01M1 Lecture
Linear and Integer Programming: an Introduction

Martin Grötschel

Beijing Block Course
"Combinatorial Optimization at Work"
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typical optimization problems

$$\max\ f(x)\ \ or\ \min\ f(x)$$
$$g_i(x) = 0,\quad i = 1, 2, \ldots, k$$
$$h_j(x) \le 0,\quad j = 1, 2, \ldots, m$$
$$x \in \mathbb{R}^n\ (and\ \ x \in S)$$

„general" (nonlinear) program NLP

$$\min c^T x$$
$$Ax = a$$
$$Bx \le b$$
$$x \ge 0$$
$$(x \in \mathbb{R}^n)$$
$$(x \in \mathbb{k}^n)$$

linear program LP

$$\min c^T x$$
$$Ax = a$$
$$Bx \le b$$
$$x \ge 0$$
$$x \in \mathbb{Z}^n$$
$$(x \in \{0,1\}^n)$$

(linear) integer program IP, MIP

program = optimization problem

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will be covered by Bob Bixby on Saturday

Linear Programming

$$\max c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$subject\ \ to$$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\cdot$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

linear program
in standard form

Linear Programming

- 1939 L. V. Kantorovitch: Foundations of linear programming (Nobel Prize 1975)
- 1947 G. B. Dantzig: Invention of the simplex algorithm

$$\max c^T x$$
$$Ax = b$$
$$x \geq 0$$

- Today: In my opinion and from an economic point of view, linear programming is the most important development of mathematics in the 20th century.

Optimal use of scarce ressources
foundation and economic interpretation of LP


Leonid V. Kantorovich   Tjalling C. Koopmans

Nobel Prize for Economics 1975

Stiglers „Diet Problem“: „The first linear program“

Min x1 + x2	costs
2x1 + x2 ≥ 3	protein
x1 + 2x2 ≥ 3	carbohydrates
x1 ≥ 0	potatoes
x2 ≥ 0	beans

minimizing the cost of food

George J. Stigler
Nobel Prize in economics 1982

Sets n nutrients / calorie thousands , protein grams , calcium grams , iron milligrams vitamin-a thousand ius, vitamin-b1 milligrams, vitamin-b2 milligrams, niacin milligrams , vitamin-c milligrams /

f foods / wheat , cornmeal , cannedmilk, margarine , cheese , peanut-b , lard liver , porkroast, salmon , greenbeans, cabbage , onions , potatoes spinach, sweet-pot, peaches , prunes , limabeans, navybeans /

Parameter b(n) required daily allowances of nutrients / calorie 3, protein 70 , calcium .8 , iron 12 vitamin-a 5, vitamin-b1 1.8, vitamin-b2 2.7, niacin 18, vitamin-c 75 /

Table a(f,n) nutritive value of foods (per dollar spent)

	calorie	protein	calcium	iron	vitamin-a	vitamin-b1	vitamin-b2	niacin	vitamin-c
	(1000)	(g)	(g)	(mg)	(1000iu)	(mg)	(mg)	(mg)	(mg)
wheat	44.7	1411	2.0	365		55.4	33.3	441	
cornmeal	36	897	1.7	99	30.9	17.4	7.9	106	
cannedmilk	8.4	422	15.1	9	26	3	23.5	11	60
margarine	20.6	17	.6	6	55.8	.2			
cheese	7.4	448	16.4	19	28.1	.8	10.3	4	
peanut-b	15.7	661	1	48		9.6	8.1	471	
lard	41.7				.2		.5	5	
liver	2.2	333	.2	139	169.2	6.4	50.8	316	525
porkroast	4.4	249	.3	37		18.2	3.6	79	
salmon	5.8	705	6.8	45	3.5	1	4.9	209	
greenbeans	2.4	138	3.7	80	69	4.3	5.8	37	862
cabbage	2.6	125	4	36	7.2	9	4.5	26	5369
onions	5.8	166	3.8	59	16.6	4.7	5.9	21	1184
potatoes	14.3	336	1.8	118	6.7	29.4	7.1	198	2522
spinach	1.1	106		138	918.4	5.7	13.8	33	2755
sweet-pot	9.6	138	2.7	54	290.7	8.4	5.4	83	1912
peaches	8.5	87	1.7	173	86.8	1.2	4.3	55	57
prunes	12.8	99	2.5	154	85.7	3.9	4.3	65	257
limabeans	17.4	1055	3.7	459	5.1	26.9	38.2	93	
navybeans	26.9	1691	11.4	792		38.4	24.6	217	

Positive Variable x(f) dollars of food f to be purchased daily (dollars)

Free Variable cost total food bill (dollars)

Equations nb(n) nutrient balance (units), cb cost balance (dollars) ;

nb(n).. sum(f, a(f,n)*x(f)) =g= b(n); cb.. cost=e= sum(f, x(f));

Model diet stiglers diet problem / nb,cb /;

Solution of the Diet Problem

Goal: find the cheapest combination of foods that will satisfy the daily requirements of a person

motivated by the army's desire to meet nutritional requirements of the soldiers at minimum cost

Army's problem had 77 unknowns and 9 constraints.
Stigler solved problem using a heuristic: $39.93/year (1939)
Laderman (1947) used simplex: $39.69/year (1939 prices)

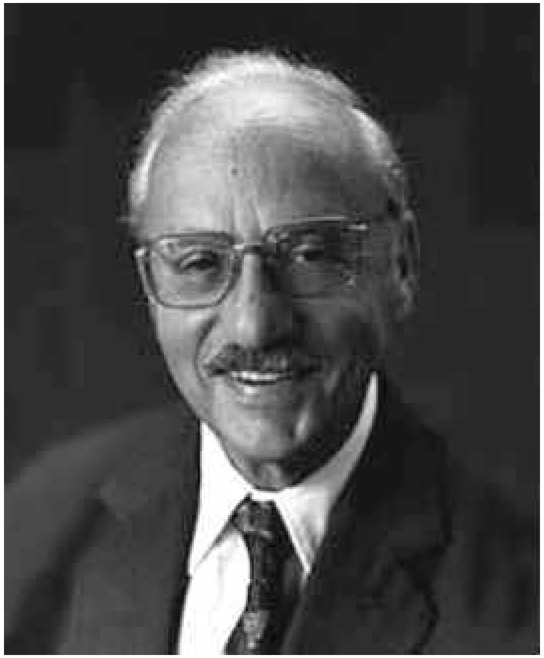
→ first "large-scale computation"
took 120 man days on hand operated
desk calculators (10 human "computers")

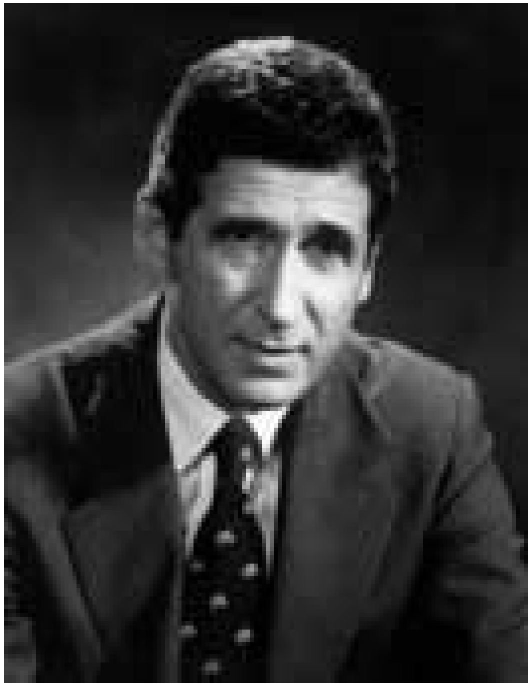
http://www.mcs.anl.gov/home/otc/Guide/CaseStudies/diet/index.html

Milton Friedman on George J. Stigler

An early example of the latter is an article on "The Cost of Subsistence" (1945), which starts, "Elaborate investigations have been made of the adequacy of diets at various income levels, and a considerable number of 'low-cost,' 'moderate,' and 'expensive' diets have been recommended to consumers. Yet, so far as I know, no one has determined the minimum cost of obtaining the amounts of calories, proteins, minerals, and vitamins which these studies accept as adequate or optimum." George then set himself to determine the minimum cost diet, in the process producing one of the earliest formulations of a linear programming problem in economics, for which he found an approximate solution, explaining that "there does not appear to be any direct method of finding the minimum of a linear function subject to linear constraints." Two years later George Dantzig provided such a direct method, the simplex method, now widely used in many economic and industrial applications.

