	114	146	134	111	85	84	86	59	52	47	51	53	49	32	24	24	30	9								
36	74	76	61	60	42	44	51	60	66	83	102	110	147	185	179	155	133	159	146	122	98	105	107	79	71	66	70	70	60	48	40	36	33	25	18							
37	57	59	46	41	25	30	36	47	52	71	93	98	136	172	172	148	126	157	147	124	120	97	99	71	65	59	63	67	62	46	38	37	43	23	13	17						
38	45	46	41	34	20	34	38	48	53	73	96	99	137	176	178	151	131	163	159	135	108	102	103	73	67	64	69	75	72	54	46	49	54	34	24	29	12					
39	35	37	35	26	18	34	36	46	51	70	93	97	134	171	176	151	129	161	163	139	118	102	101	71	65	65	70	84	78	58	50	56	62	41	32	38	21	9				
40	29	33	30	21	18	35	33	40	45	65	87	91	117	166	171	144	125	157	156	139	113	95	97	67	60	62	67	79	82	62	53	59	66	45	38	45	27	15	6			
41	3	11	41	37	47	57	55	58	63	83	105	109	147	186	188	164	144	176	182	161	134	119	116	86	78	84	88	101	108	88	80	86	92	71	64	71	54	41	32	25		
42	5	12	55	41	53	64	61	61	66	84	111	113	150	186	192	166	147	180	188	167	140	124	119	90	87	90	94	107	114	77	86	92	98	80	74	77	60	48	38	32	6	

Distance table hand-written by D. R. Fulkerson (from the preprint Bob Bland owns)

The Optimal Solution

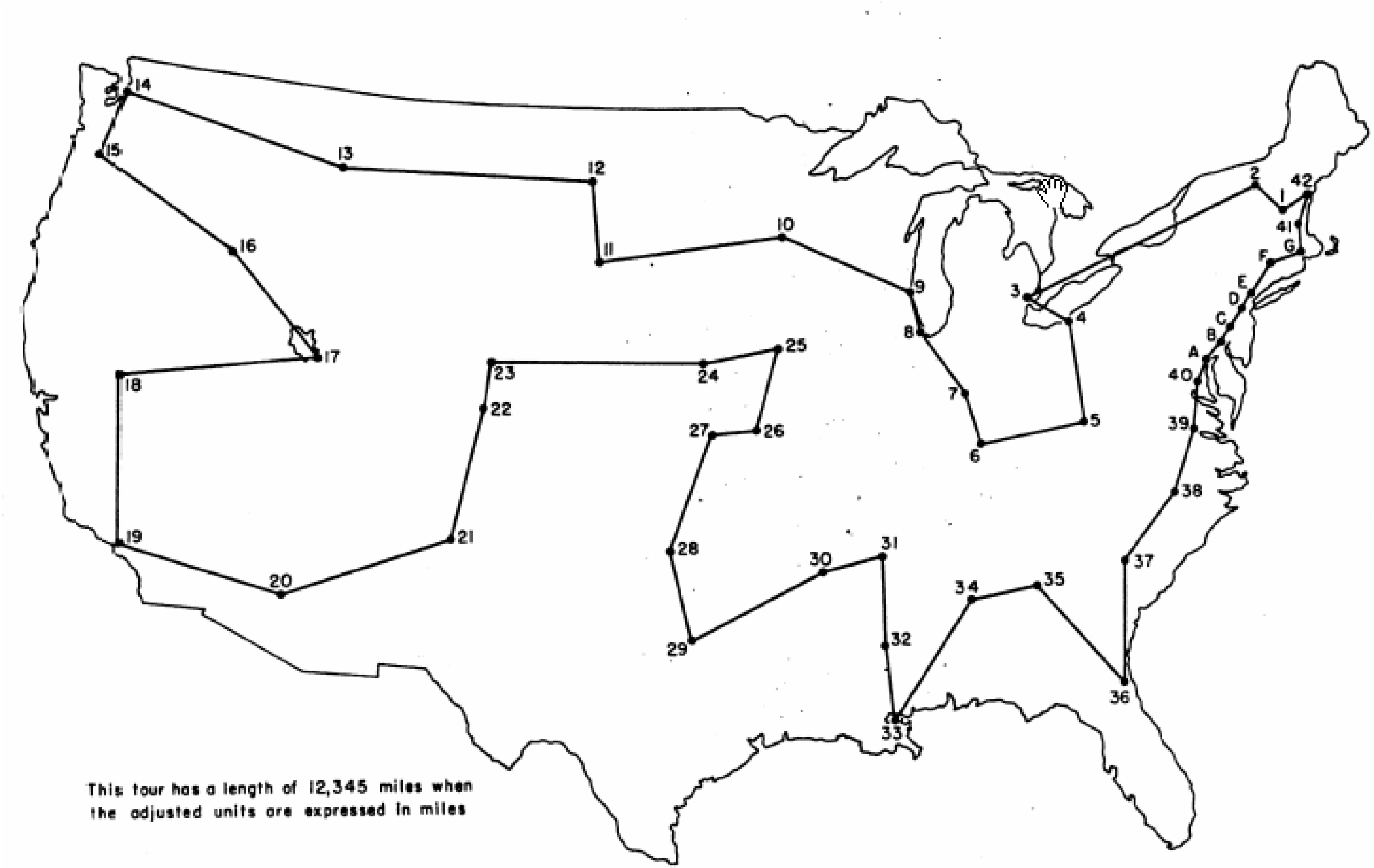

FIG. 16. The optimal tour of 49 cities.

Martin Grötschel

Contents

The Travelling Salesman Problem and Some of its Variants

- The symmetric TSP
- The asymmetric TSP
- The TSP with precedences or time windows
- The online TSP
- The symmetric and asymmetric m-TSP
- The price collecting TSP
- The Chinese postman problem
  (undirected, directed, mixed)
- Bus, truck, vehicle routing
- Edge/arc & node routing with capacities
- Combinations of these and more

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http://www.densis.fee.unicamp.br/~moscato/TSPBIB_home.html

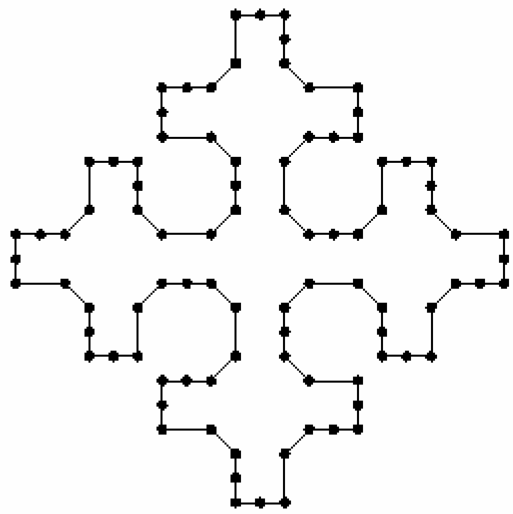
TSPBIB Home Page

This page intends to be a comprehensive listing of papers, source code, preprints, technical reports, etc, available on the Internet about the Traveling Salesman Problem (TSP) and some associated problems.

Please send us information about any other work you consider it should be included in this page.

Pablo Moscato

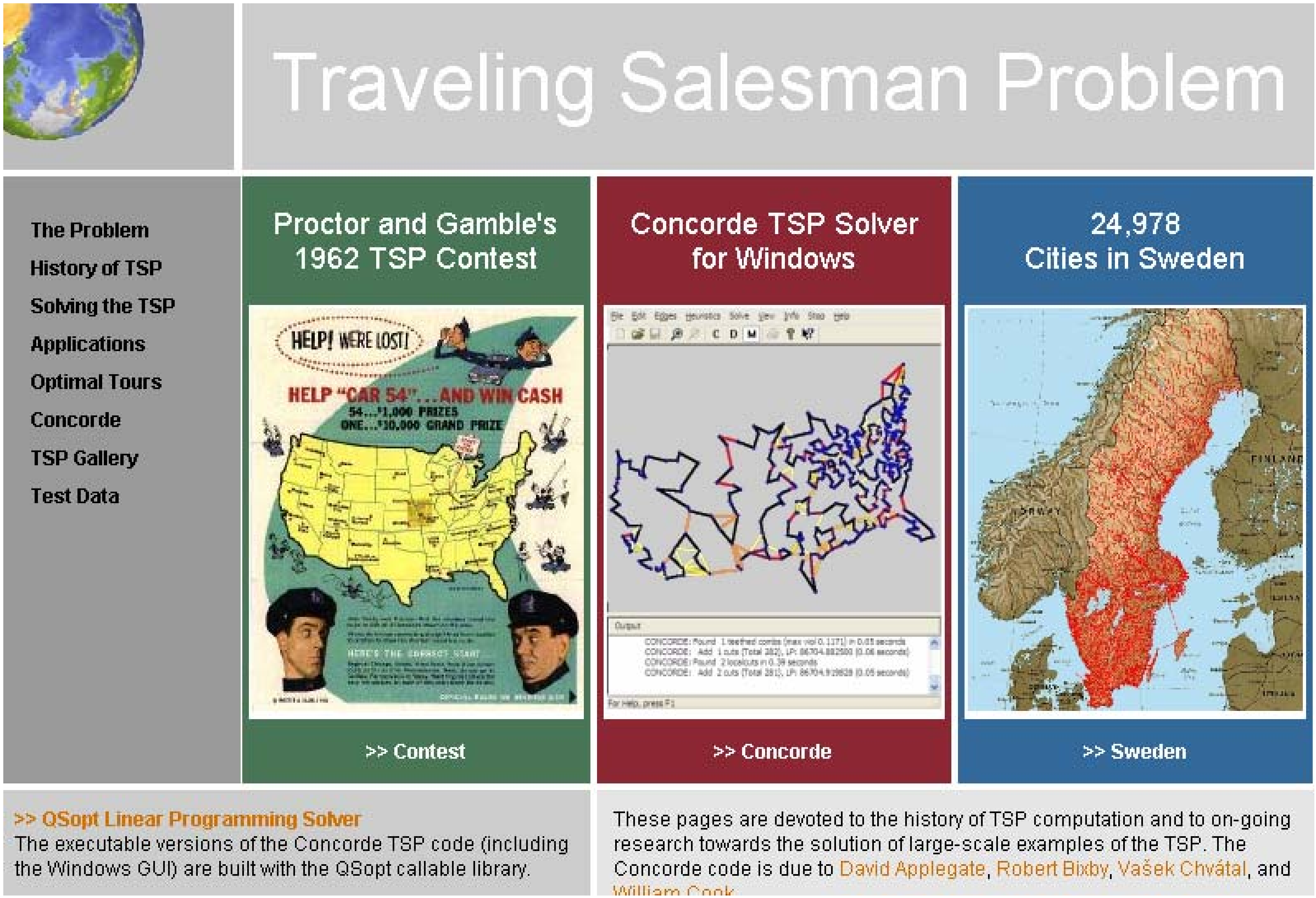
email: *moscato@cacr.caltech.edu*
email: *moscato@densis.fee.unicamp.br*



The picture above shows an instance of the Euclidean, Planar TSP and the optimal curve among the set of cities. This instance has been named MNPeano Order 2.

Contents

An excellent TSP Web site
http://www.tsp.gatech.edu/index.html



Application list from http://www.tsp.gatech.edu/index.html

Applications

- Genome
- Starlight
- Scan Chains
- DNA
- Whizzkids
- Baseball
- Coin Collection
- Airport Tours
- USA Trip
- Sonet Rings
- Power Cables

We will see many TSP applications.

Contents

Need for Heuristics

- Many real-world instances of hard combinatorial optimization problems are (still) too large for exact algorithms.
- Or the time limit stipulated by the customer for the solution is too small.
- Therefore, we need heuristics!
- Exact algorithms usually also employ heuristics.
- What is urgently needed is a decision guide:

Which heuristic will most likely work well on what problem ?

Primal and Dual Heuristics

- Primal Heuristic: Finds a (hopefully) good feasible solution.
- Dual Heuristic: Finds a bound on the optimum solution value (e.g., by finding a feasible solution of the LP-dual of an LP-relaxation of a combinatorial optimization problem).

Minimization:

dual heuristic value $\leq$ optimum value $\leq$ primal heuristic value

(In maximization the inequalities are the other way around.)

quality guarantee
in practice and theory

Primal and Dual Heuristics

Primal and Dual Heuristics give rise to worst-case guarantee:

Minimization:

optimum value ≤ primal heuristic value ≤ $(1+\varepsilon)$ optimum value

dual heuristic value ≤ primal heuristic value ≤ $(1+\varepsilon)$ dual heuristic value

(In maximization the inequalities are the other way around.)

quality guarantee

in practice and theory

Heuristics: A Survey

- Greedy Algorithms
- Exchange & Insertion Algorithms
- Neighborhood/Local Search
- Variable Neighborhood Search, Iterated Local Search
- Random sampling
- Simulated Annealing
- Taboo search
- Great Deluge Algorithms
- Simulated Tunneling
- Neural Networks
- Scatter Search
- Greedy Randomized Adaptive Search Procedures

Heuristics: A Survey

- Genetic, Evolutionary, and similar Methods
- DNA-Technology
- Ant and Swarm Systems
- (Multi-) Agents
- Population Heuristics
- Memetic Algorithms (Meme are the "missing links" gens and mind)
- Space Filling Curves
- Fuzzy Logic Based...
- Fuzzy Genetics-Based Machine Learning
- Fast and Frugal Method (Psychology)
- Ecologically rational heuristic (Sociology)
- Method of Devine Intuition (Psychologist Thorndike)
- .....

An Unfortunate Development

- There is a marketing battle going on with unrealistic, or even ideological, claims about the quality of heuristics – just to catch attention
- Linguistic Overkill:
- Vodoo Approach

A Quote

Quote:
Genetic Programming is an evolutionary computation technique which searches for those computer programs that best solve a given problem.

(Will this also solve P = NP?)

Kalyanmoy Deb: „Multi-objective optimization using evolutionary algorithms“ (Wiley, 2001)

from the Preface

- Optimization is a procedure of finding and comparing feasible solutions until no better solution can be found.

- Evolutionary algorithms (EAs), on the other hand, can find multiple optimal solutions in one single simulation run due to their population-approach. Thus, EAs are ideal candidates for solving…

?