George Dantzig and Ralph Gomory





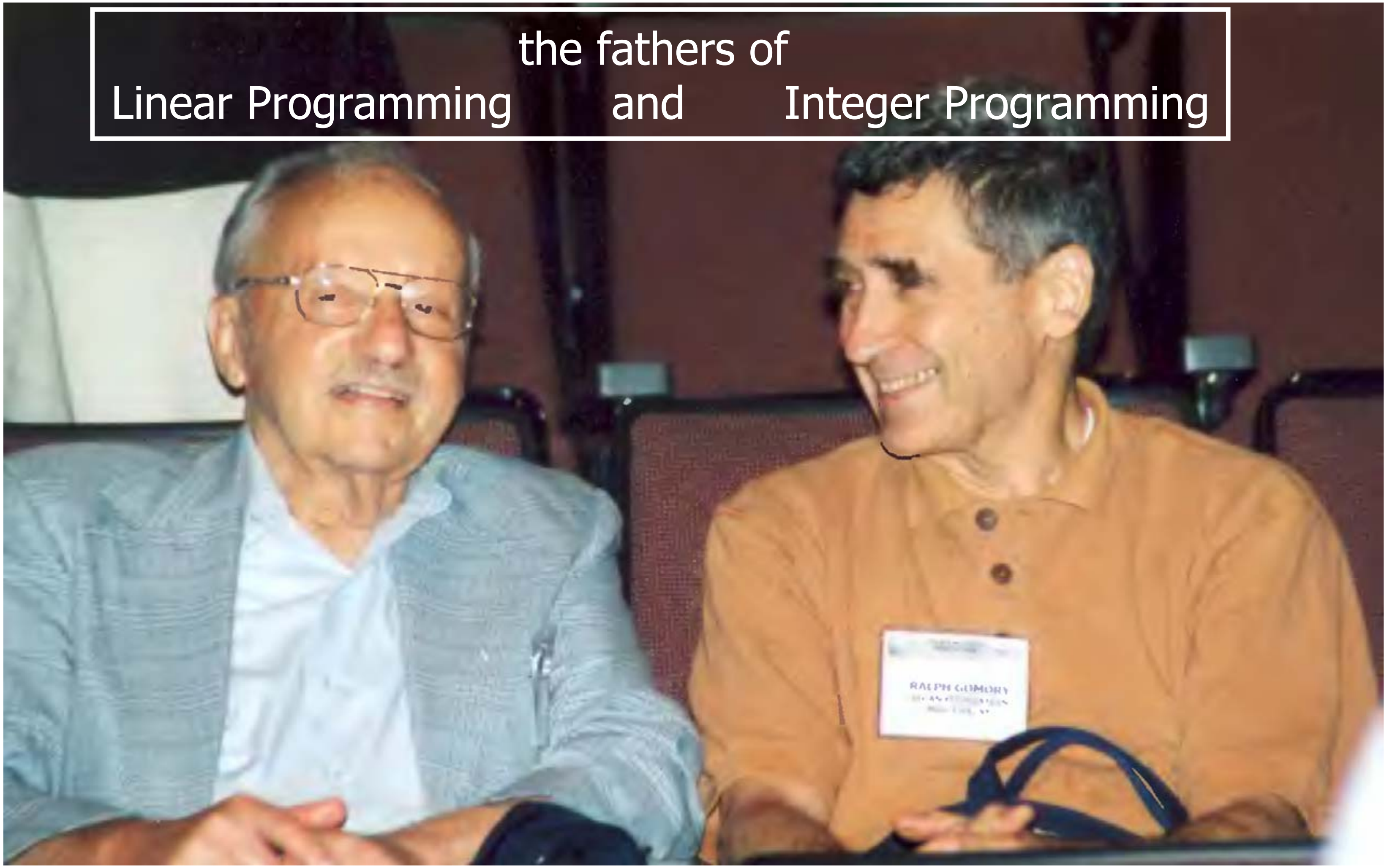
„founding fathers"

~1950
linear programming

~1960
integer programming

George Dantzig and Ralph Gomory

ISMP Atlanta 2000



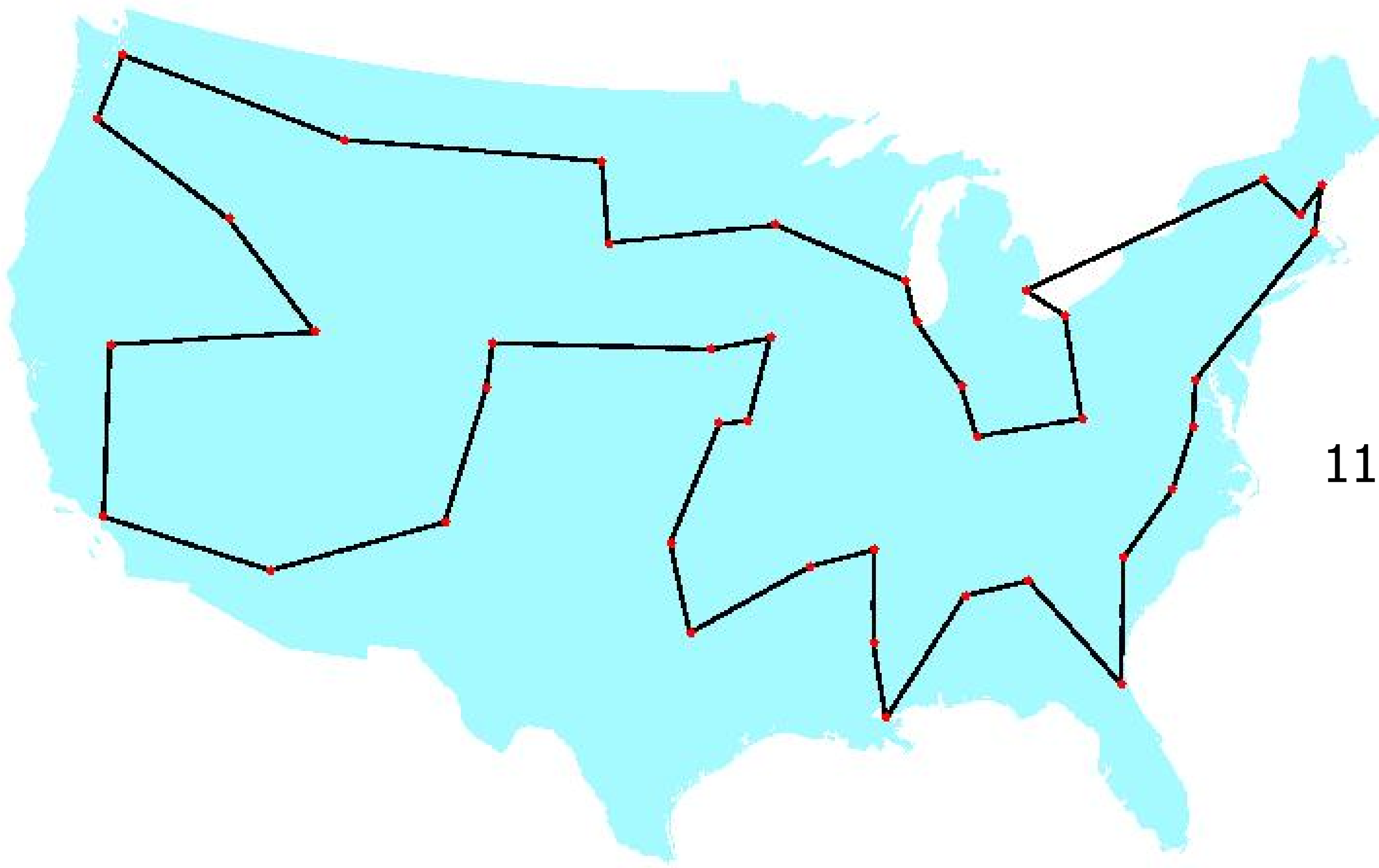
Dantzig and Bixby

George Dantzig and Bob Bixby

at the International Symposium on Mathematical Programming,

Atlanta, August 2000

1954, the Beginning of IP
G. Dantzig, D.R. Fulkerson, S. Johnson



USA
49 cities
1146 variables

1954

Commercial software

William Orchard-Hayes (in the period 1953-1954)

The first commercial LP-Code was on the market in 1954 (i.e., ~50 years ago) and available on an IBM CPC (card programmable calculator):

Code: Simplex Algorithm with explicit basis inverse, that was recomputed in each step.

Shortly after, Orchard-Hayes implemented a version with product form of the inverse (idea of A. Orden),
Record: 71 variables, 26 constraints, 8 h running time

Contents

Optimizers' dream: Duality theorems

- Max-Flow Min-Cut Theorem
  The maximal (s,t)-flow in a capacitated network is equal to the minimal capacity of an (s,t)-cut.
- The Duality Theorem of linear programming

$$\begin{array}{ccc} \max c^T x & & \min y^T b \\ Ax \le b & = & y^T A \ge c^T \\ x \ge 0 & & y \ge 0 \end{array}$$

Optimizers' dream: Duality theorems for integer programming

- The Max-Flow Min-Cut Theorem does not hold if several source-sink relations are given (multicommodity flow).
- The Duality Theorem of linear programming does not hold if integrality conditions are added

$$\begin{array}{ccc} \max c^T x & \le & \min y^T b \\ Ax \le b & & y^T A \ge c^T \\ x \ge 0 & & y \ge 0 \\ x \in \mathbb{Z}^n & & y \in \mathbb{Z}^m \end{array}$$

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Important theorems

- complementary slackness theorems
- redundancy characterizations

LP Solvability

- Linear programs can be solved in polynomial time with
  - the Ellipsoid Method (Khachiyan, 1979)
  - Interior Points Methods (Karmarkar, 1984, and others)
- Open: is there a strongly polynomial time algorithm for the solution of LPs?
- Certain variants of the Simplex Algorithm run – under certain conditions – in expected polynomial time (Borgwardt, 1977…)
- Open: Is there a polynomial time variant of the Simplex Algorithm?