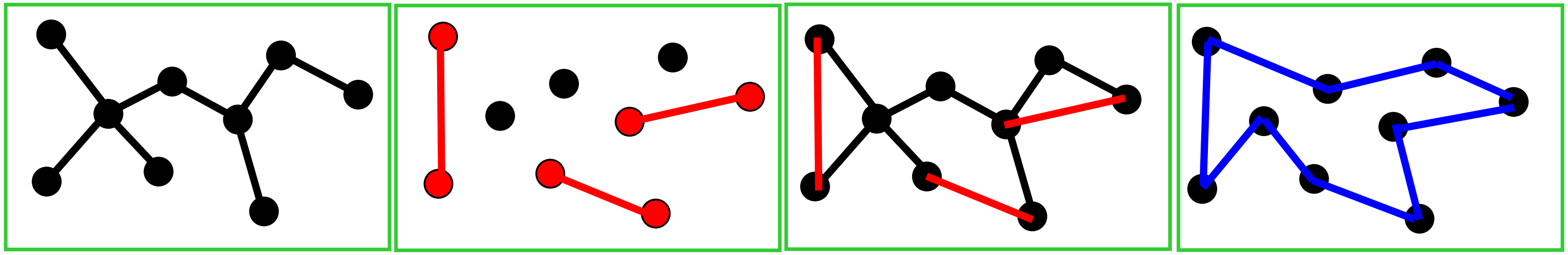
Heuristics: A Survey

Currently best heuristic with respect to worst-case guarantee:

Christofides heuristic

- compute a shortest spanning tree
- compute a minimum perfect 1-matching of the graph induced by the odd nodes of the minimum spanning tree
- the union of these edge sets is a connected Eulerian graph
- turn this graph into a TSP-tour by making short-cuts.

For distance functions satisfying the triangle inequality, the resulting tour is at most 50% above the optimum value



Martin Grötschel

Understanding Heuristics, Approximation Algorithms

- worst case analysis
  - There is no polynomial time approx. algorithm for STSP/ATSP.
  - Christofides algorithm for the STSP with triangle inequality
- average case analysis
  - Karp's analysis of the patching algorithm for the ATSP
- probabilistic problem analysis
  - for Euclidean STSP in unit square: TSP constant 1.714..$n^{1/2}$
- polynomial time approximation schemes (PAS)
  - Arora's polynomial-time approximation schemes for Euclidean STSPs
- fully-polynomial time approximation schemes (FPAS)
  - not for TSP/ATSP but, e.g., for knapsack (Ibarra&Kim)

- These concepts – unfortunately – often do not really help to guide practice.
- experimental evaluation
  - Lin-Kernighan for STSP (DIMACS challenges))

Contents

Polyhedral Theory (of the TSP)

STSP-, ATSP-,TSP-with-side-constraints-

Polytope:= Convex hull of all incidence vectors of feasible tours

$$Q_T^n := conv\{\chi^T \in \mathbf{Z}^E \mid T \text{ tour in } K_n\} \qquad (\chi_{ij}^T = 1 \text{ if } ij \in T, \text{ else } = 0)$$

To be investigated:

- Dimension
- Equation system defining the affine hull
- Facets
- Separation algorithms

The symmetric travelling salesman polytope

$$Q_T^n := conv\{\chi^T \in \mathbf{Z}^E \mid T \text{ tour in } K_n\} \qquad (\chi_{ij}^T = 1 \text{ if } ij \in T, \text{ else } = 0)$$

$$\begin{aligned} \subseteq \{x \in \mathbf{R}^E \mid x(\delta(i)) &= 2 && \forall i \in V \\ x(E(W)) &\leq |W| - 1 && \forall W \subset V \setminus \{1\}, 3 \leq |W| \leq n-3 \\ 0 \leq x_{ij} &\leq 1 && \forall ij \in E\} \end{aligned}$$

- IP formulation

$$\begin{aligned} &\min c^T x \\ &x(\delta(i)) = 2 && \forall i \in V \\ &x(E(W)) \leq |W| - 1 && \forall W \subset V \setminus \{1\}, 3 \leq |W| \leq n-3 \\ &x_{ij} \in \{0,1\} && \forall ij \in E \end{aligned}$$

- The LP relaxation is solvable in polynomial time.

Dimension of the sym TSP polytope

- Proof

Relation between IP and LP-relaxation

Open Problem:

- If costs satisfy the triangle inequality, then

  IP-OPT <= 4/3 LP-SEC

  IP-OPT <= 3/2 LP-SEC (Wolsey)

Facets of the TSP polytope

- Finding facets!
- Proving that an inequality defines a facet!
- Finding exact or heuristic separation algorithms to be used in a cutting plane algorithm!

Why are facets important?

An integer programming formulation from a textbook:

$$\min c^T x$$

$$x(\delta(i)) = 2 \qquad \forall i \in V$$

$$X(E) = n$$

$$x(C) \leq |C| - 1 \quad \forall C \subset E, C \text{ a nonhamiltonian cycle}$$

$$x_{ij} \in \{0,1\} \qquad \forall ij \in E$$

What would you say?

Subtour elimination constraints: equivalent versions

- SEC constraints
- cut constraints

General cutting plane theory: Gomory Cut (the „rounding trick“)

Let $P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$ be a polyhedron, and we suppose that A and b are integral.

We would like to describe the convex hull P_I of all integral points in P.

Observation: For any $y \in \mathbb{R}^m$

$$y^T Ax \leq y^T b$$

is a valid inequality for P.

Observation: For any $y \in \mathbb{R}^m$

$$\lfloor y^T Ax \rfloor \leq \lfloor y^T b \rfloor$$

is a valid inequality for P_I.

$$\lfloor y^T Ax \rfloor = \sum_{j=}^{n} \sum_{i=1}^{m} \lfloor y_i a_{ij} \rfloor x_j \leq \left\lfloor \sum_{i=1}^{m} y_i b_j \right\rfloor$$

Choose y so that $y_i a_{ij}$ is integral: $\lfloor y^T Ax \rfloor = \sum_{j=}^{n} \sum_{i=1}^{m} y_i a_{ij} x_j \leq \left\lfloor \sum_{i=1}^{m} y_i b_j \right\rfloor$

Martin Grötschel

Chvátal-Gomory Procedure

- Does the rounding procedure deliver P_I?
- How many rounds of rounding do we need?
- Other better methods?

General cutting plane theory: Gomory Mixed-Integer Cut

- Given $y, x_j \in \text{¢}_+$, and

$$y + \sum a_{ij} x_j = d = \lfloor d \rfloor + f,\; f > 0$$

- Rounding: Where $a_{ij} = \lfloor a_{ij} \rfloor + f_j$, define

$$t = y + \sum \left(\lfloor a_{ij} \rfloor x_j : f_j \le f \right) + \sum \left(\lceil a_{ij} \rceil x_j : f_j > f \right) \in \text{¢}$$

- Then

$$\sum \left(f_j x_j : f_j \le f \right) + \sum \left(f_j - 1 \right) x_j : f_j > f = d - t$$

- Disjunction:

$$t \le \lfloor d \rfloor \Rightarrow \sum \left(f_j x_j : f_j \le f \right) \ge f$$

$$t \ge \lceil d \rceil \Rightarrow \sum \left(\left(1 - f_j \right) x_j : f_j > f \right) \ge 1 - f$$

- Combining

$$\sum \left(\left(f_j / f \right) x_j : f_j \le f \right) + \sum \left(\left[\left(1 - f_j \right) / \left(1 - f \right) \right] x_j : f_j > f \right) \ge 1$$

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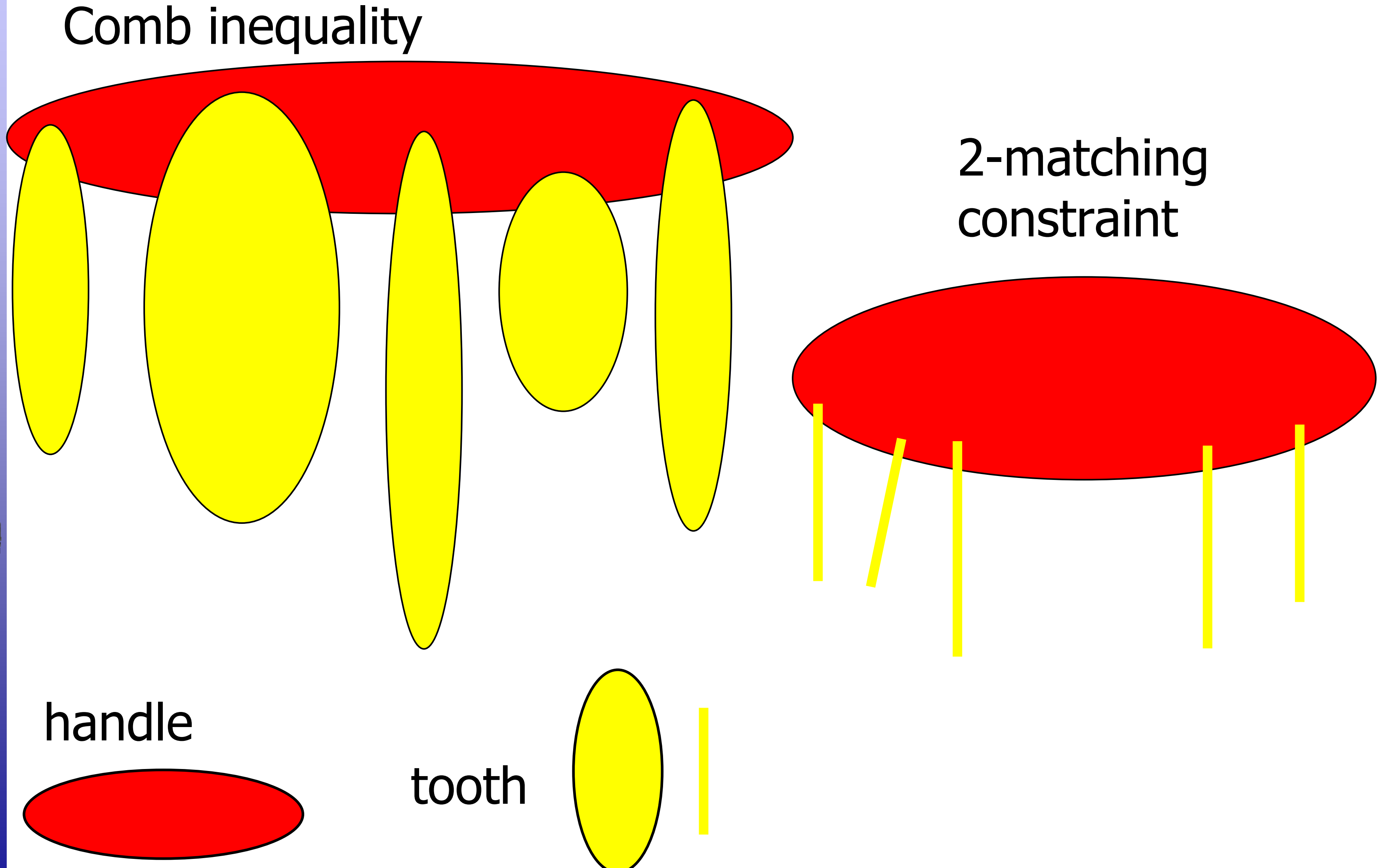
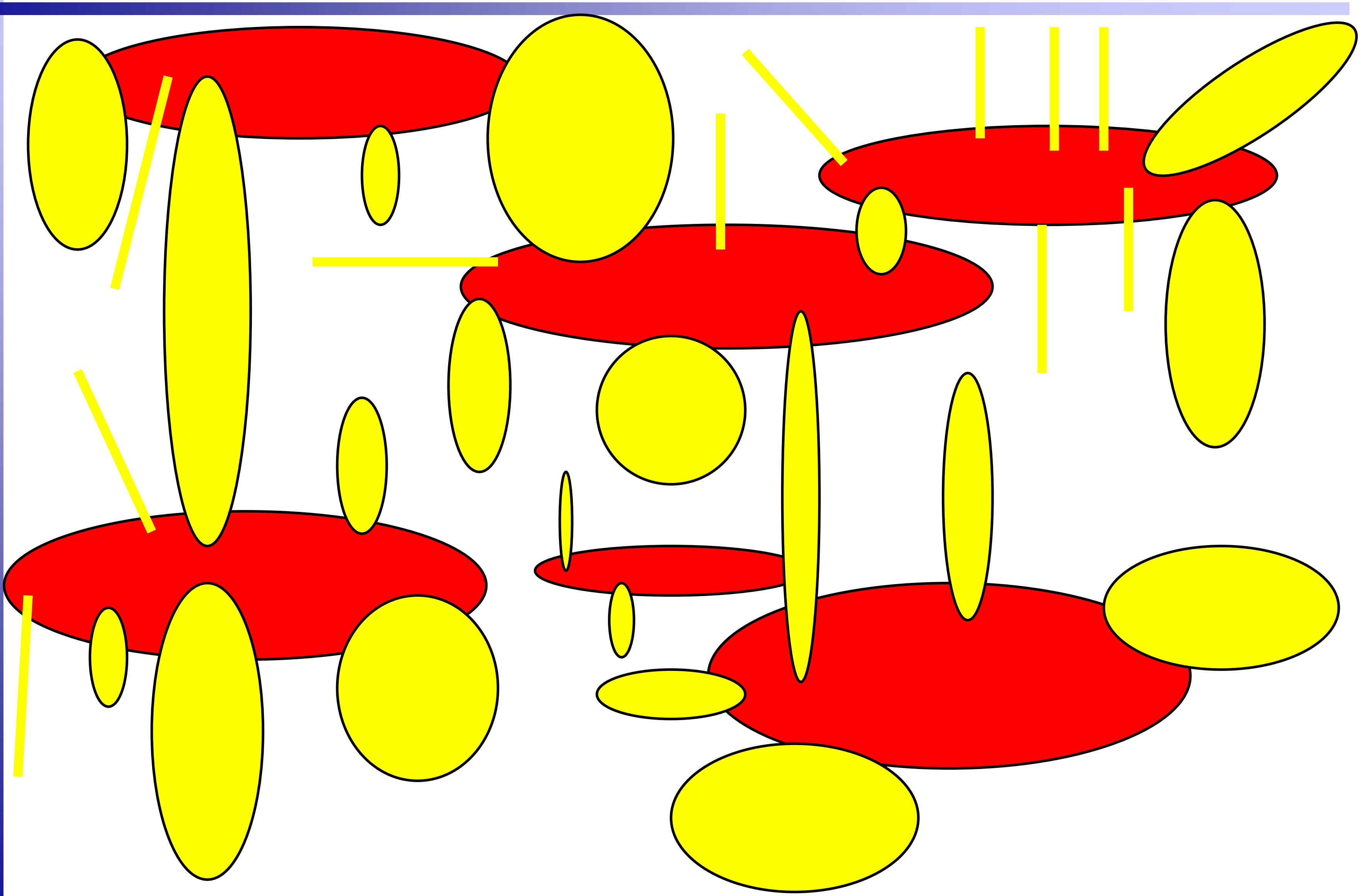
From SECs to

- 2-matching constraints
- combs
- clique tree inequalities
- etc.

Polyhedral Theory of the TSP



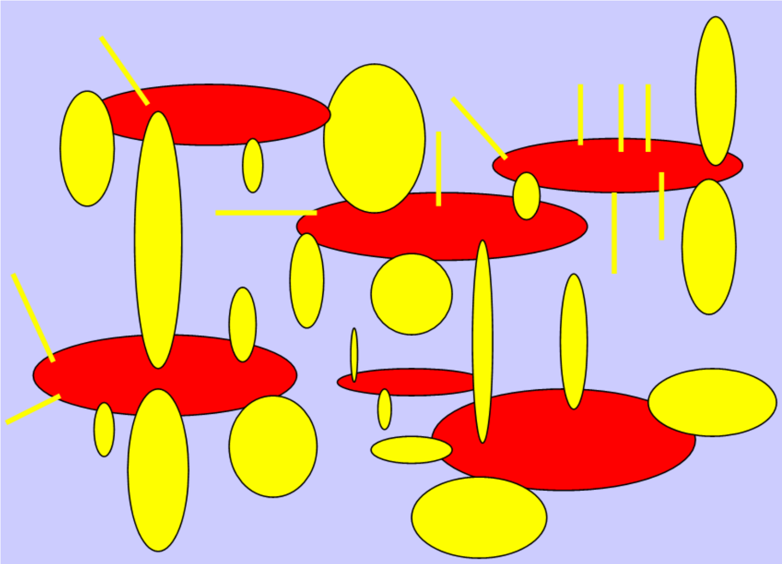
Clique Tree Inequalities



Clique Tree Inequalities

$$\sum_{i=1}^{h} x(\partial(H_i)) + \sum_{j=1}^{t} x(\partial(T_j)) \geq \sum_{i=1}^{h} |H_i| + h + 2t$$

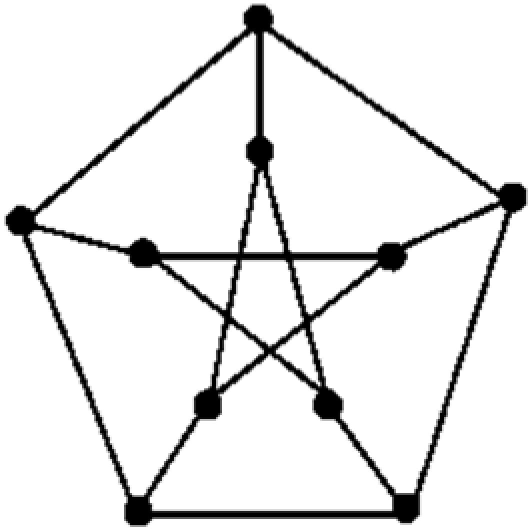
$$\sum_{i=1}^{h} x(E(H_i)) + \sum_{j=1}^{t} x(E(T_j)) \leq \sum_{i=1}^{h} |H_i| + \sum_{i=1}^{t} (|T_j| - t_j) - \frac{t+1}{2}$$



Valid Inequalities for STSP

- Trivial inequalities
- Degree constraints
- Subtour elimination constraints
- 2-matching constraints, comb inequalities
- Clique tree inequalities (comb)
- Bipartition inequalities (clique tree)
- Path inequalities (comb)
- Star inequalities (path)
- Binested Inequalities (star, clique tree)
- Ladder inequalities (2 handles, even # of teeth)
- Domino inequalities
- Hypohamiltonian, hypotraceable inequalities
- etc.

A very special case



Petersen graph, G = (V, F),
the smallest hypohamiltonian graph

$$x(F) \leq 9 \ \textit{defines a facet of } Q_T^{10}$$

$$\textit{but not a facet of } Q_T^n, n \geq 11$$

M. Grötschel & Y. Wakabayashi

Valid and facet defining inequalities for STSP: Survey articles

- M. Grötschel, M. W. Padberg (1985 a, b)
- M. Jünger, G. Reinelt, G. Rinaldi (1995)
- D. Naddef (2002)

Counting Tours and Facets

n	# tours	# different facets	# facet classes
3	1	0	0
4	3	3	1
5	12	20	2
6	60	100	4
7	360	3,437	6
8	2520	194,187	24
9	20,160	42,104,442	192
10	181,440	>= 52,043.900.866	>=15,379

Separation Algorithms

- Given a system of valid inequalities (possibly of exponential size).
- Is there a polynomial time algorithm (or a good heuristic) that,
  - given a point,
  - checks whether the point satisfies all inequalities of the system, and
  - if not, finds an inequality violated by the given point?

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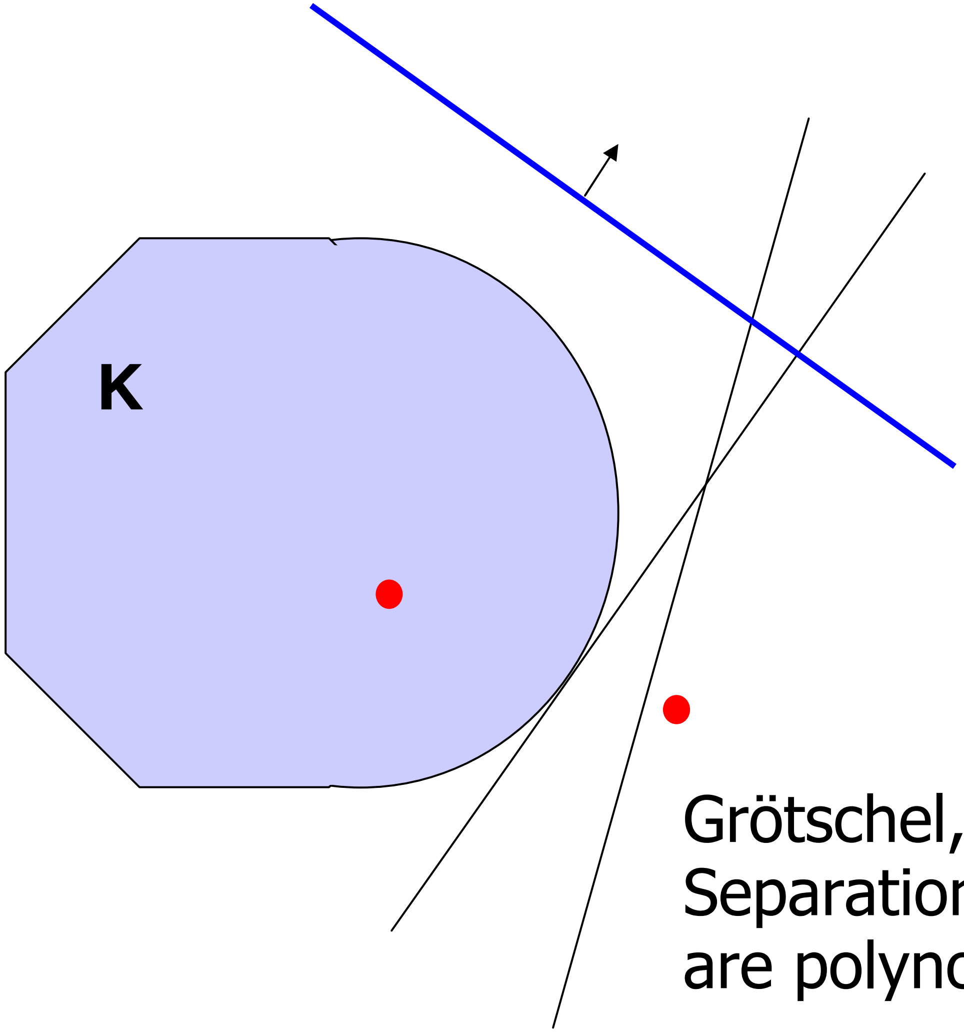
Separation



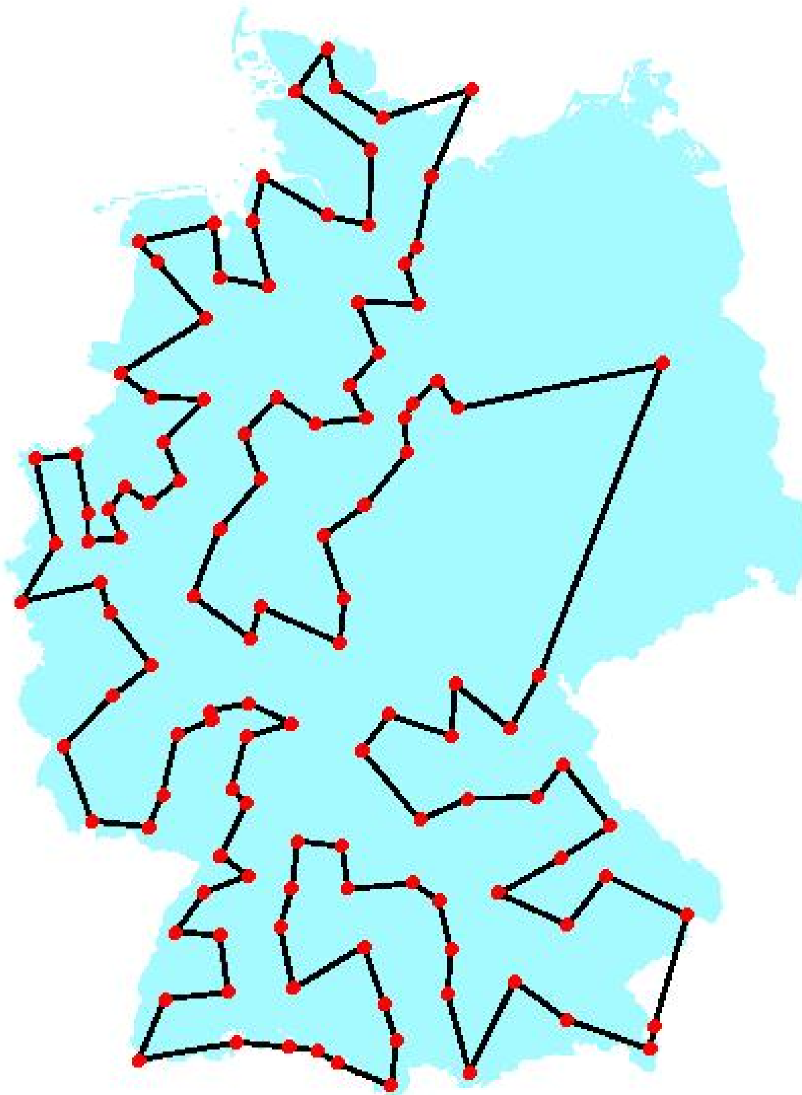
Grötschel, Lovász, Schrijver: Separation and optimization are polynomial time equivalent.

Separation Algorithms

- There has been great success in finding exact polynomial time separation algorithms, e.g.,
  - for subtour-elimination constraints
  - for 2-matching constraints (Padberg&Rao, 1982)
- or fast heuristic separation algorithms, e.g.,
  - for comb constraints
  - for clique tree inequalities
- and these algorithms are practically efficient