LP Solvability: Generalizations

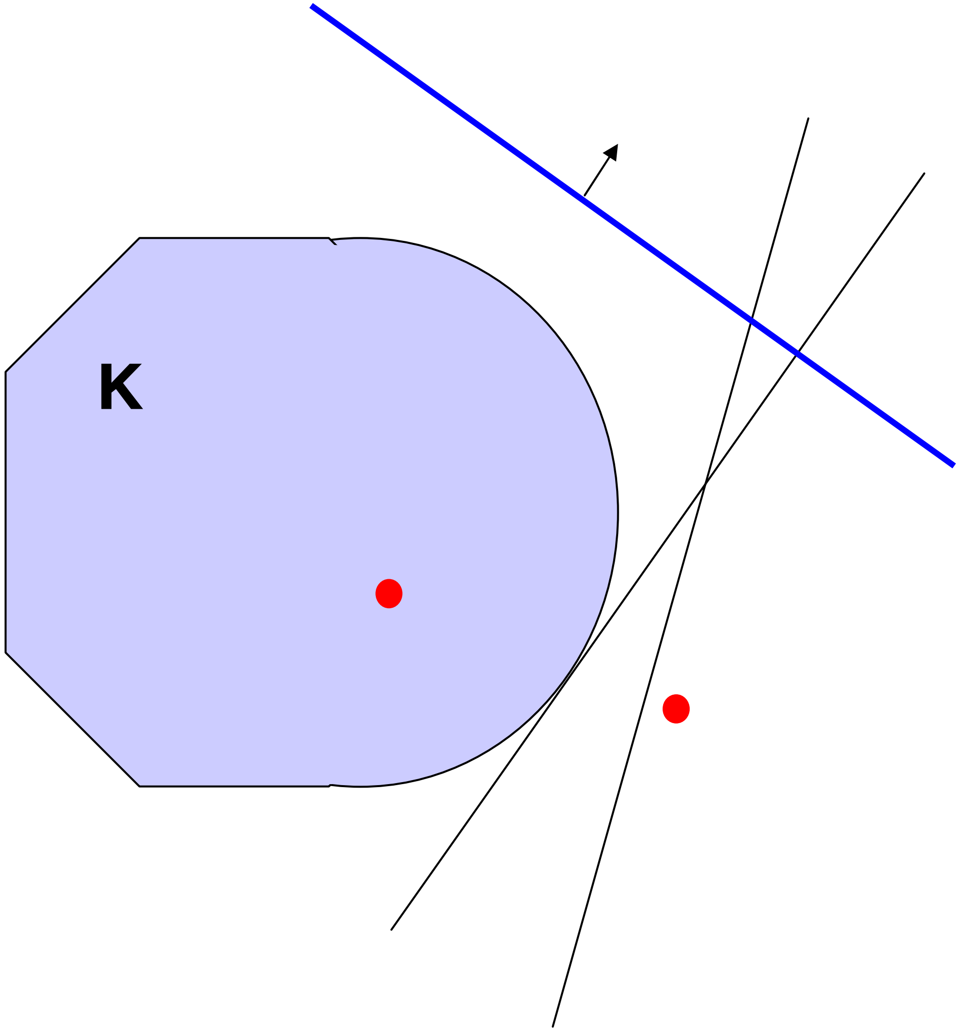
Theorem (GLS 1979, 1988) (modulo technical details) **:**

There exists a polynomial time algorithm to minimize convex functions (e.g., linear functions) over the elements of a class of convex bodies **K** (e. g. polyhedra) if and only if, there exists a polynomial time algorithm that decides, for any given point **x**, whether **x** is in **K**, and that, when **x** is not in **K**, finds a hyperplane that separates **x** from **K.**

Short version: Optimization and Separation are polynomial-time equivalent.

Consequence: Theoretical Foundation of cutting plane algorithms.

Separation



IP Solvability

Theorem

Integer, 0/1, and mixed integer programming are NP-hard.

Consequence

- special cases
- special purposes
- heuristics

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Algorithms for nonlinear programming

- Iterative methods that solve the equation and inequality sytems representing the necessary local optimality conditions.

$$x_{i+1} = x_i + \lambda_i d_i$$
$$d_i \sim \text{"descent direction"}$$
$$\lambda_i \sim \text{"steplength"}$$

$$d_i = -\nabla f(x_i)$$ Steepest descent

$$d_i = -(H(x_i))^{-1} \nabla f(x_i)$$ Newton

(Quasi-Newton, conjugate-gradient-, SQP-, subgradient...methods)

- Sufficient conditions are rarely checked.

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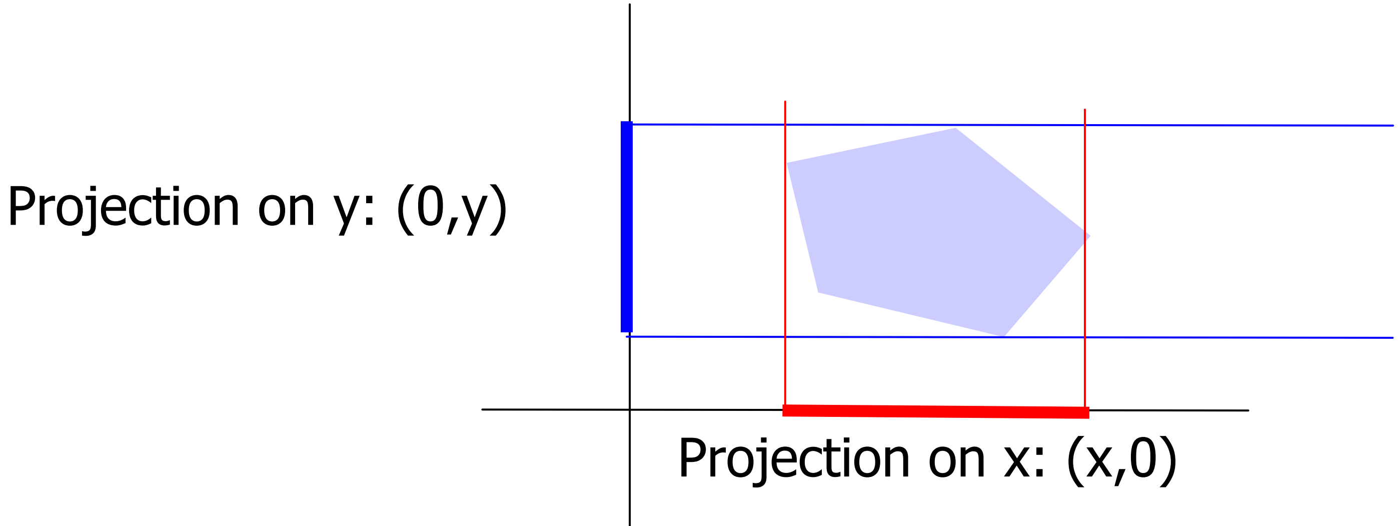
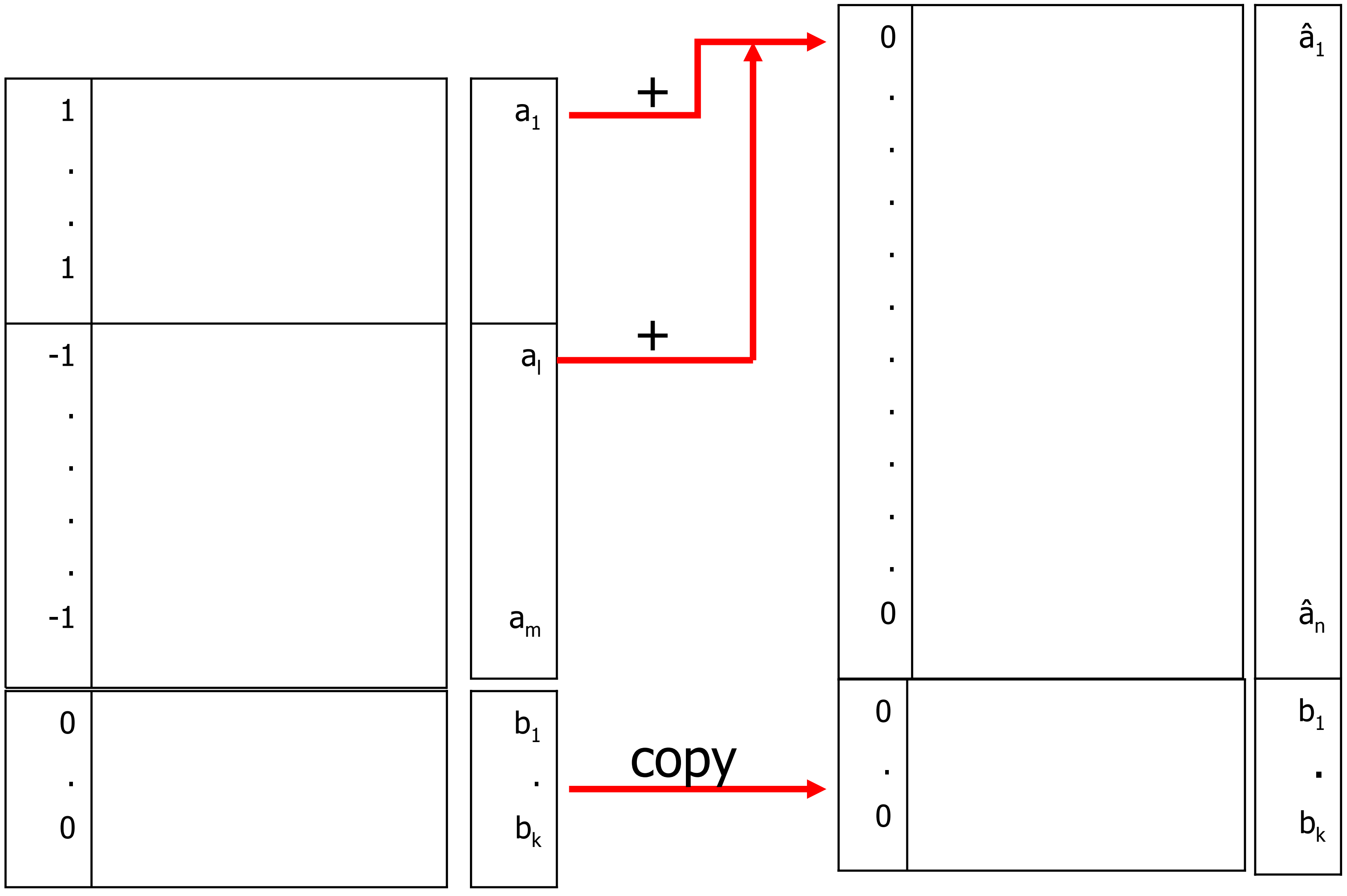
Fourier-Motzkin Elimination

- Fourier, 1847
- Motzkin, 1938
- **Method**: successive projection of a polyhedron in n-dimensional space into a vector space of dimension n-1 by elimination of one variable.