SEC Separation

West-Deutschland und Berlin



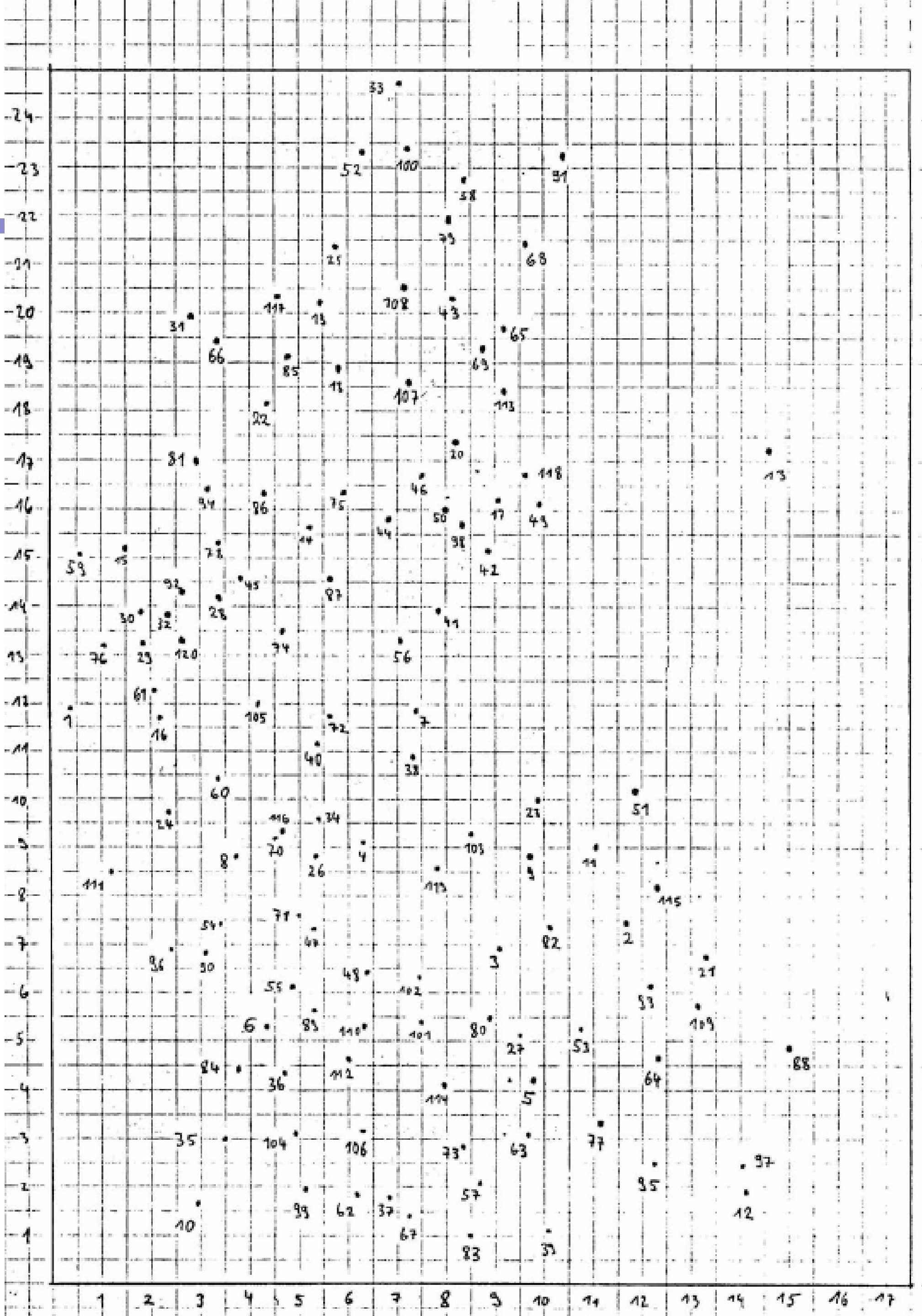
120 Städte
7140 Variable

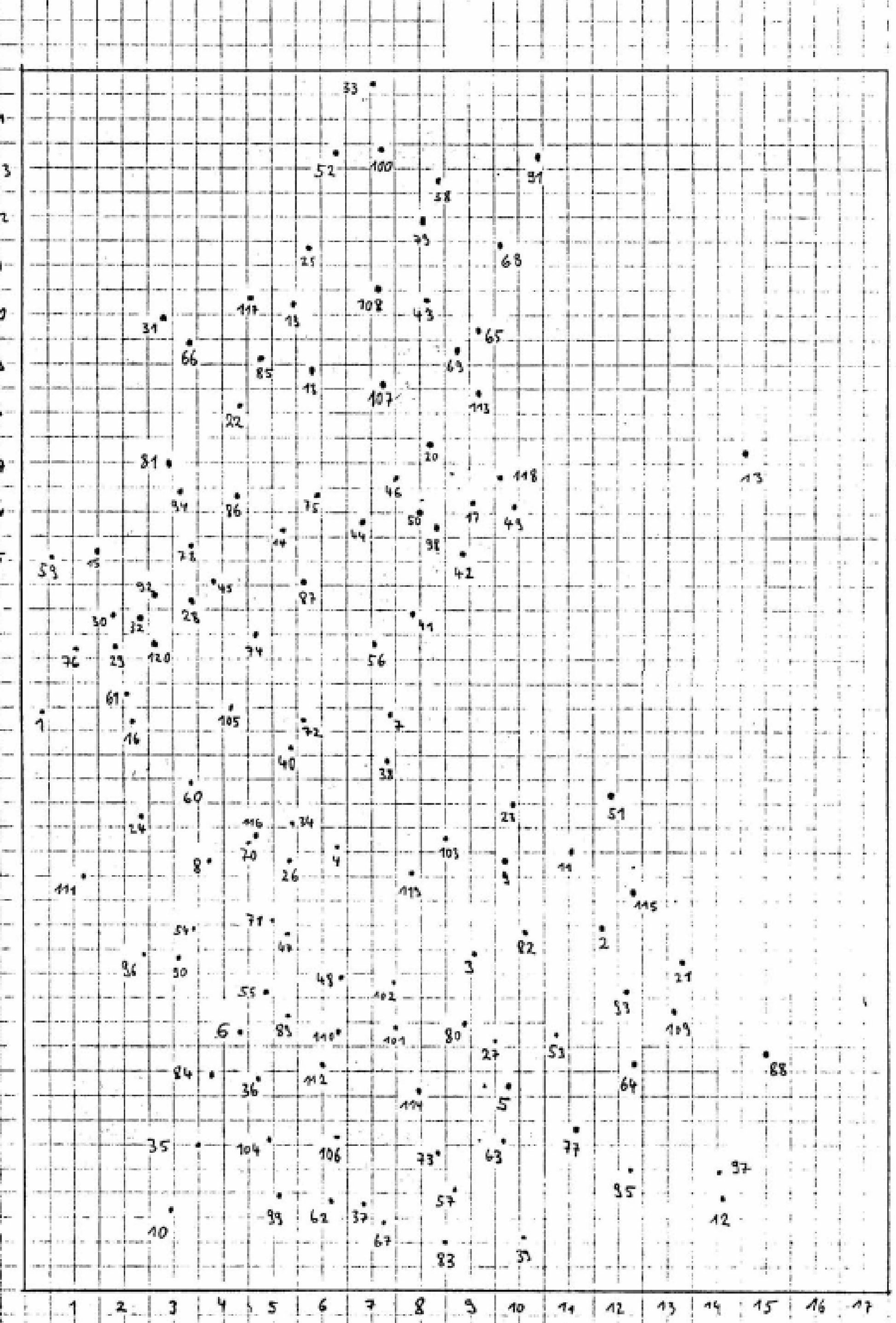
1975/1977/1980

M. Grötschel

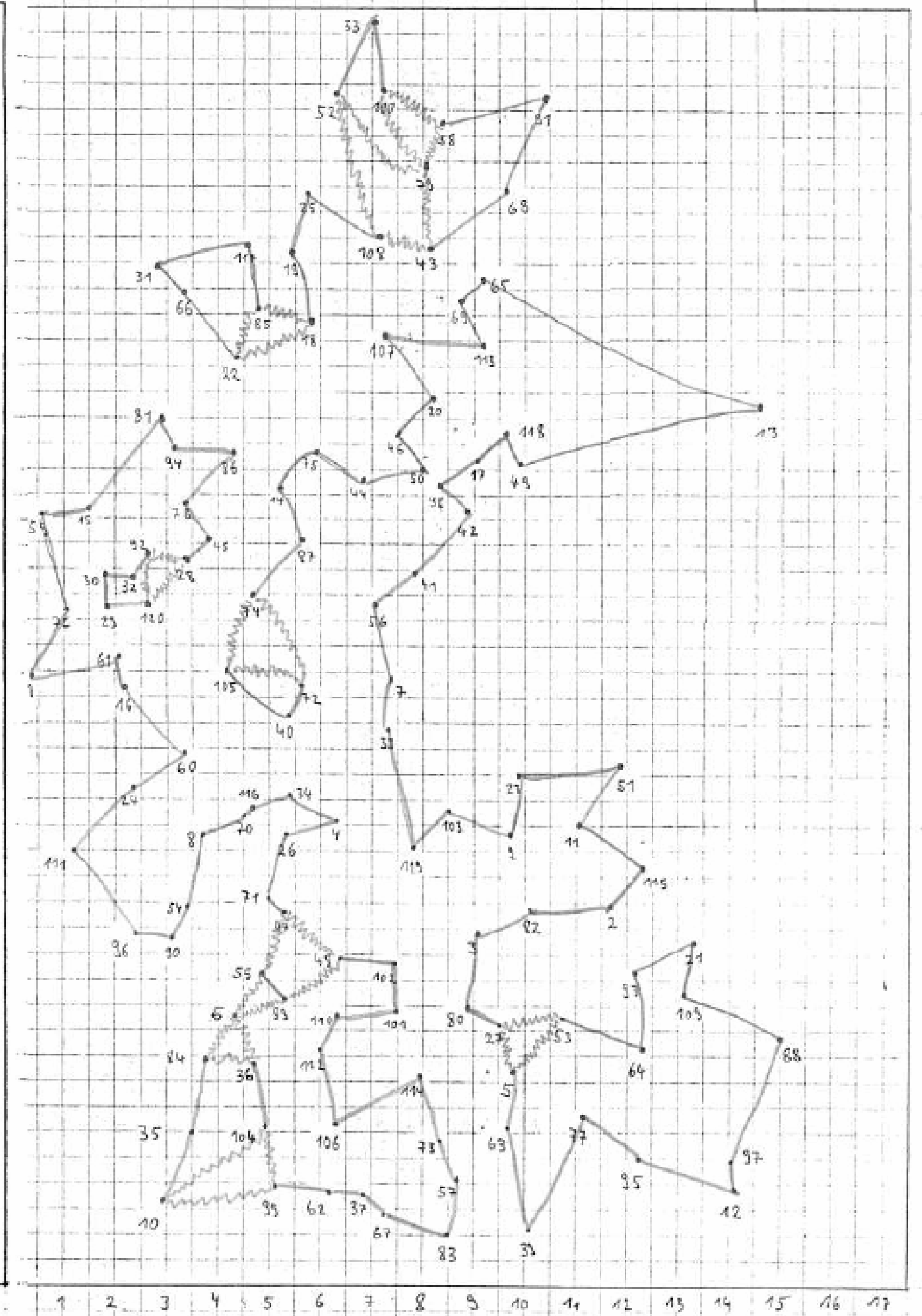
In the old days: 1975, TSP 120

- my drawing of Germany



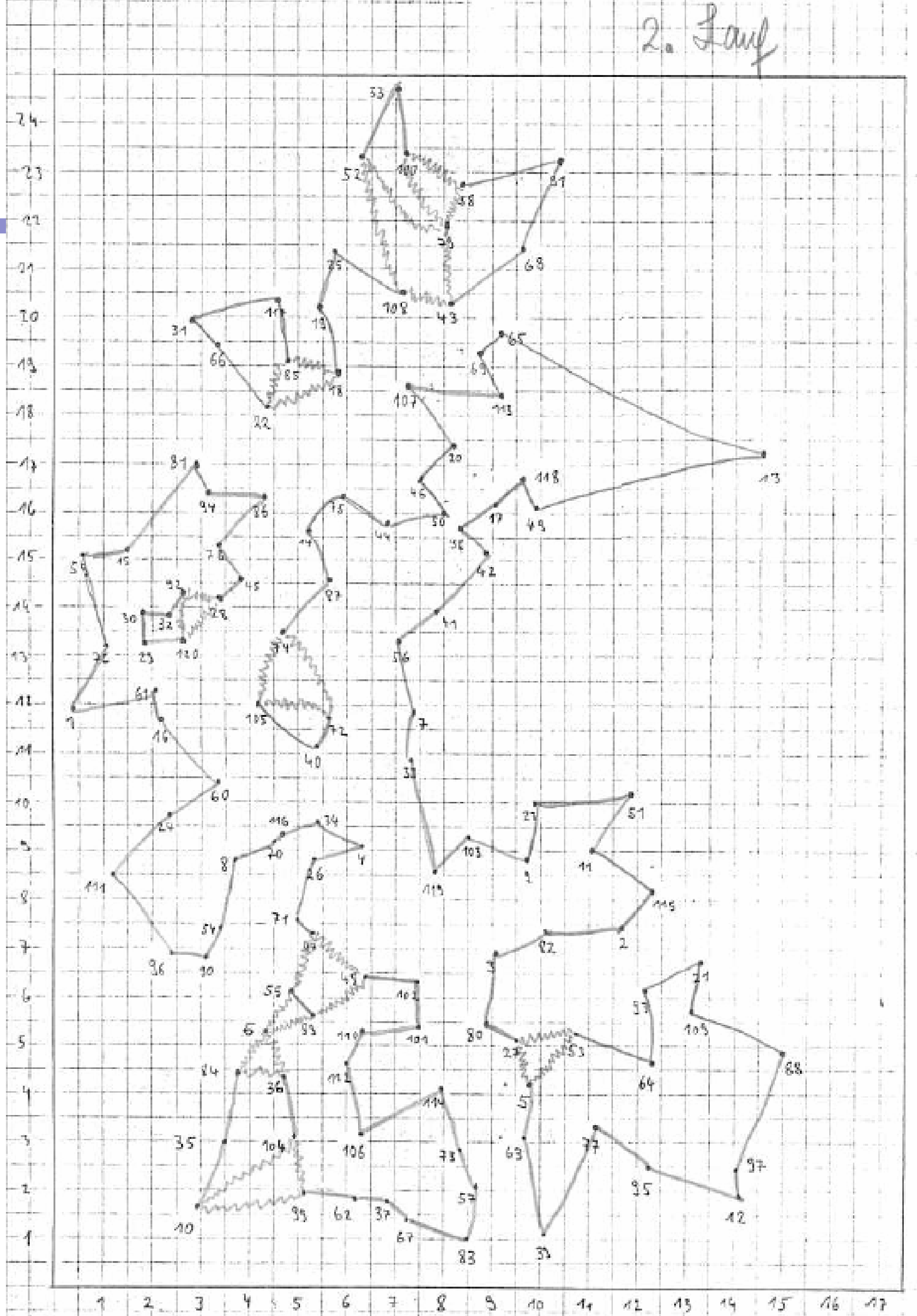


2. Lauf

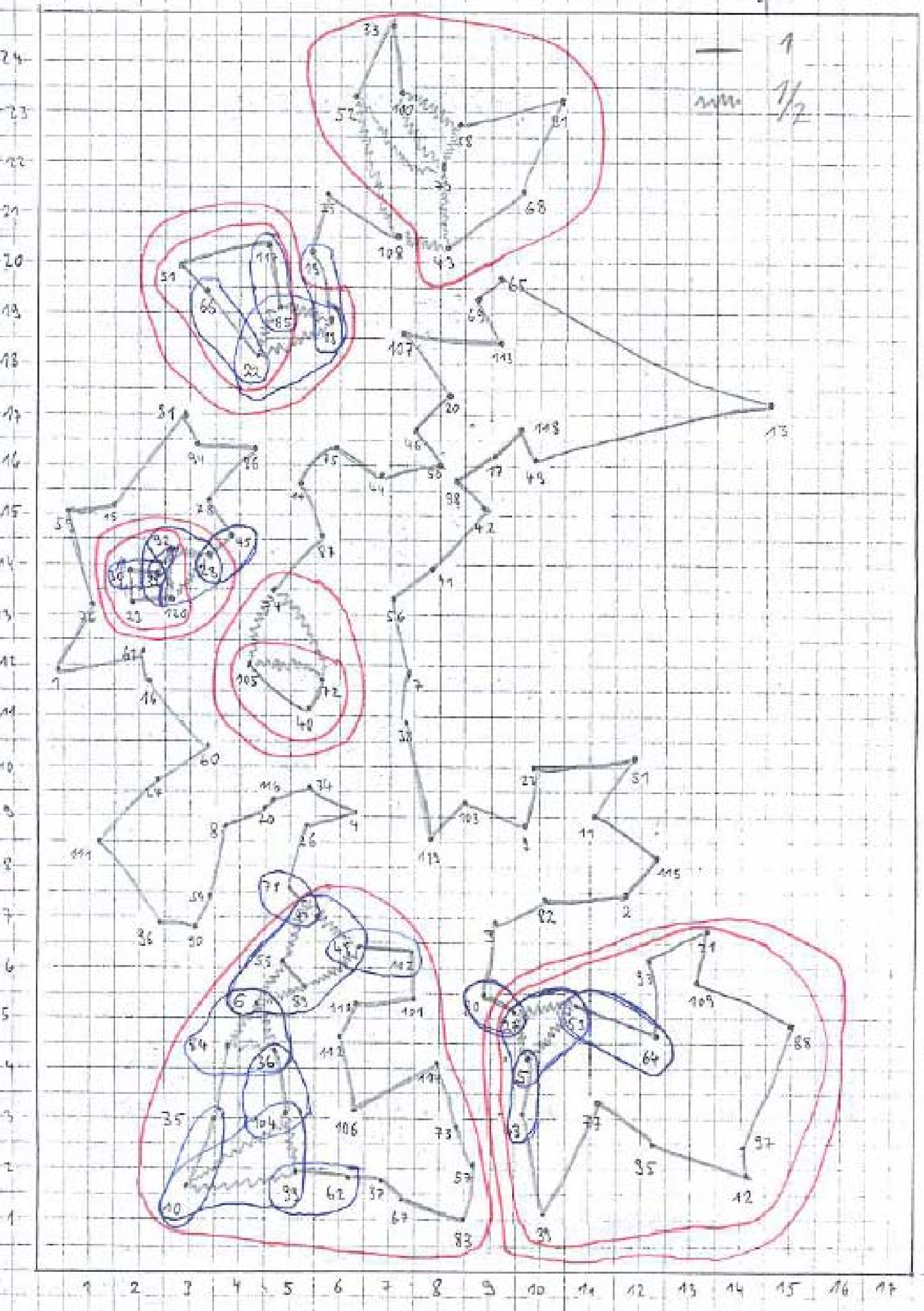


In the old days: 1975 TSP 120

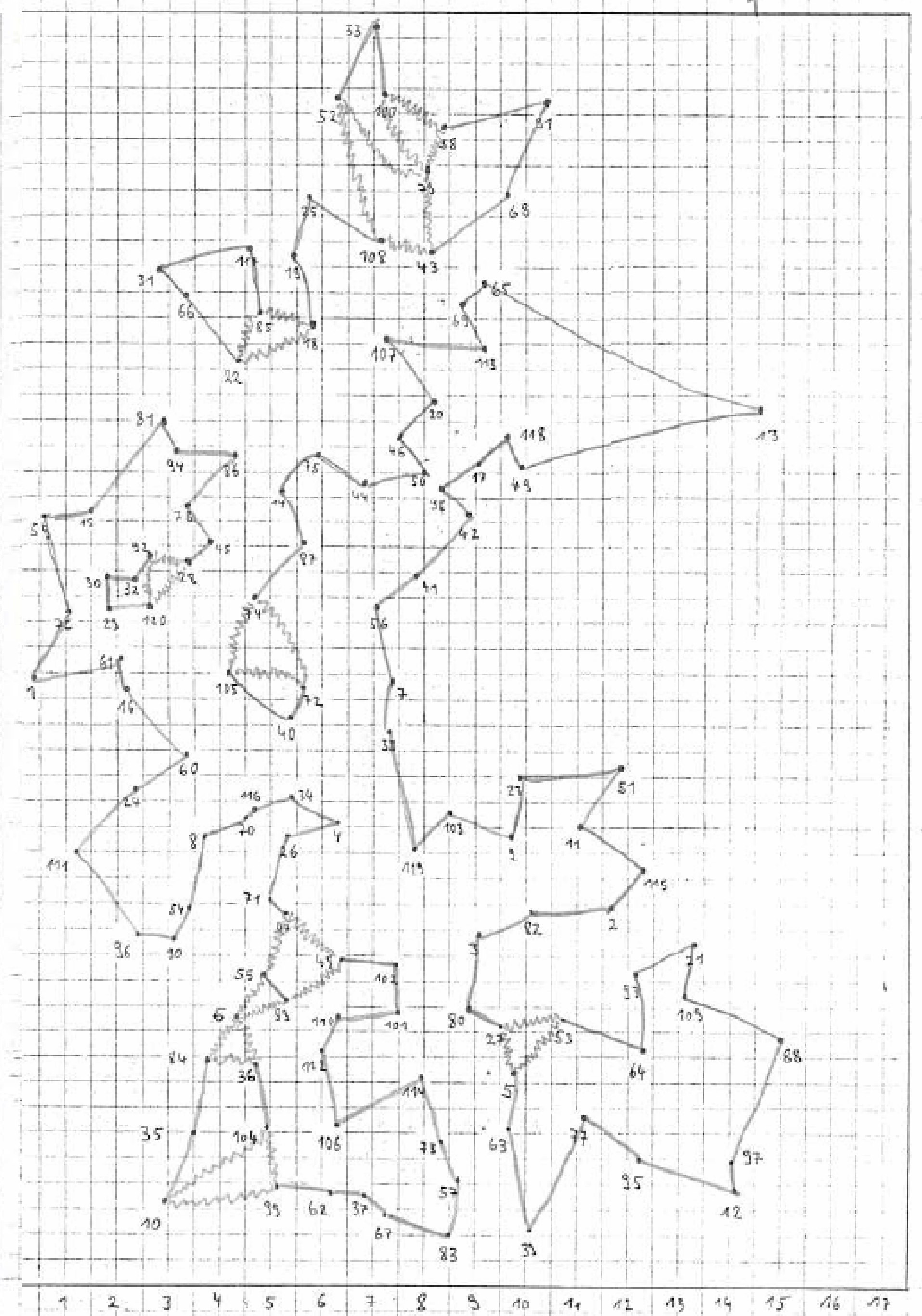
- optimal LP solution after second run



2. Lauf



2. Lauf

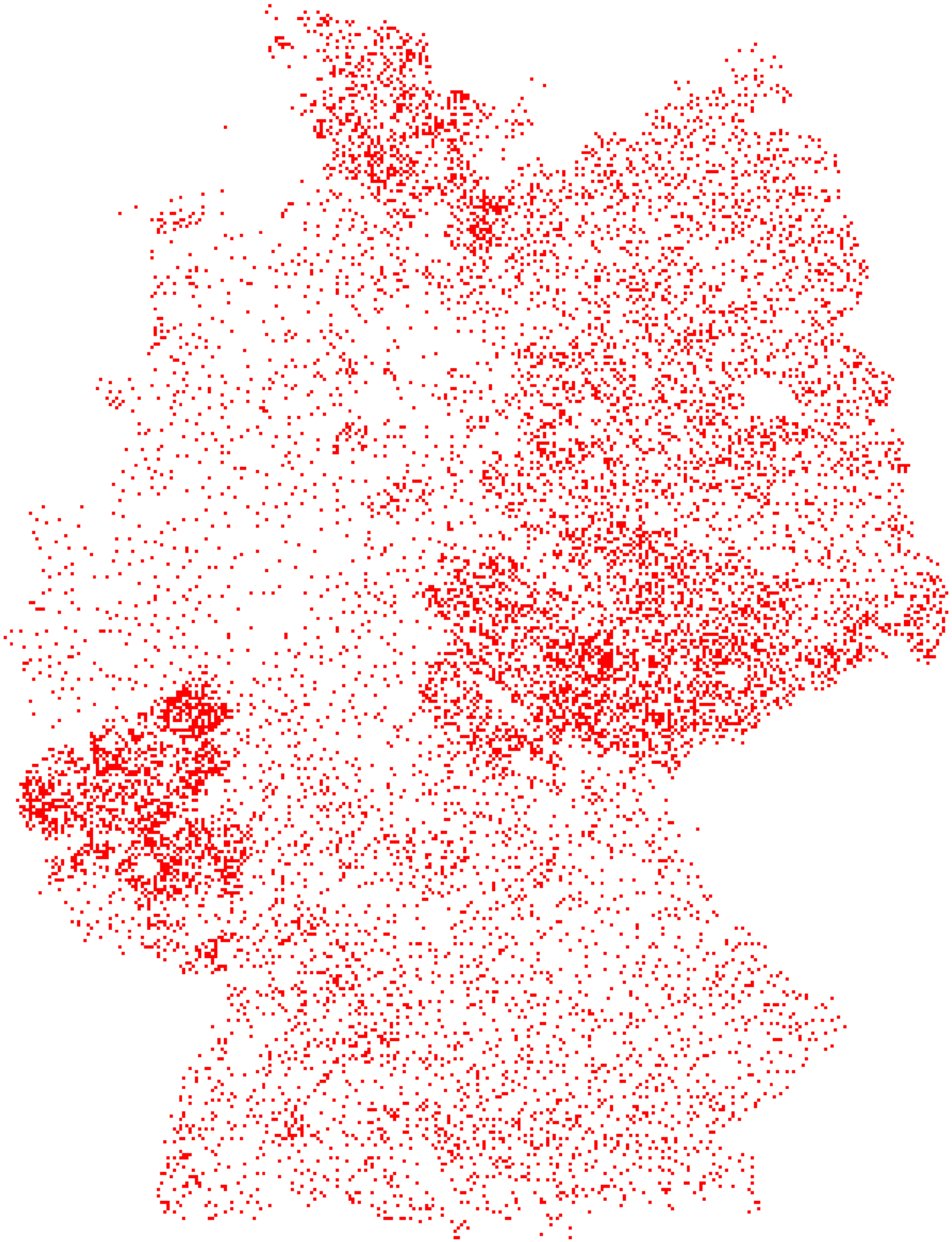


Polyhedral Combinatorics

This line of research has resulted in powerful cutting plane algorithms for combinatorial optimization problems.

They are used in practice to solve exactly or approximately (including branch & bound) large-scale real-world instances.

Deutschland 15,112



D. Applegate, R.Bixby, V. Chvatal, W. Cook

15,112

cities

114,178,716

variables

2001

How do we solve a TSP like this?

- **Upper bound:**

  Heuristic search

  - Chained Lin-Kernighan

- **Lower bound:**
  - Linear programming
  - Divide-and-conquer
  - Polyhedral combinatorics
  - Parallel computation
  - Algorithms & data structures

The **LOWER BOUND** is the mathematically and algorithmically hard part of the work

Work on LP relaxations of the symmetric travelling salesman polytope

$$Q_T^n := conv\{\chi^T \in \mathbf{Z}^E \mid T \text{ tour in } K_n\}$$

$$\begin{array}{ll}
\min c^T x & \\
x(\delta(i)) = 2 & \forall i \in V \\
x(E(W)) \leq |W| - 1 & \forall W \subset V \setminus \{1\}, 3 \leq |W| \leq n-3 \\
0 \leq x_{ij} \leq 1 & \forall ij \in E \\
~~x_{ij} \in \{0,1\}~~ & ~~\forall ij \in E~~
\end{array}$$

- Integer Programming Approach

cutting plane technique for integer and mixed-integer programming

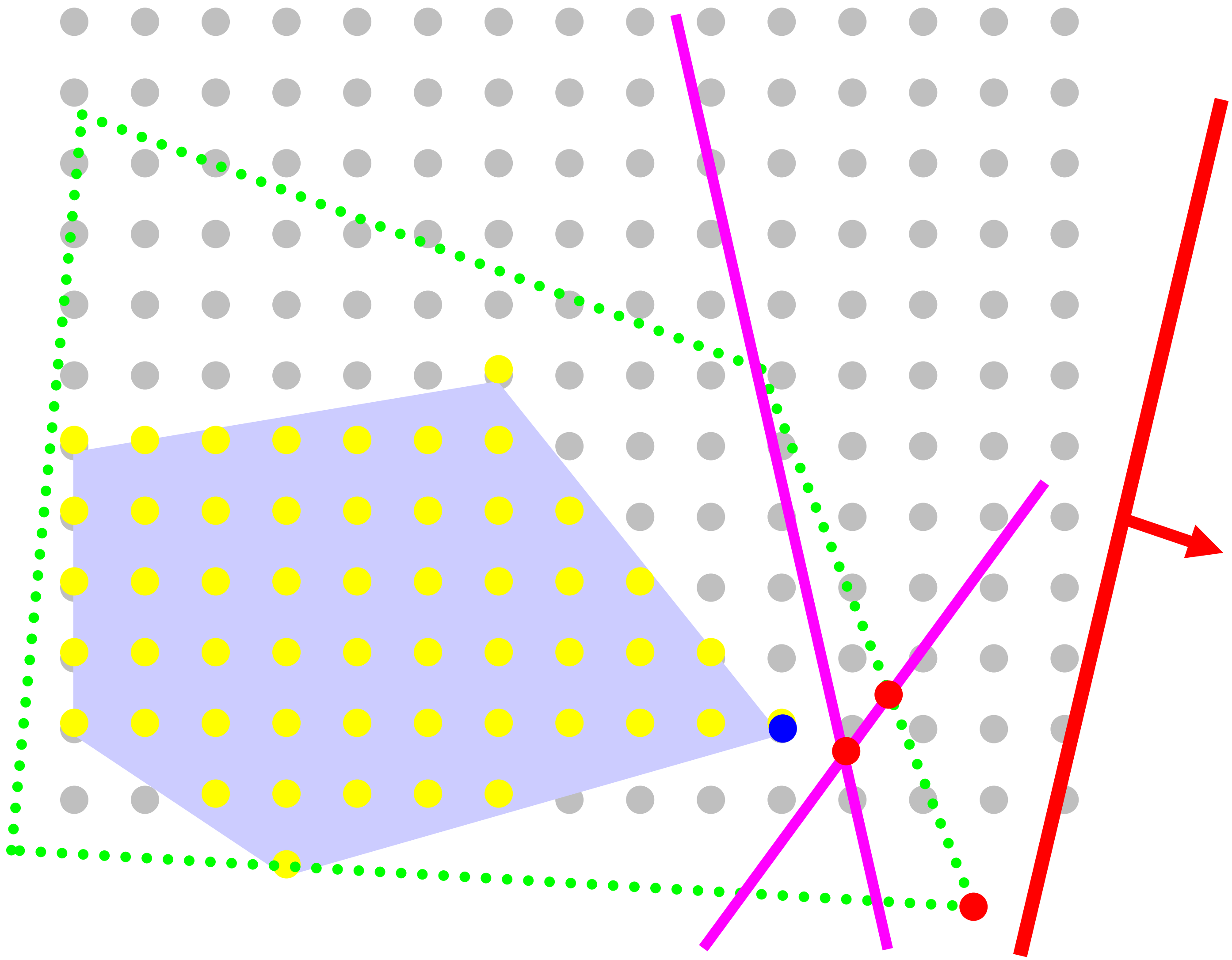
Feasible integer solutions

Objective function

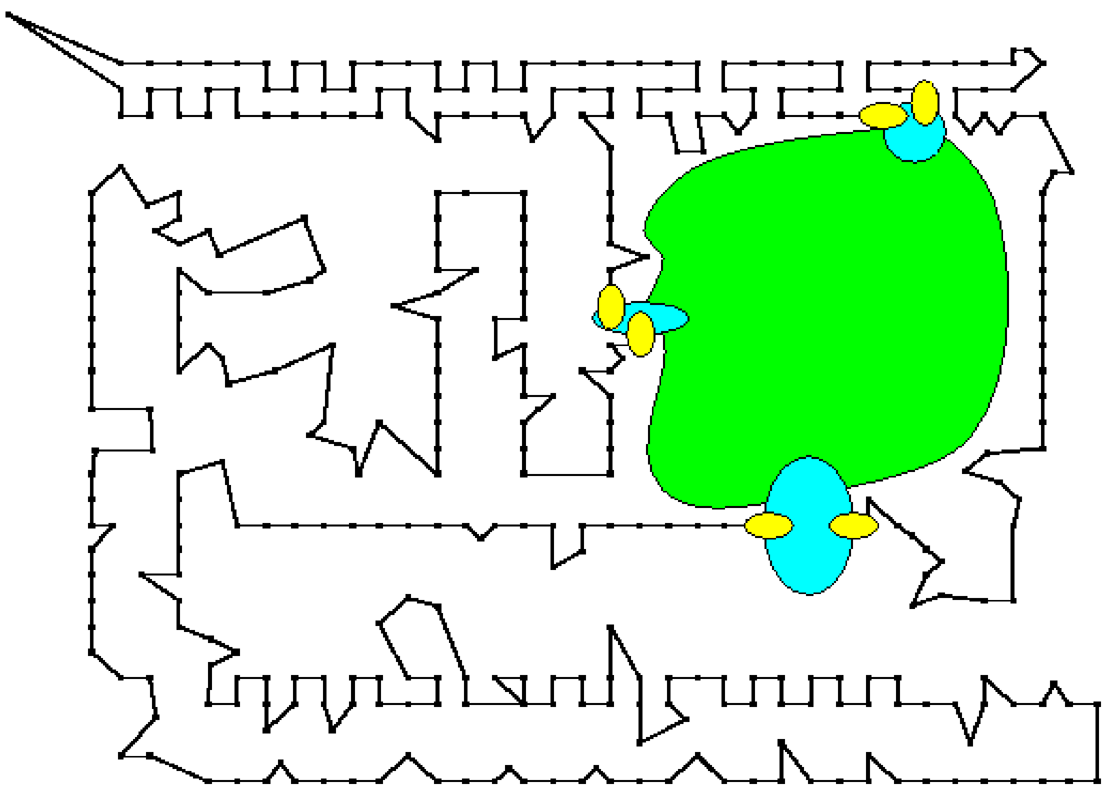
Convex hull

LP-based relaxation

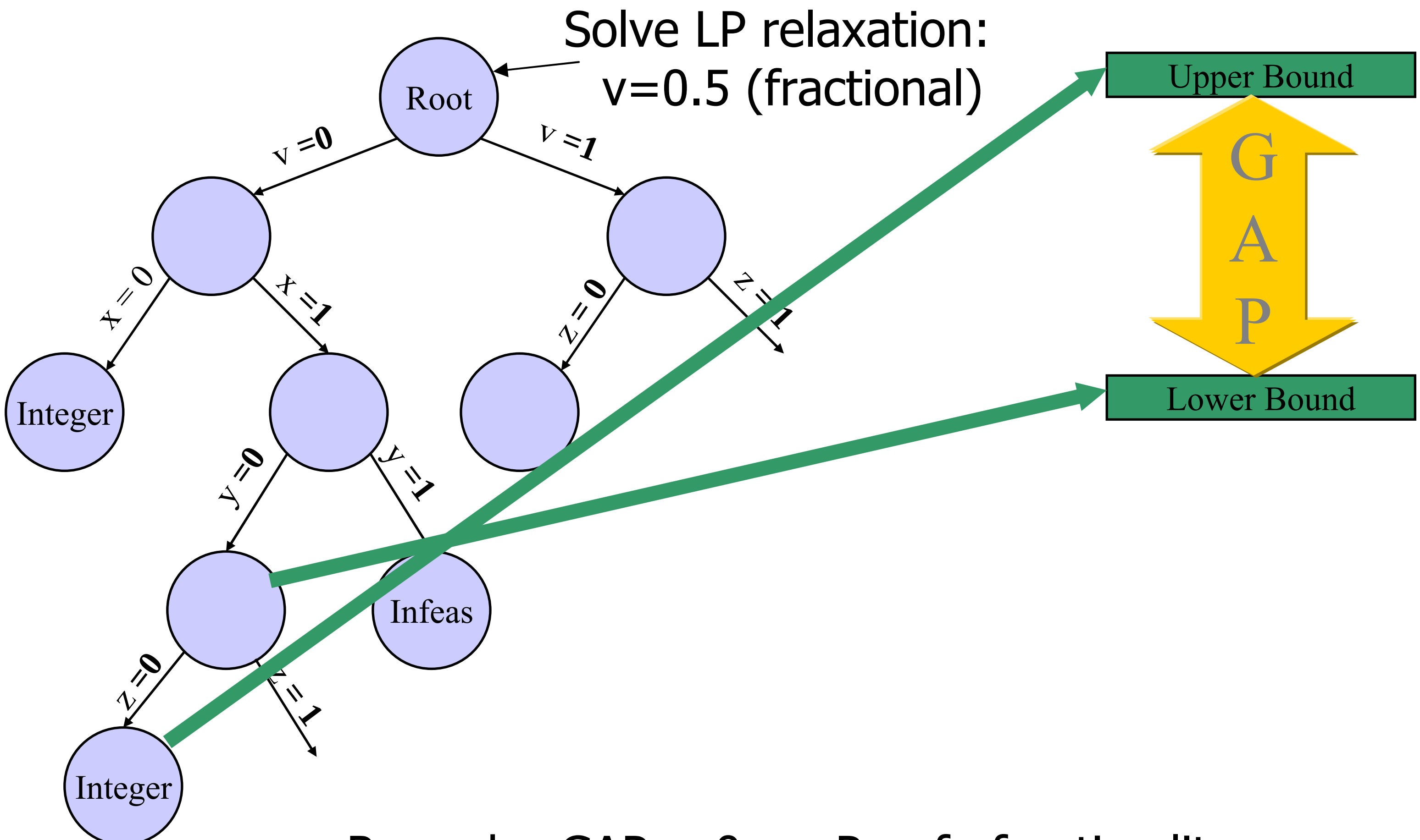
Cutting planes



Clique-tree cut for pcb442



LP-based Branch & Bound



Remark: GAP = 0 $\Rightarrow$ Proof of optimality

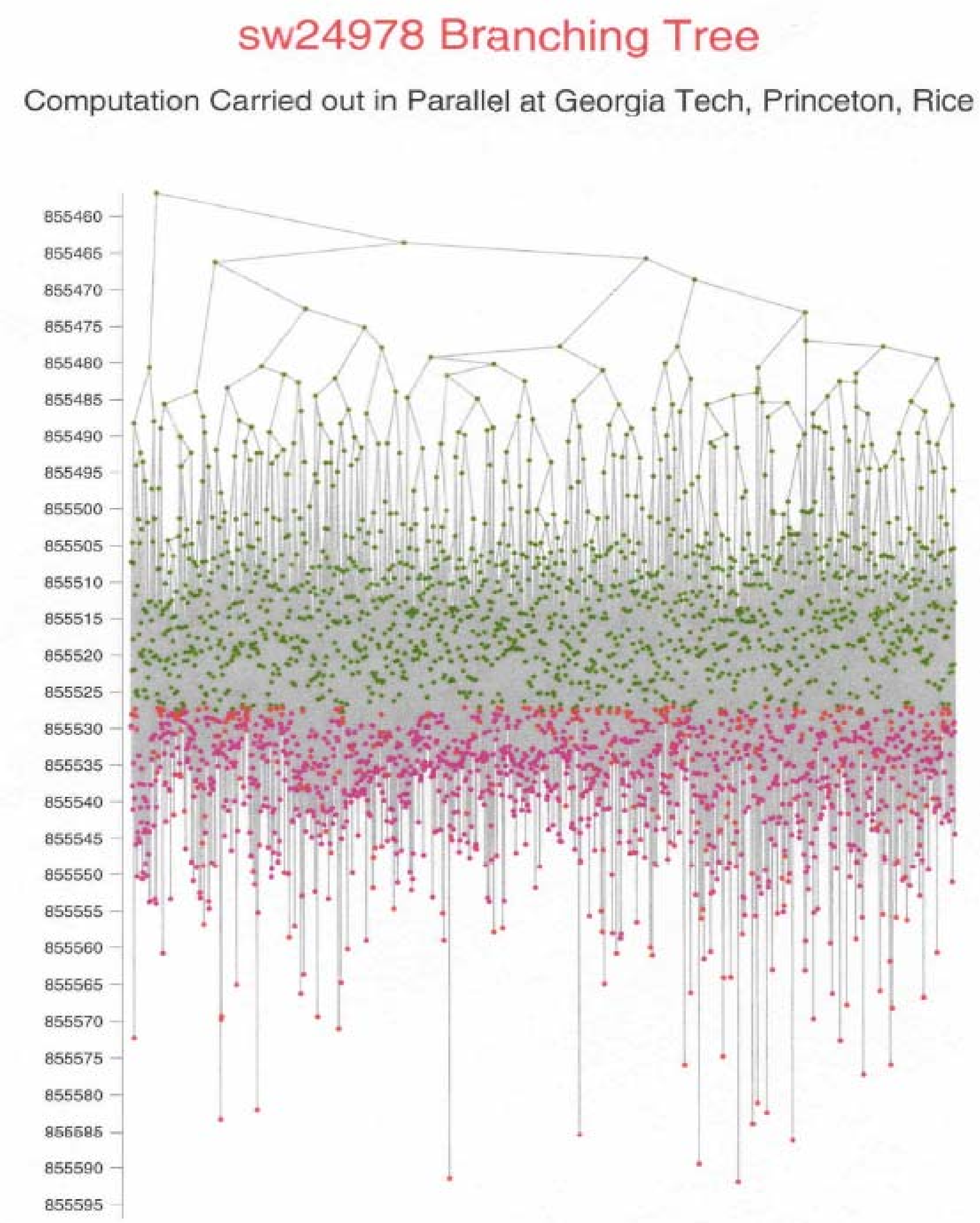
A Branching Tree

Applegate

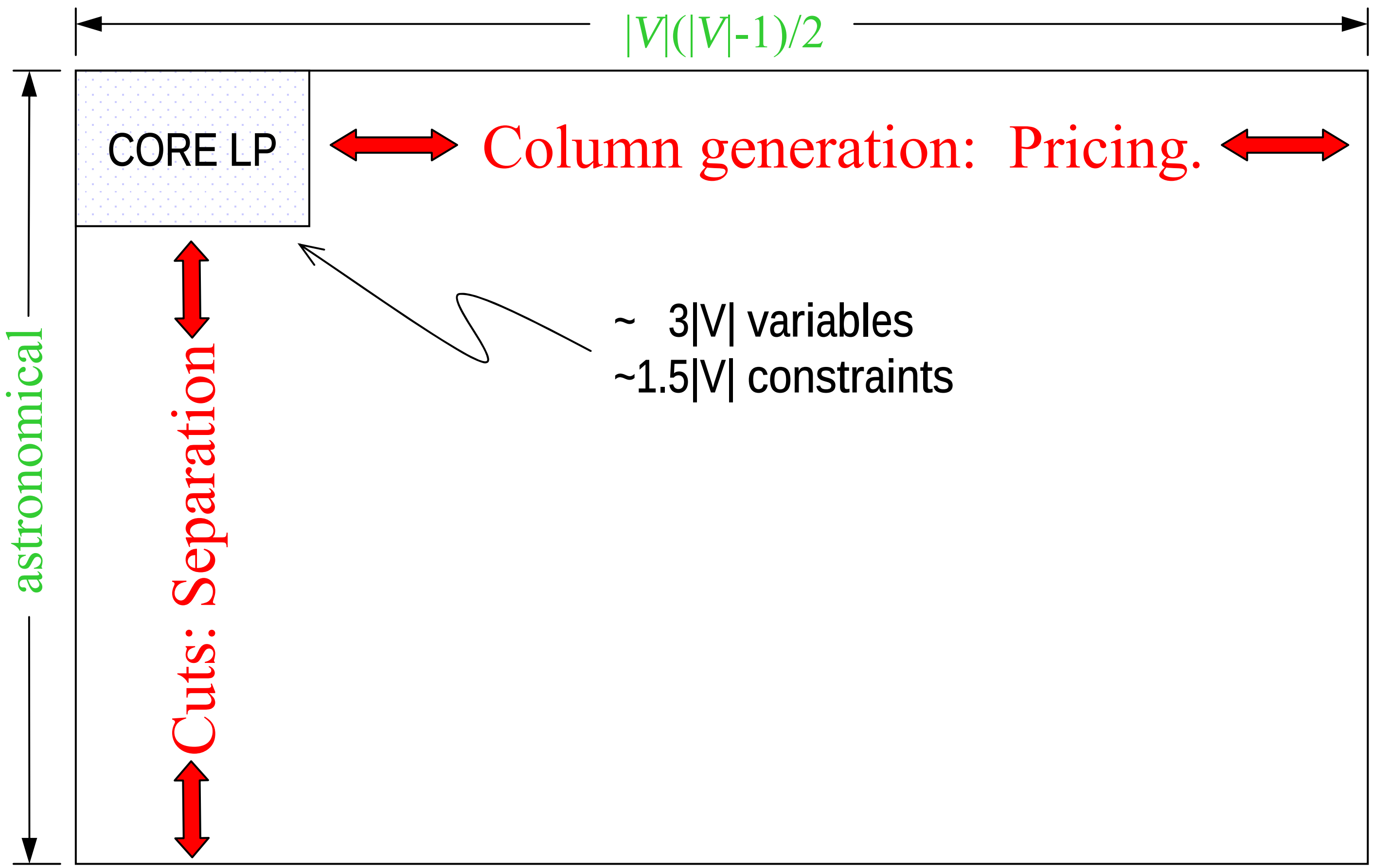
Bixby

Chvátal

Cook



Managing the LPs of the TSP



Martin Grötschel

A Pictorial History of Some TSP World Records

Some TSP World Records

number of cities 700x increase

500,000 times problem size increase

in 51 years

year	authors	# cities	# variables
1954	DFJ	42/49	1146
1977	G	120	7140
1987	PR	532	141,246
1988	GH	666	221,445
1991	PR	2,392	2,859,636
1992	ABCC	3,038	4,613,203
1994	ABCC	7,397	27,354,106
1998	ABCC	13,509	91,239,786
2001	ABCC	15,112	114,178,716
2004	ABCC	24,978	311,937,753