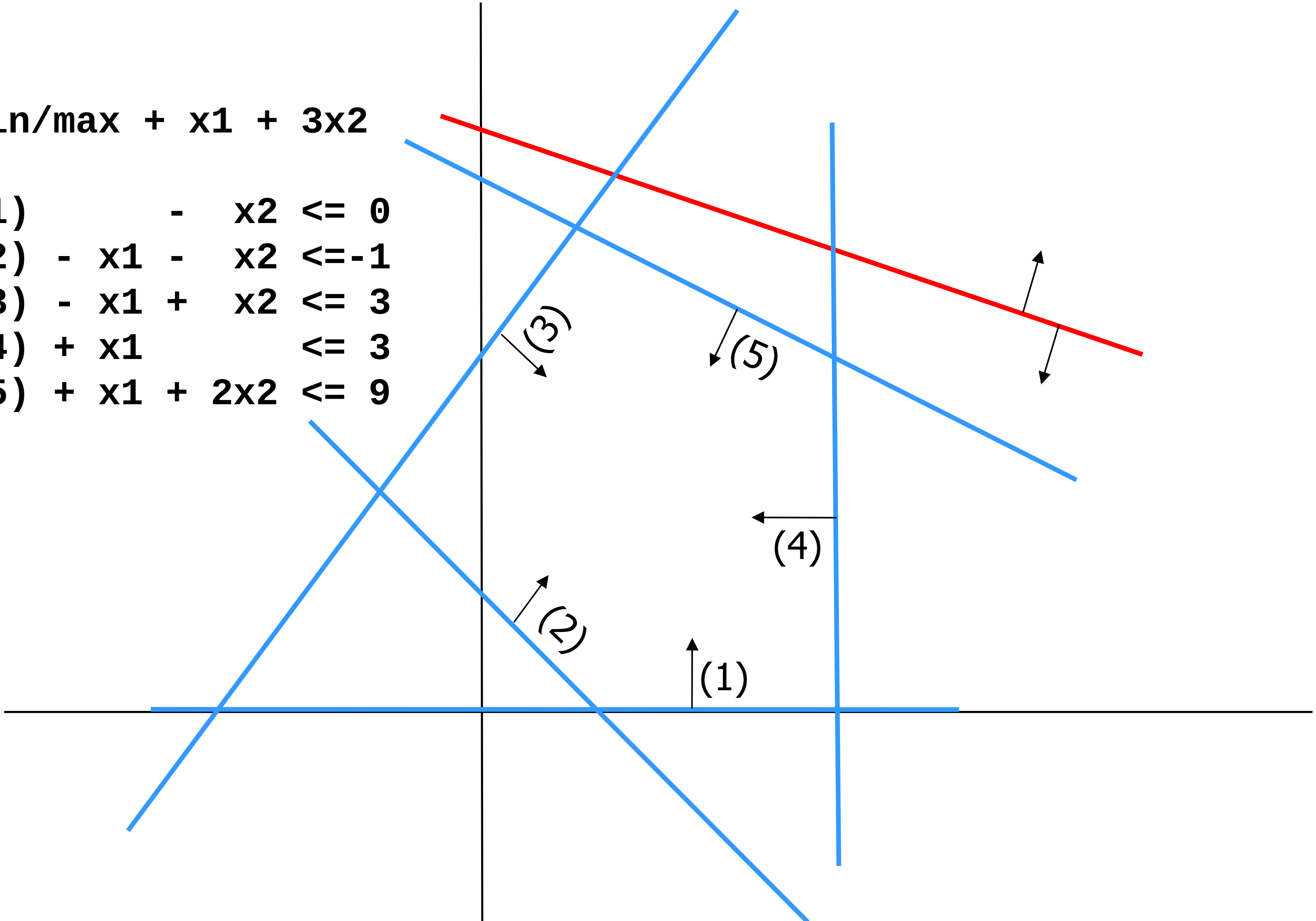
A Fourier-Motzkin step



Fourier-Motzkin Elimination: an example

```
min/max + x1 + 3x2

(1)        -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1          <= 3
(5) + x1 + 2x2 <= 9
```



Fourier-Motzkin Elimination: an example, call of PORTA

```
min/max + x1 + 3x2

(1)       -   x2 <= 0
(2) - x1 -    x2 <=-1
(3) - x1 +    x2 <= 3
(4) + x1         <= 3
(5) + x1 + 2x2 <= 9
```

→

```
DIM = 3

INEQUALITIES_SECTION

(1)       -   x2         <= 0
(2) - x1 -    x2         <=-1
(3) - x1 +    x2         <= 3
(4) + x1                 <= 3
(5) + x1 + 2x2           <= 9
(6) + x1 + 3x2 - x3 <= 0
(7) - x1 - 3x2 + x3 <= 0


ELIMINATION_ORDER
1 0 0
```

Fourier-Motzkin Elimination: an example

```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2        <=  0
(2,4) (2) - x2        <=  2
(2,5) (3) + x2        <=  8
(2,6) (4) +2x2 - x3 <= -1
(3,4) (5) + x2        <=  6
(3,5) (6) + x2        <=  4
(3,6) (7) +4x2 - x3 <=  3
(7,4) (8) -3x2 + x3 <=  3
(7,5) (9) - x2 + x3 <=  9
(7,6)
```

```
DIM = 3

INEQUALITIES_SECTION

(1)        -  x2       <= 0
(2) - x1 -  x2       <=-1
(3) - x1 +  x2       <= 3
(4) + x1               <= 3
(5) + x1 + 2x2       <= 9
(6) + x1 + 3x2 - x3 <= 0
(7) - x1 - 3x2 + x3 <= 0

ELIMINATION_ORDER
1 0 0
```

Fourier-Motzkin Elimination: an example

```
DIM = 3

INEQUALITIES_SECTION

(1)   (1) - x2          <=  0
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(3,4) (5) + x2          <=  6
(3,5) (6) + x2          <=  4
(3,6) (7) +4x2 - x3 <=  3
(7,4) (8) -3x2 + x3 <=  3
(7,5) (9) - x2 + x3 <=  9
(7,6)

ELIMINATION_ORDER
0 1 0
```

```
(1,4) (  1) -x3 <=  -1
(1,7) (  2) -x3 <=   3
(2,4) (  3) -x3 <=   3
(2,7) (  4) -x3 <=  11
(8,3) (  5) +x3 <=  27
(8,4) (  6) -x3 <=   3
(8,5) (  7) +x3 <=  21
(8,6) (  8) +x3 <=  15
(8,7) (  9) +x3 <=  21
(9,3) ( 10) +x3 <=  17
(9,4) ( 11) +x3 <=  17
(9,5) ( 12) +x3 <=  15
(9,6) ( 13) +x3 <=  13
(9,7) ( 14)+3x3 <=  39
```

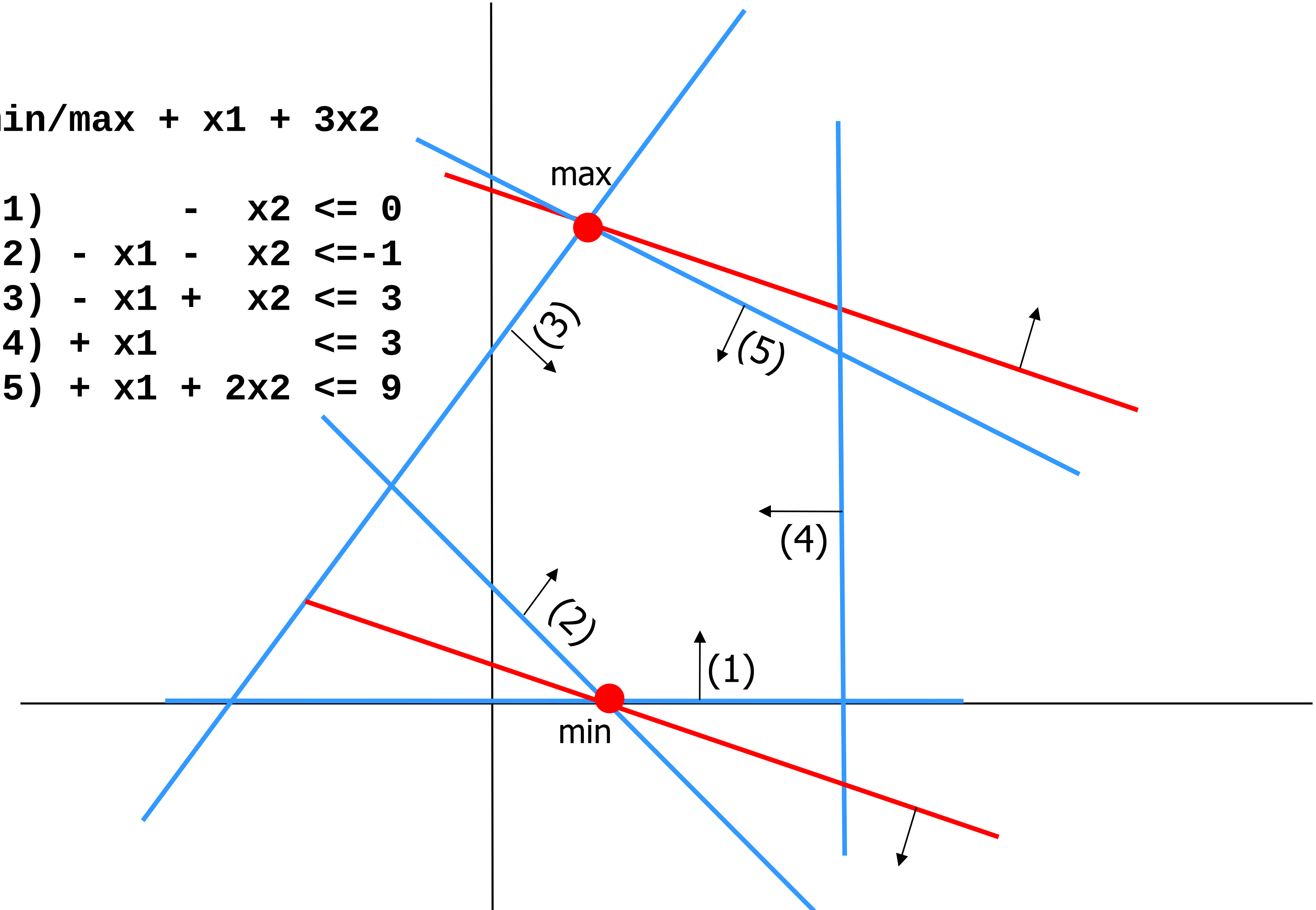
min = 1 <= x3 <= 13 = max

x1 = 1 x1 = 1
x2 = 0 x2 = 4

Fourier-Motzkin Elimination: an example

```
min/max + x1 + 3x2

(1)       -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
```



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will be covered by Bob Bixby on Saturday

The Simplex Method

- Dantzig, 1947: primal Simplex Method
- Lemke, 1954; Beale, 1954: dual Simplex Method
- Dantzig, 1953: revised Simplex Method
- ....
- **Underlying Idea**: Find a vertex of the set of feasible LP solutions (polyhedron) and move to a better neighbouring vertex, if possible.