2005 W. Cook, D. Epsinoza, M. Goycoolea 33,810 571,541,145

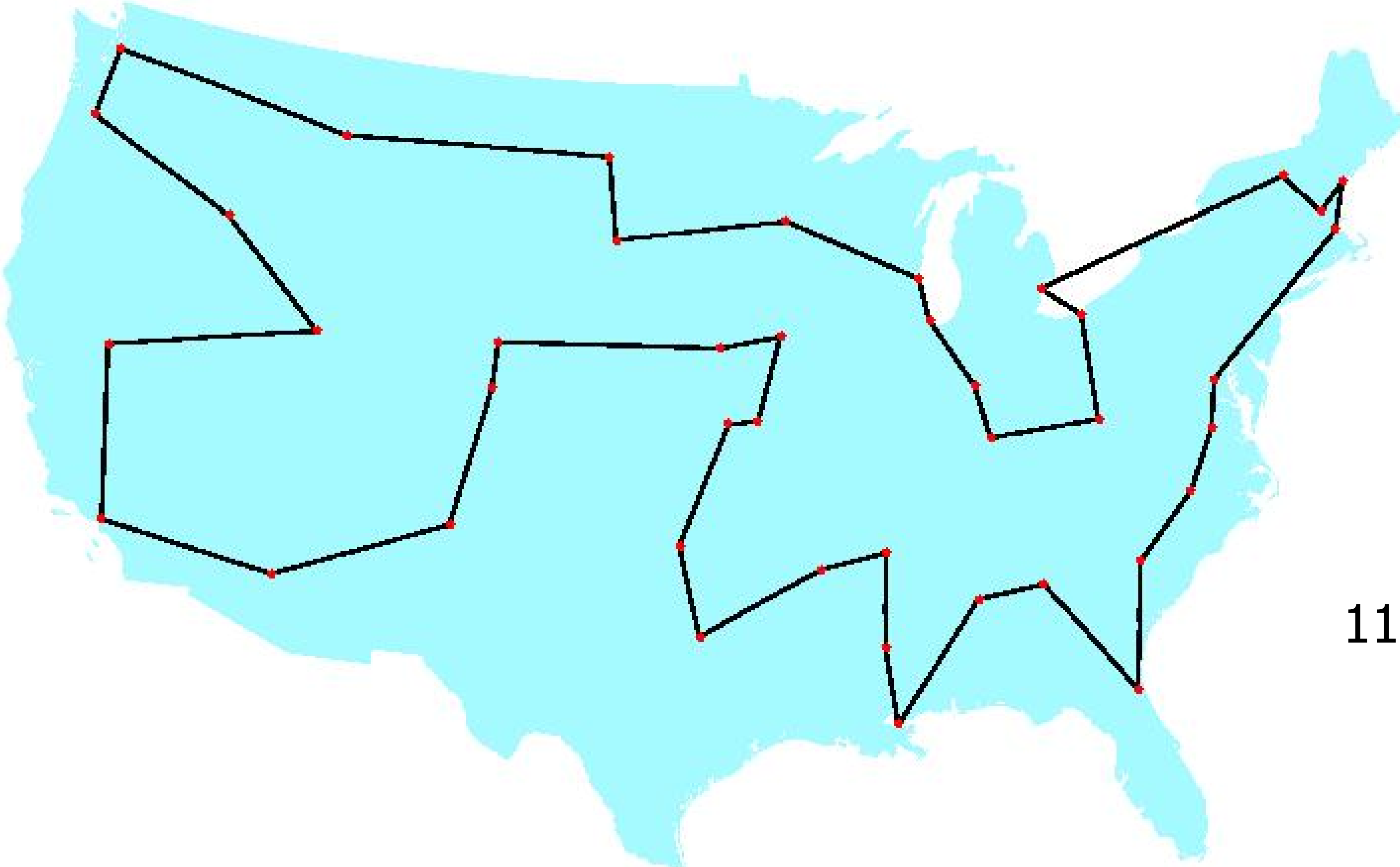
The current champions

ABCC stands for

D. Applegate, B. Bixby, W. Cook, V. Chvátal

- almost 15 years of code development
- presentation at ICM'98 in Berlin, see proceedings
- have made their code CONCORDE available in the Internet