The Simplex Method: an example

```
min/max + x1 + 3x2

(1)        -   x2 <= 0
(2) - x1 -   x2 <=-1
(3) - x1 +   x2 <= 3
(4) + x1        <= 3
(5) + x1 + 2x2 <= 9
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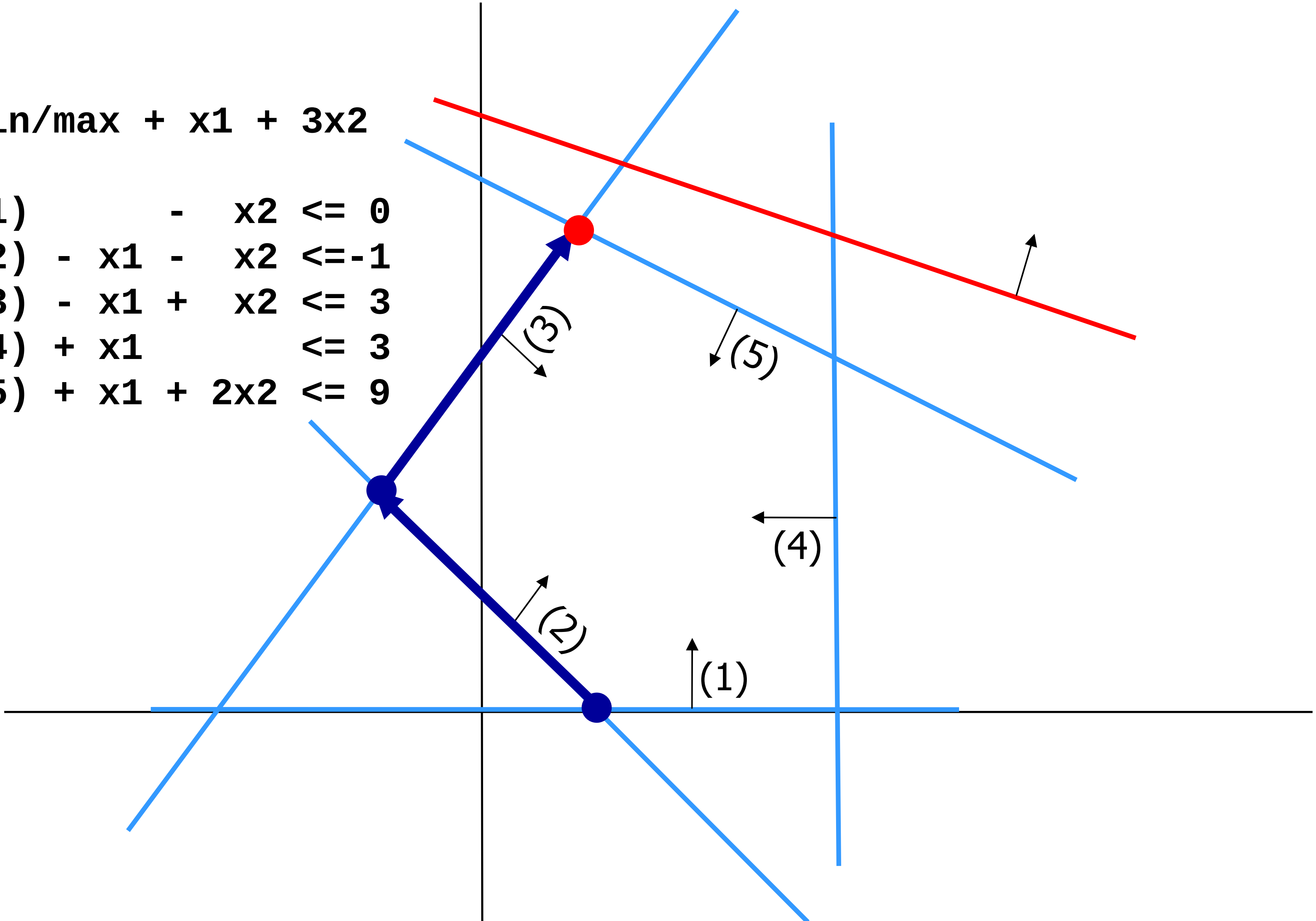

The Simplex Method: an example

```
min/max + x1 + 3x2

(1)        -   x2 <= 0
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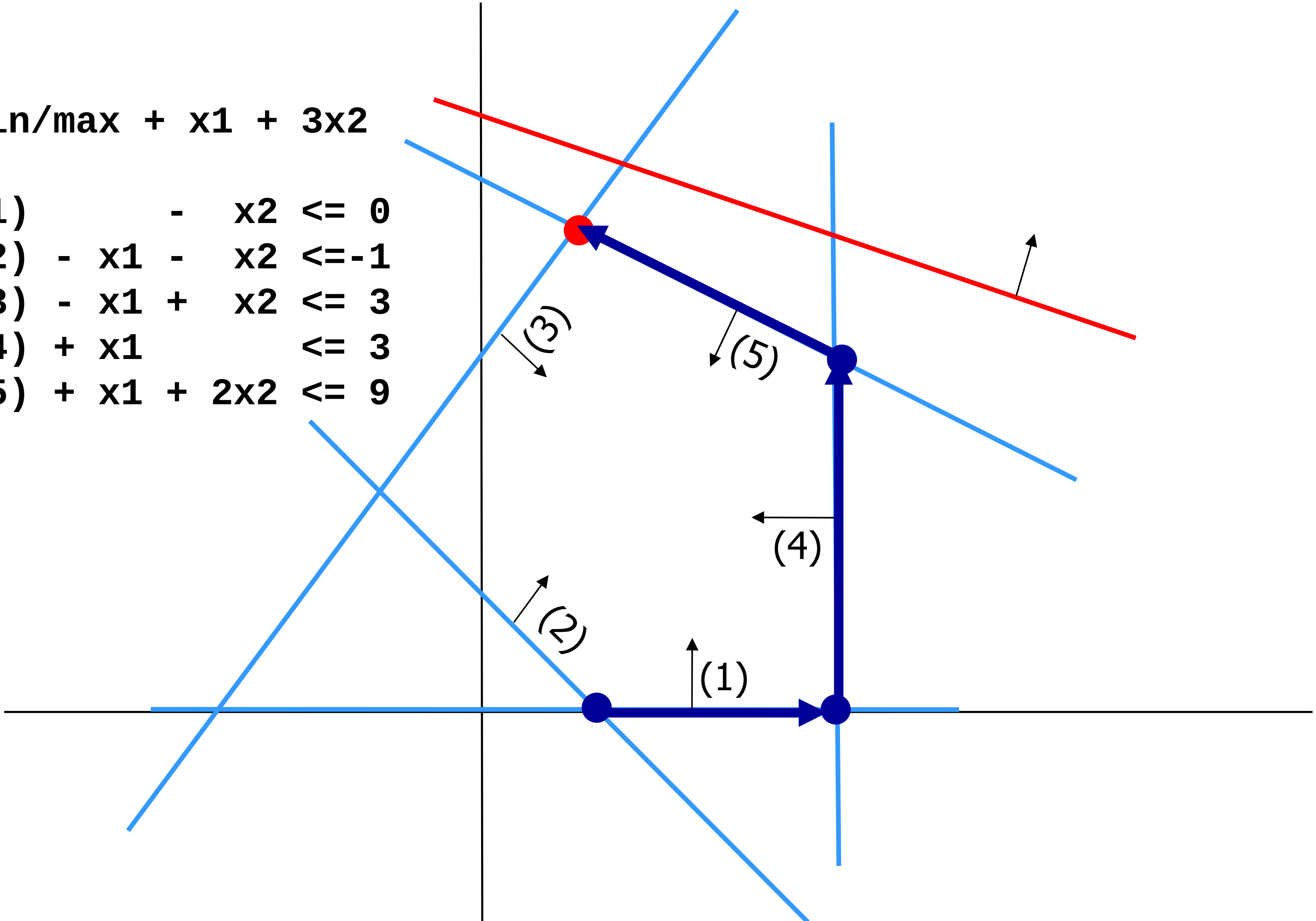

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Hirsch Conjecture

If P is a polytope of dimension n with m facets then every vertex of P can be reached from any other vertex of P on a path of length at most m-n.

In the example before: m=5, n=2 and m-n=3, conjecture is true.

At present, not even a polynomial bound on the path length is known.

Best upper bound: Kalai, Kleitman (1992): The diameter of the graph of an n-dimensional polyhedron with m facets is at most $m^{(\log n+1)}$.

Lower bound: Holt, Klee (1997): at least m-n (m, n large enough).

computationally important idea of the Simplex Method

Let a (m,n)-Matrix A with full row rank m, an m-vector b and an n-vector c with m<n be given. For every vertex y of the polyhedron of feasible solutions of the LP,

$$\max c^T x \quad Ax = b \quad x \geq 0$$

A = | B | N |
|---|---|

there is a non-singular (m,m)-submatrix B (called basis) of A representing the vertex y (basic solution) as follows

$$y_B = B^{-1}b, \quad y_N = 0$$

Many computational consequences:

Update-formulas, reduced cost calculations, number of non-zeros of a vertex,…

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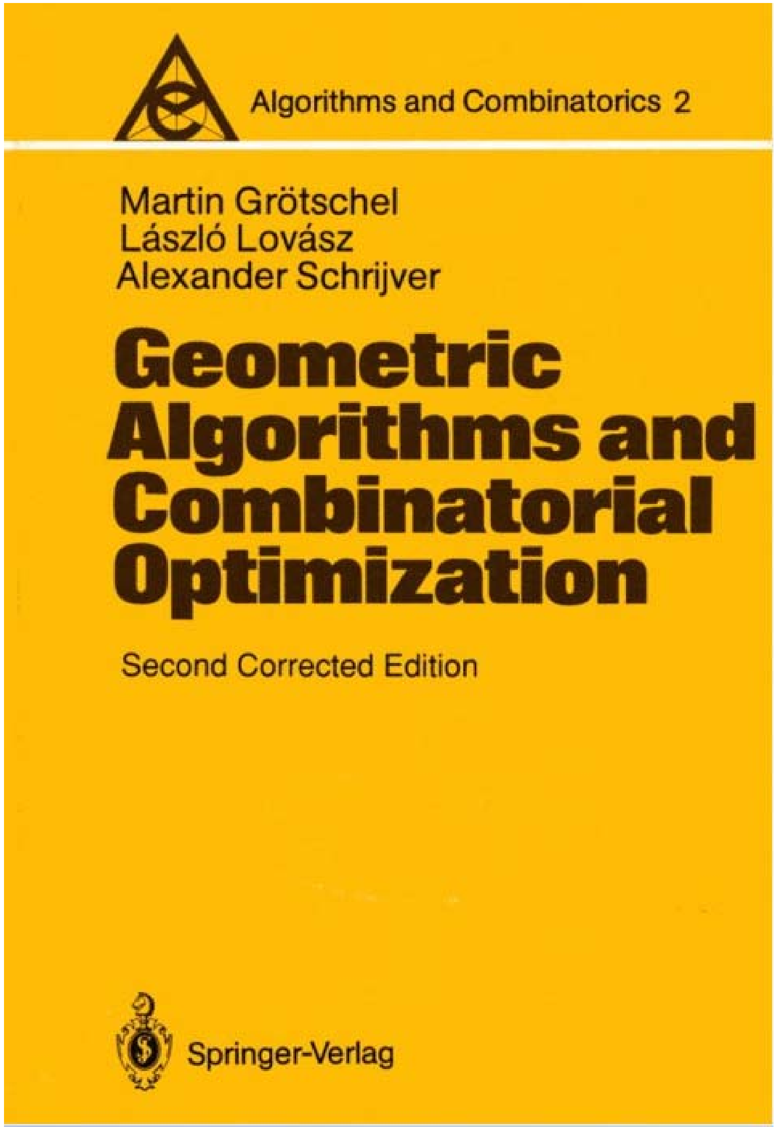
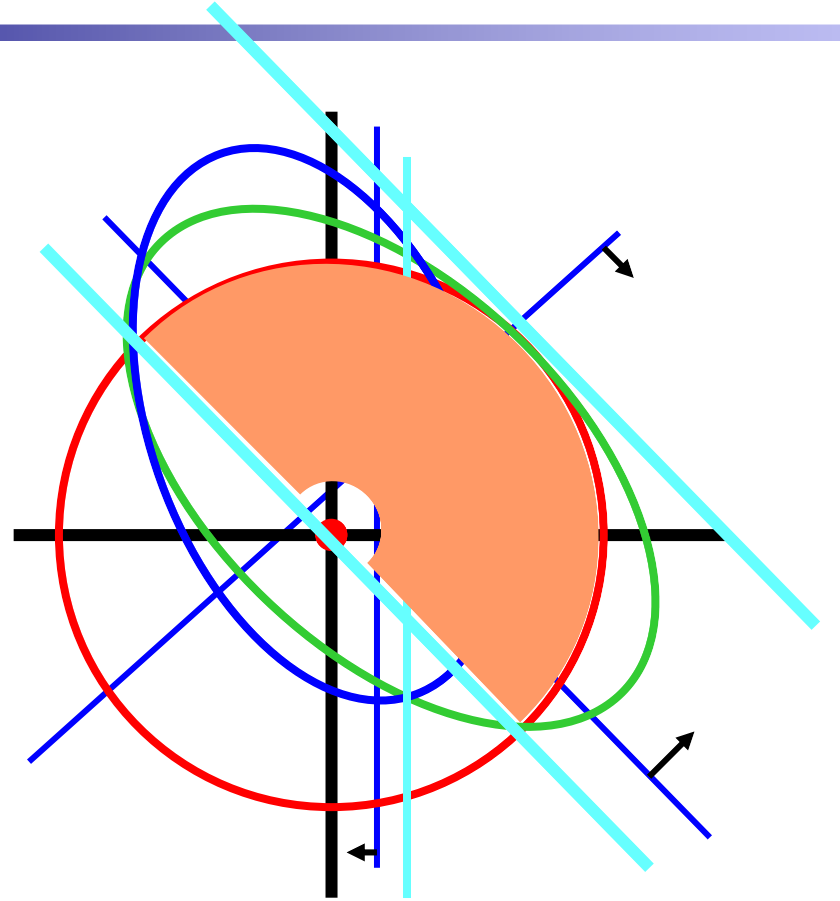
The Ellipsoid Method

- Shor, 1970 - 1979
- Yudin & Nemirovskii, 1976
- Khachiyan, 1979
- M. Grötschel, L. Lovász, A. Schrijver,
  Geometric Algorithms and Combinatorial Optimization
  Algorithms and Combinatorics 2, Springer, 1988



There is a Chinese version

The Ellipsoid Method: an example



$$k := 0,$$
$$N := 2n((2n+1)\langle C \rangle + n\langle d \rangle - n^3)$$
$$A_0 := R^2 I \quad \text{with } R := \sqrt{n}\, 2^{\langle C,d \rangle - n^2}$$

$$P := \{x \mid Cx \le d\}$$

Initialization

$$a_0 := 0$$

If $k = N$, *STOP!* (Declare P empty.)

If $a_k \in P$, *STOP!* (A feasible solution is found.)

If $a_k \notin P$, *then choose an inequality, say* $c^T x \le \gamma$ *of the system* $Cx \le d$ *that is violated by* a_k.

Stopping criterion

Feasibility check

Cutting plane choice

$$b := \frac{1}{\sqrt{c^T A_k c}} A_k c$$

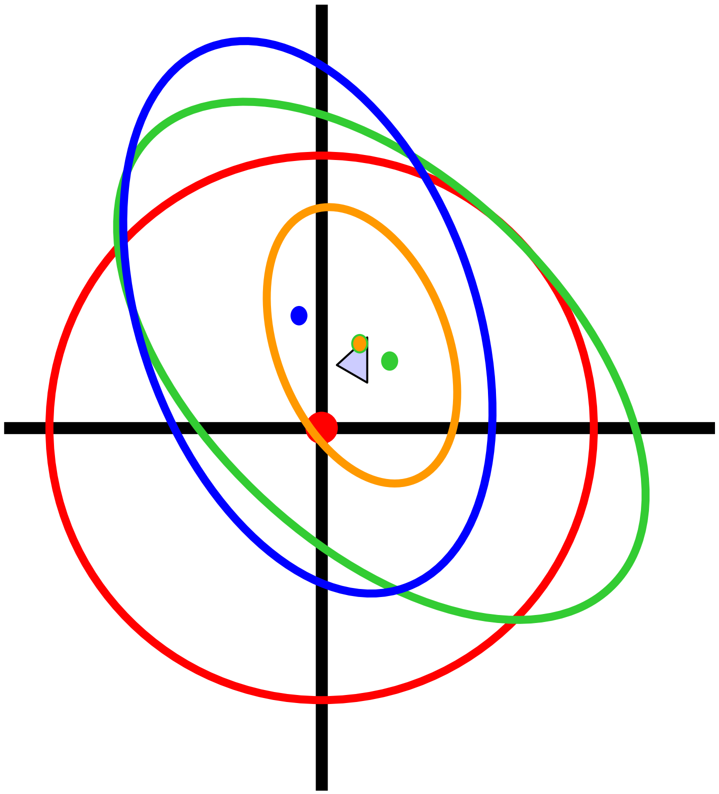
$$a_{k+1} := a_k - \frac{1}{n+1} b$$

Update

$$A_{k+1} := \frac{n^2}{n^2-1}\left(A_k - \frac{2}{n+1} b b^T\right)$$

The Ellipsoid Method

Ellipsoid Method



a(0)

a(1)

a(2)

a(7)