USA 49



49 cities

1146 variables

1954

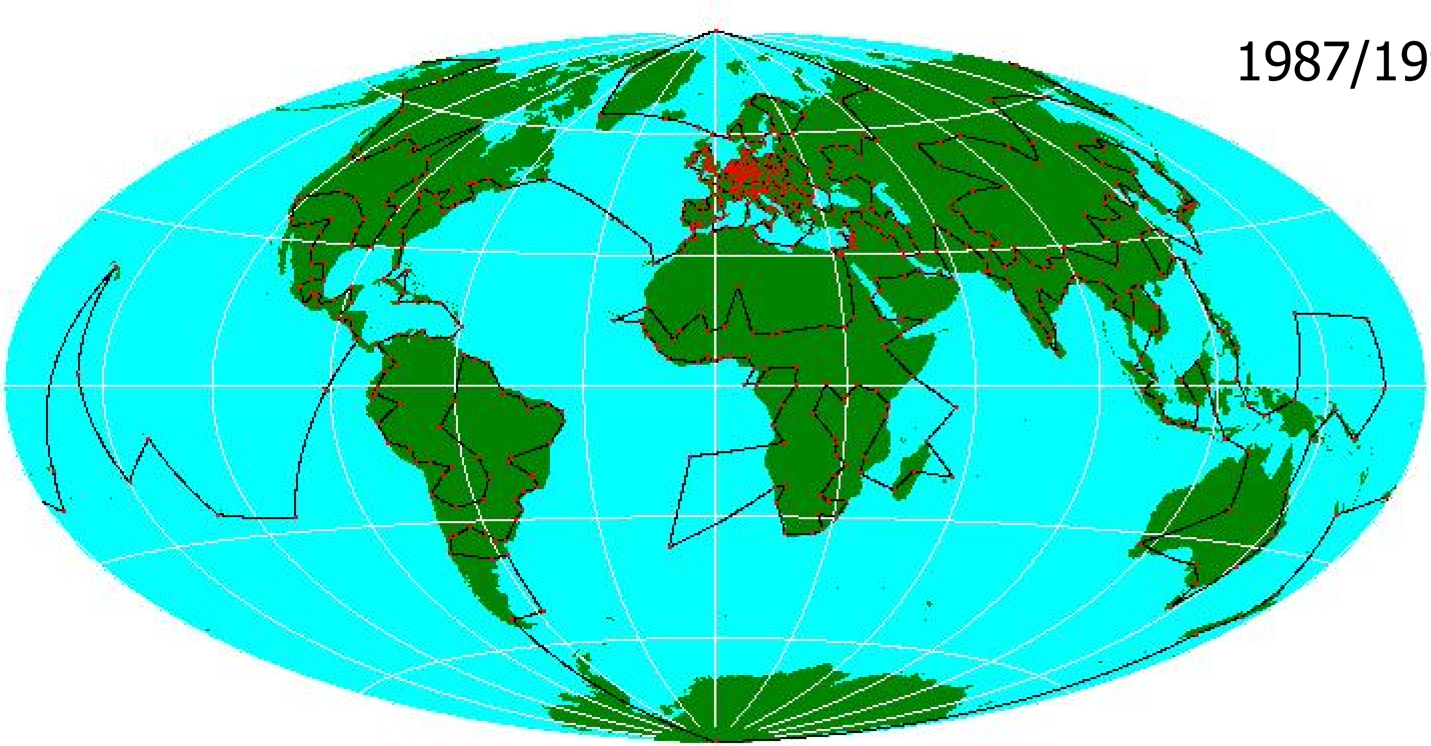
G. Dantzig, D.R. Fulkerson, S. Johnson

Die Reise um die Welt

666 Städte
221.445 Variable

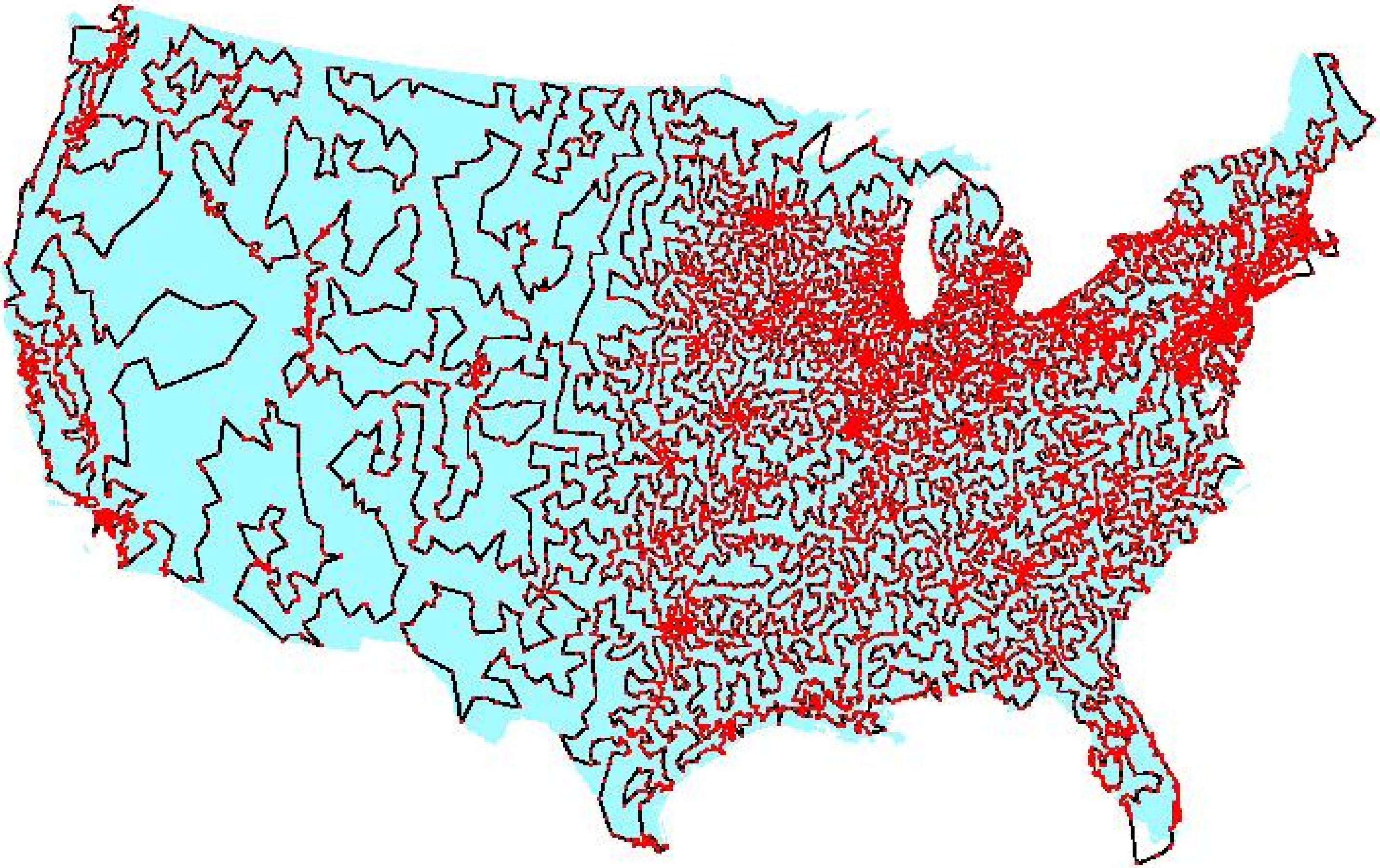
city list

1987/1991



M. Grötschel, O. Holland

USA cities with population >500

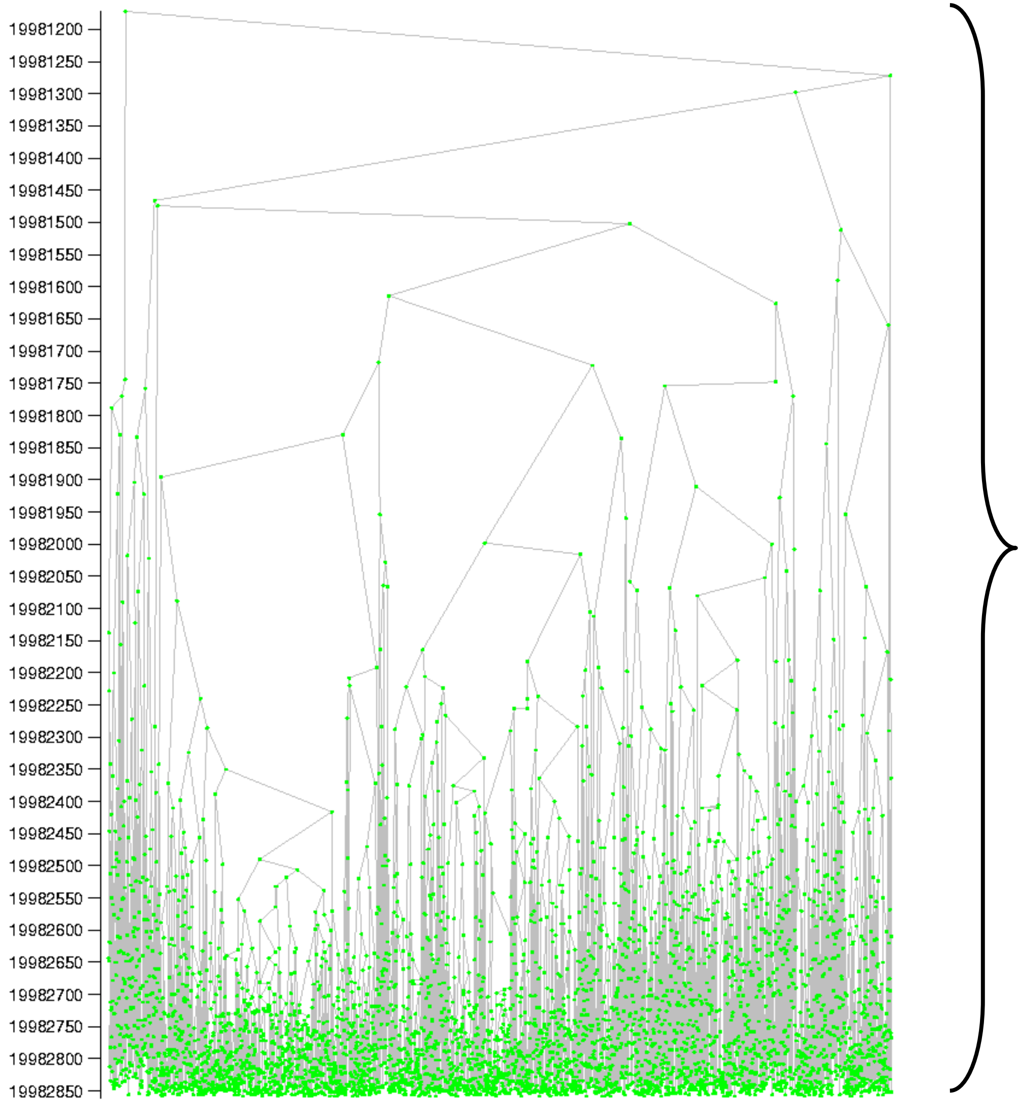


13,509 cities
91,239,786 Variables

1998

D. Applegate, R.Bixby, V. Chvátal, W. Cook

usa13509: The branching tree

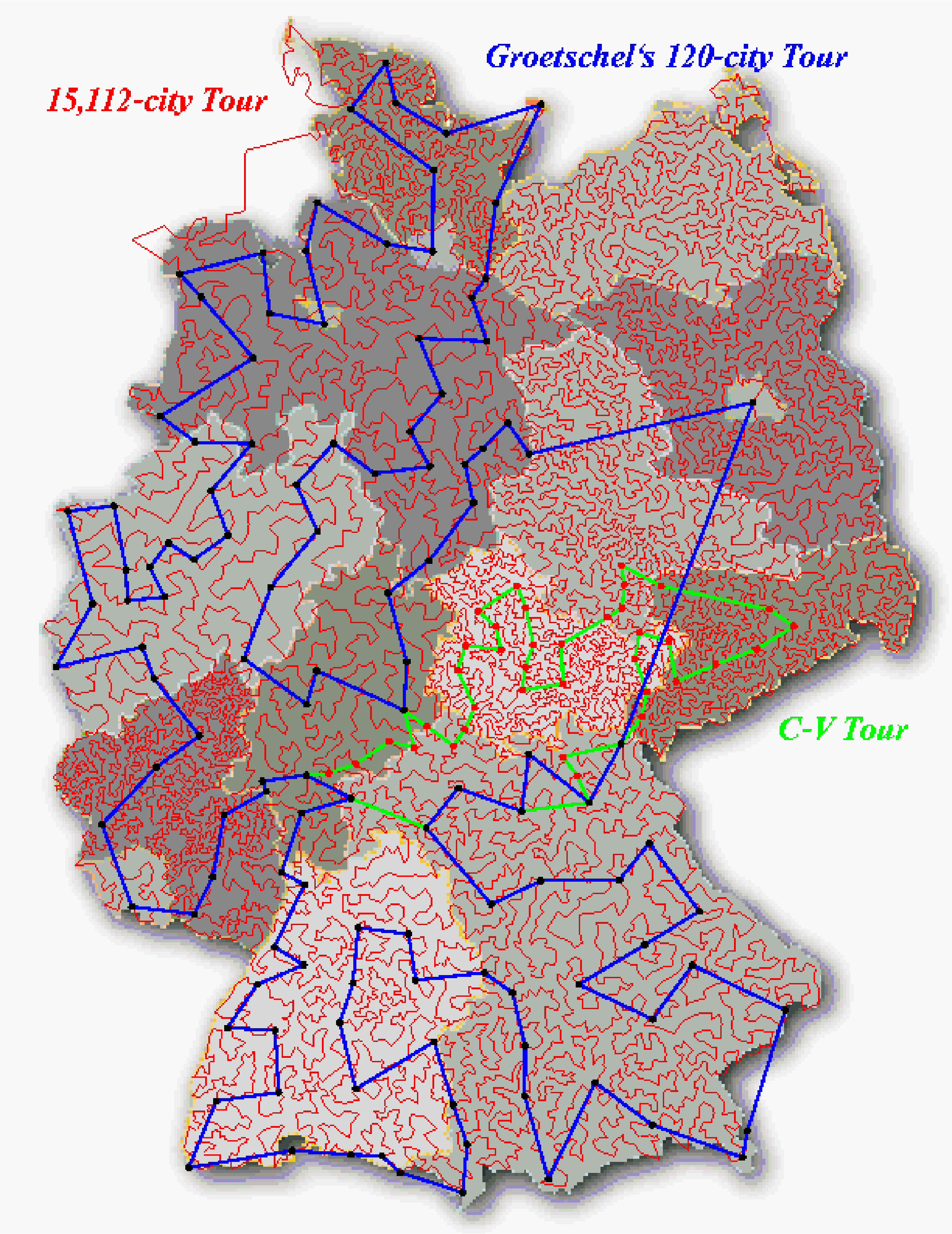


0.01% initial gap

Summary: usa13509

- 9539 nodes branching tree
- 48 workstations (Digital Alphas, Intel Pentium IIs, Pentium Pros, Sun UntraSparcs)
- Total CPU time: **4 cpu years**

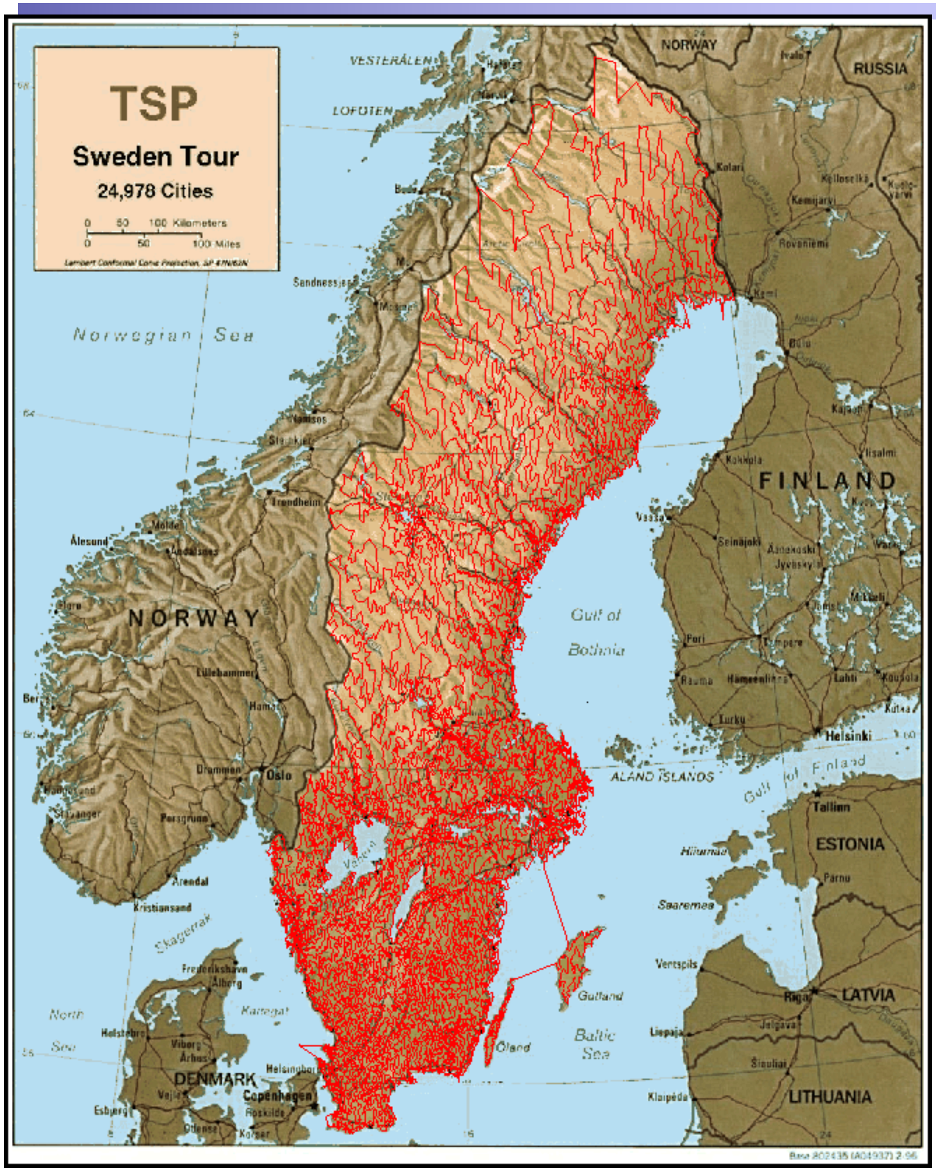
Overlay of 3 Optimal Germany tours



from
ABCC 2001

http://www.math.princeton.edu/tsp/d15sol/dhistory.html

Optimal Tour of Sweden

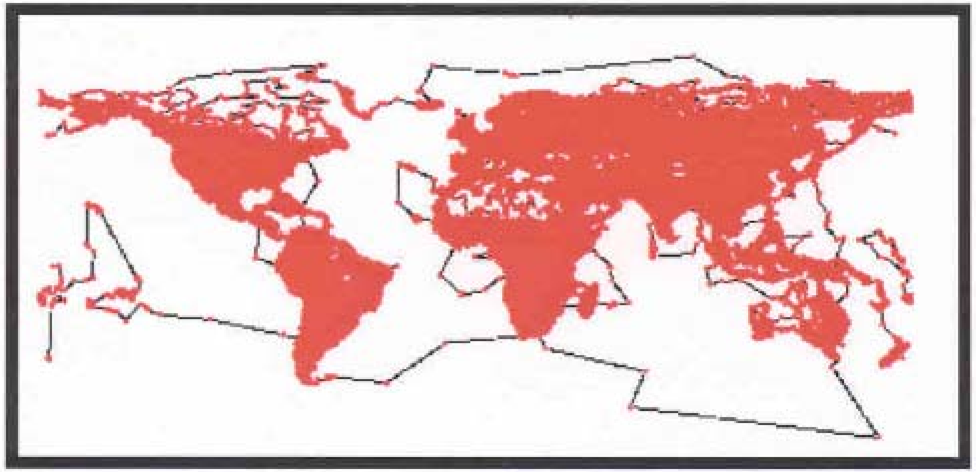


311,937,753 variables

ABCC
plus
Keld Helsgaun
Roskilde Univ.
Denmark.

The importance of LP in IP solving
(slide from Bill Cook)

1,904,711–City World TSP, 2001

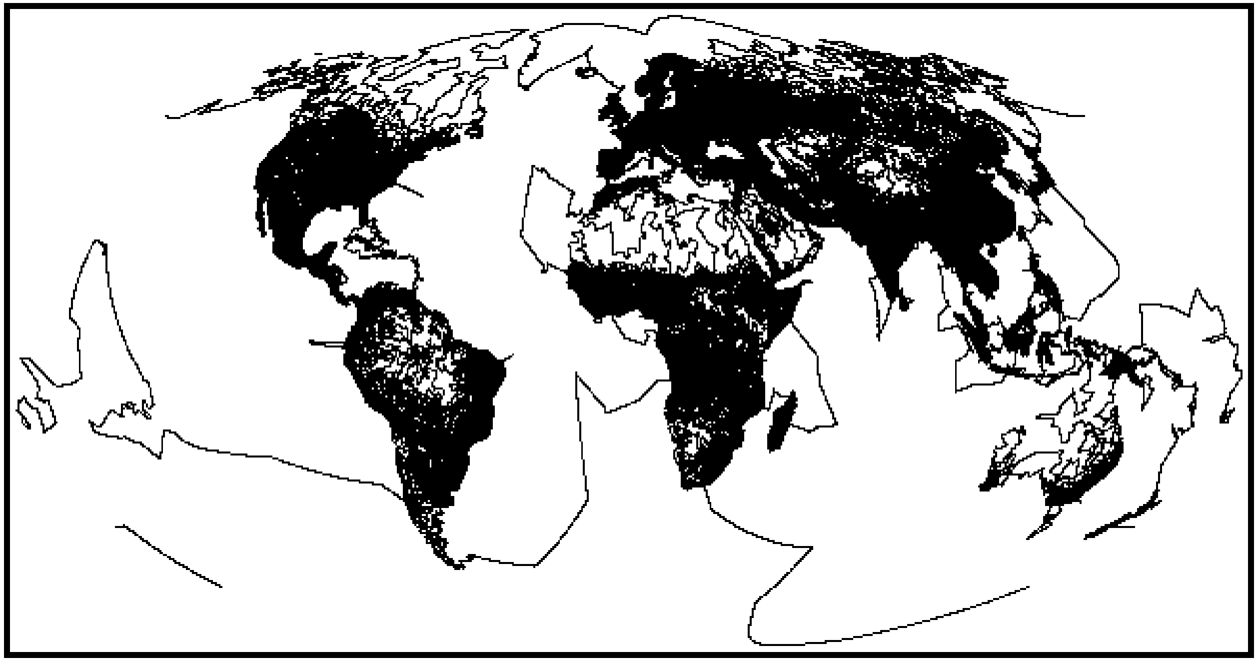


K	Optimality Gap
0	0.235%
8	0.190%
12	0.135%
14	0.111%
16	0.103%

Solution of LP Problems takes over 99% of CPU time

World Tour, current status

http://www.tsp.gatech.edu/world/



We give links to several images of the World TSP tour of length **7,516,353,779** found by Keld Helsgaun in December 2003. A lower bound provided by the Concorde TSP code shows that this tour is at most **0.076%** longer than an optimal tour through the 1,904,711 cities.

02M1 Lecture
The Travelling Salesman Problem and some Applications

The End

Martin Grötschel
- Institut für Mathematik, Technische Universität Berlin (TUB)
- DFG-Forschungszentrum "Mathematik für Schlüsseltechnologien" (MATHEON)
- Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)

groetschel@zib.de
http://www.zib.de/groetschel