feasible solution found

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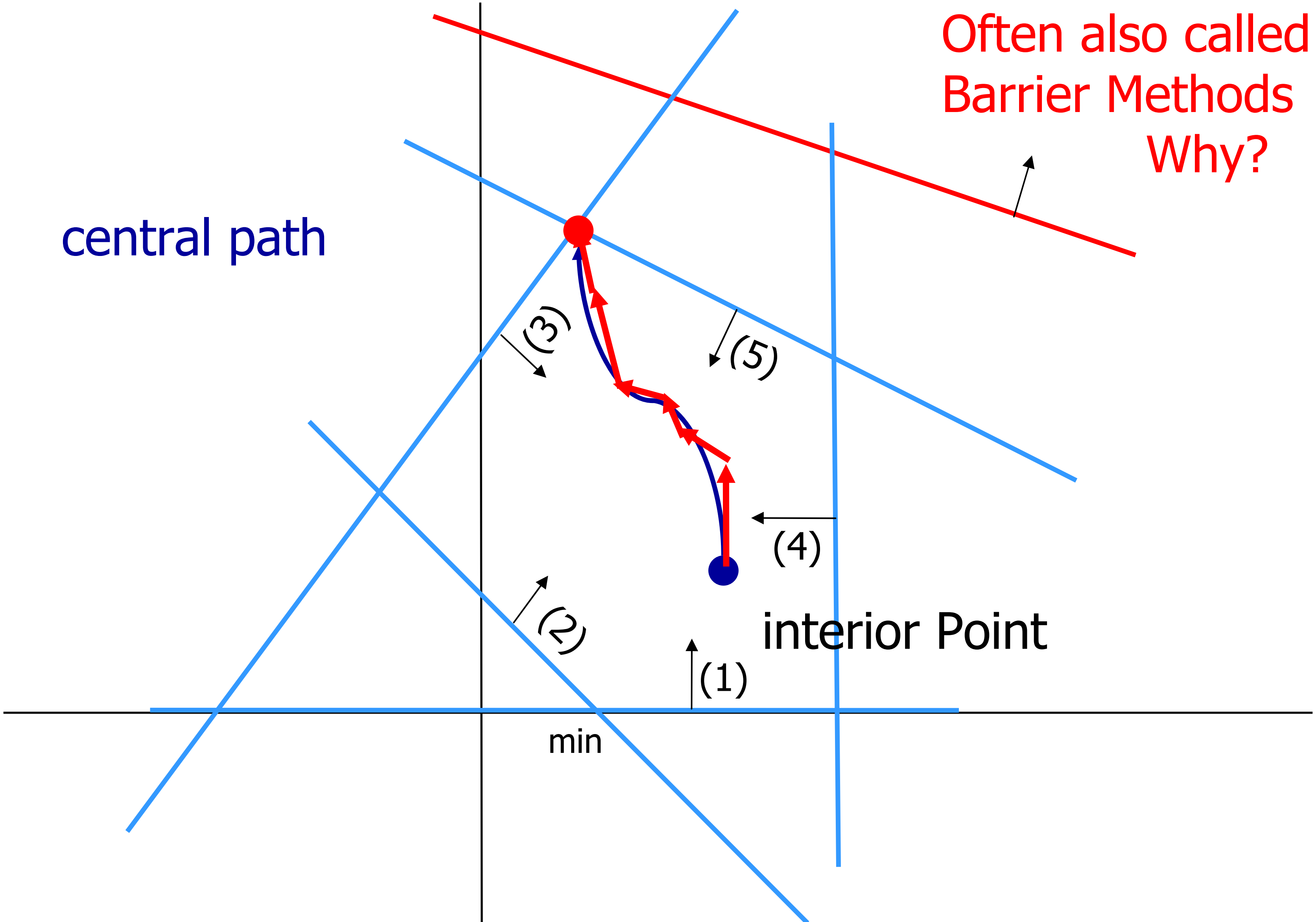
Interior-Point Methods: an example

Often also called
Barrier Methods
Why?



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Another approach

- Lagrangean Relaxation
  (for very large scale and structured LPs)
- plus
  - subgradient
  - bundle
  - bundle trust region

or any other nondifferentiable NLP method that looks promising

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special „simple“ combinatorial optimization problems

Finding a

- minimum spanning tree
- shortest path
- maximum matching
- maximal flow through a network
- cost-minimal flow
- …

solvable in polynomial time by special purpose algorithms

special „hard“ combinatorial optimization problems

- travelling salesman problem
- location und routing
- set-packing, partitioning, -covering
- max-cut
- linear ordering
- scheduling (with a few exceptions)
- node and edge colouring
- ...

NP-hard (in the sense of complexity theory)

The most successful solution techniques employ linear programming.

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The Branch&Bound Technique: An Example

$$\begin{aligned} &\min c^T x \\ &Ax = a \\ &Bx \le b \\ &x \ge 0 \\ &x \in \{0,1\}^n \end{aligned}$$

0/1-program

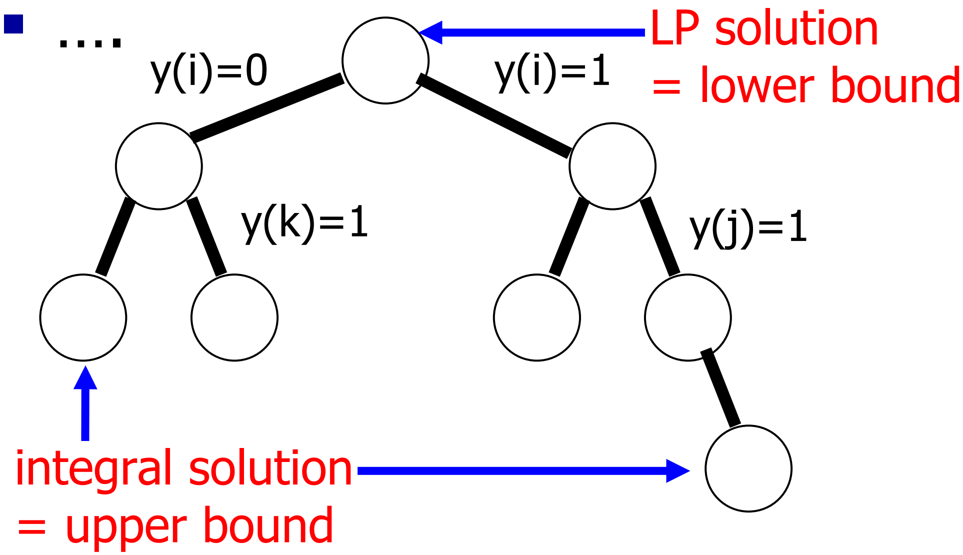
$$\begin{aligned} &\min c^T x \\ &Ax = a \\ &Bx \le b \\ &x \ge 0 \end{aligned}$$

~~$x \in \{0,1\}^n$~~

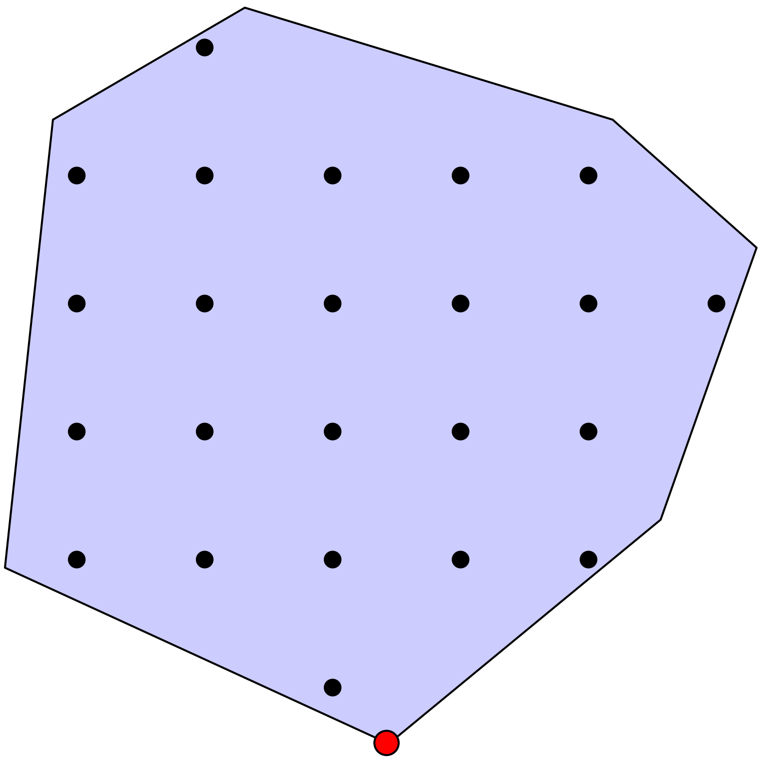
$$x \le 1$$

LP-relaxation

- Solve the LP-relaxation and get optimal solution y. (lower bound)
- If y integral, DONE!
- Otherwise pick fractional component y(i).
- Create two new subproblems by adding y(i)=1 and y(i)=0, resp.
- ....



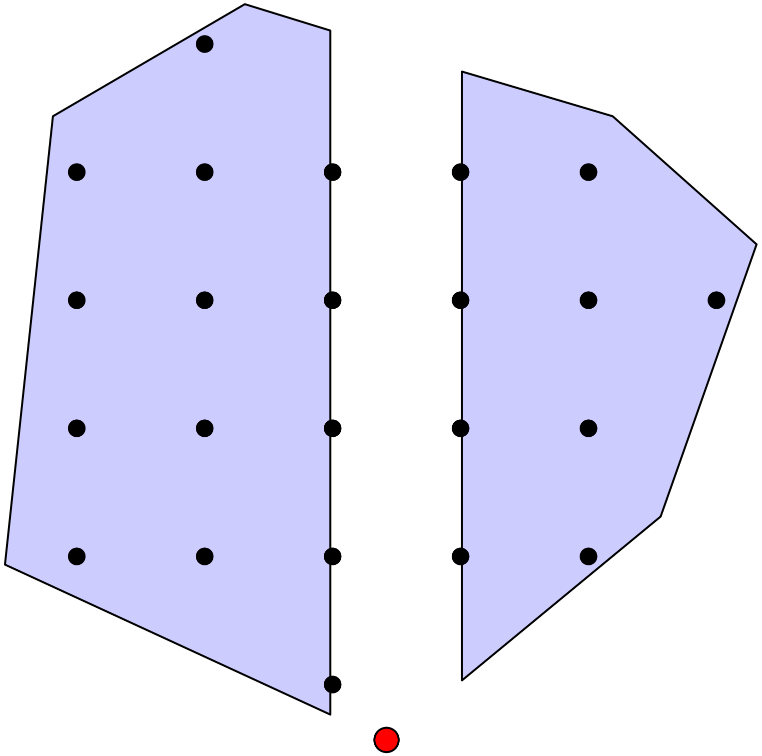
Branching (in general)



- Current solution is infeasible

Branching (in general)

- Rounding a fractional component up and down



- Decomposition into subproblems removes infeasible solution

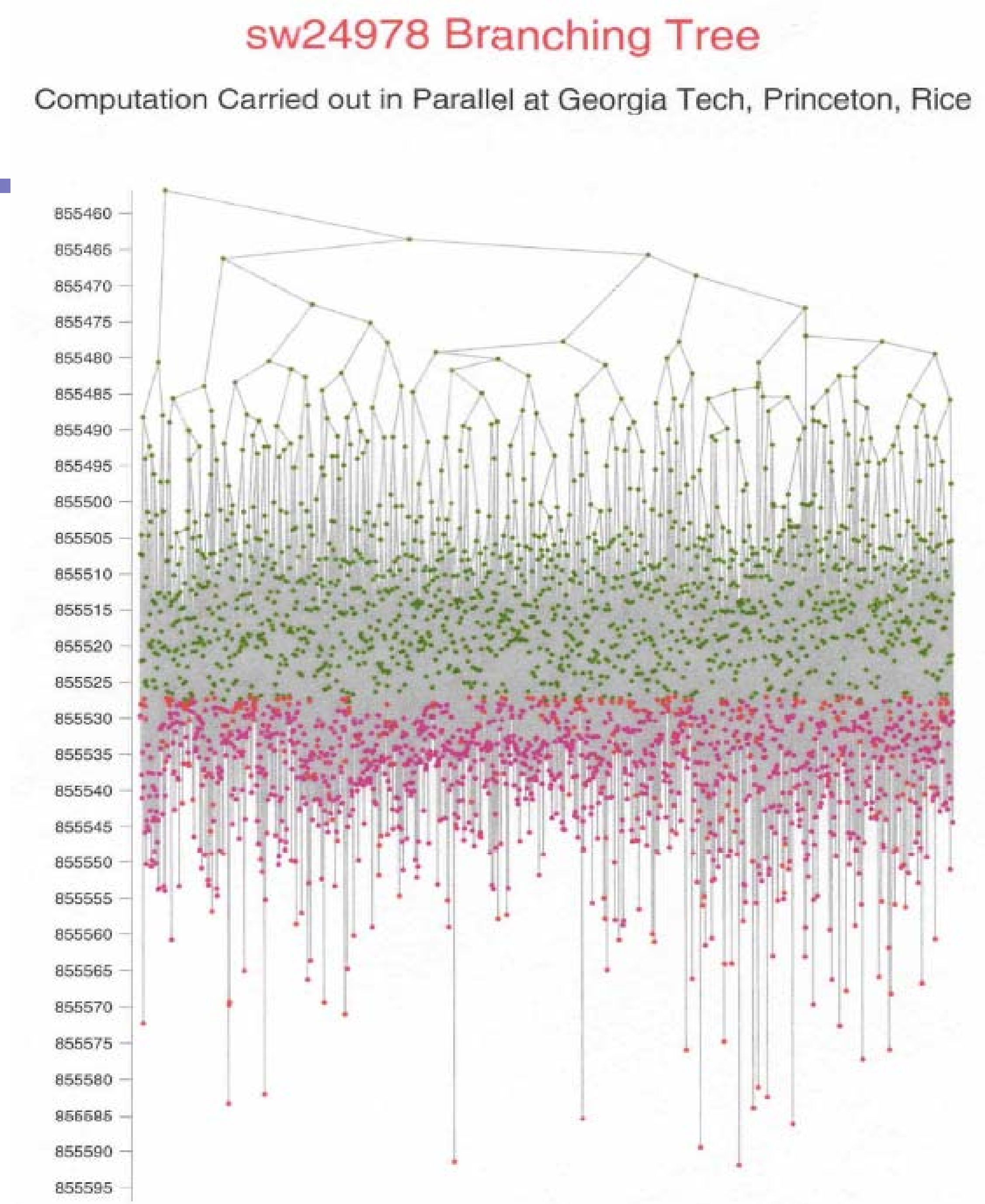
A Branching Tree

Applegate

Bixby

Chvátal

Cook



Contents

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cutting plane technique for integer and mixed-integer programming

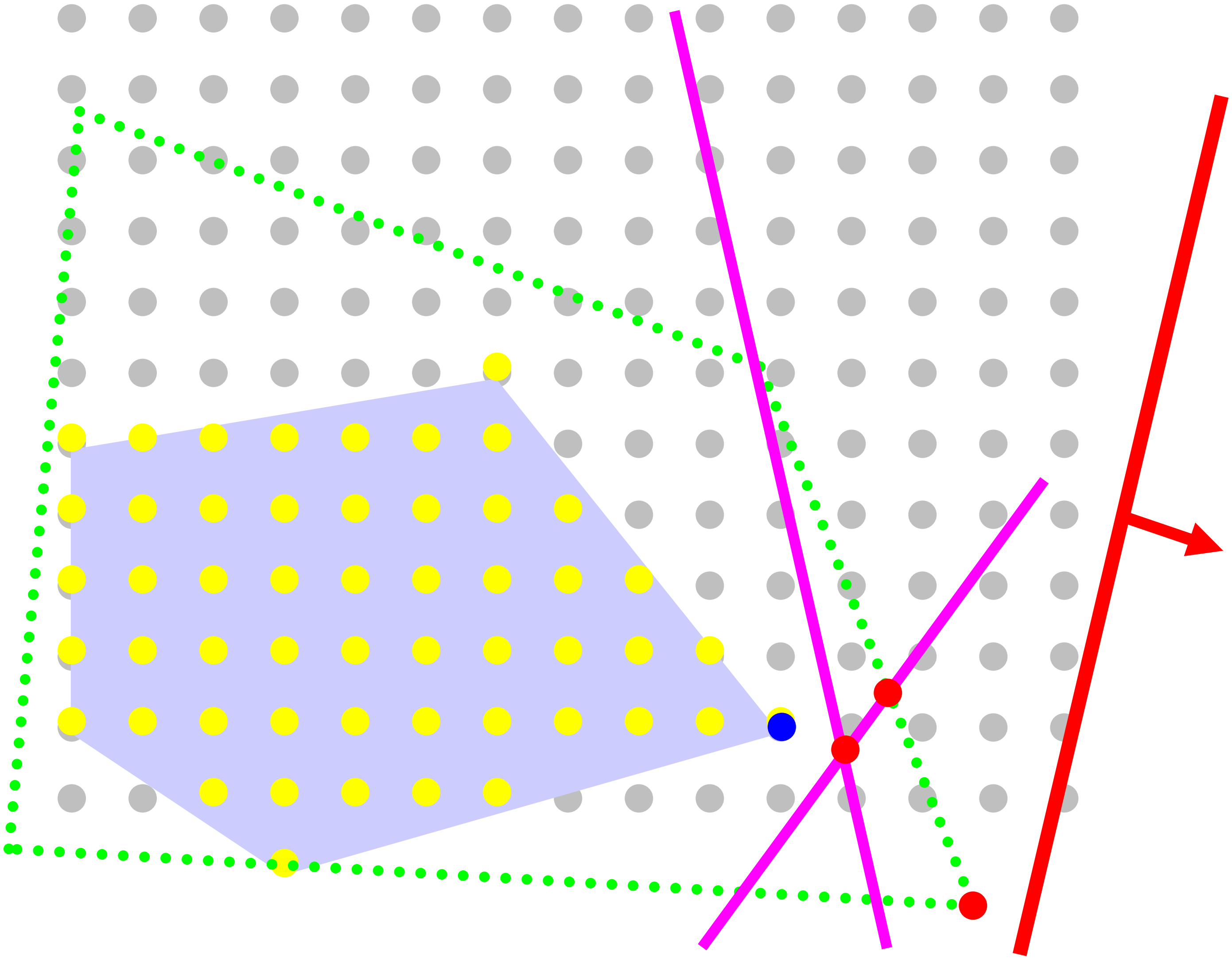
Feasible integer solutions

Objective function

Convex hull

LP-based relaxation

Cutting planes



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Other Names

- Branch & Cut
- Branch & Price
- Branch & Cut & Price
- etc.

Contents

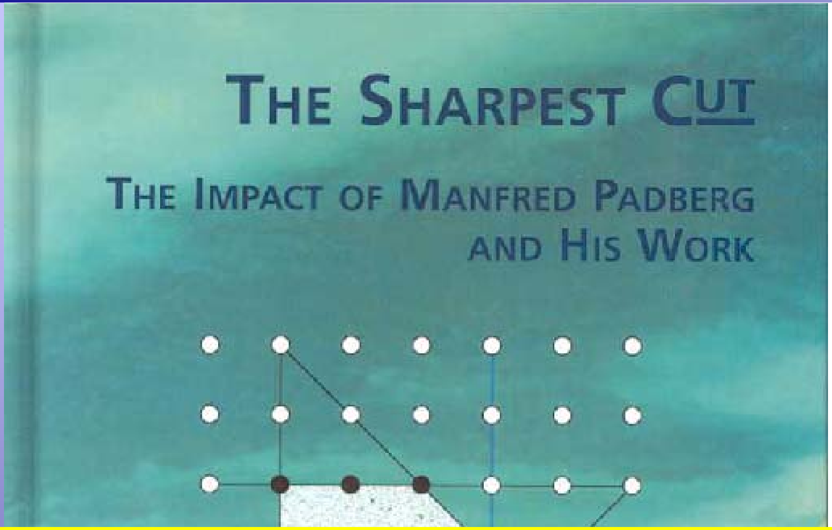
The most recent LP survey

Robert E. Bixby,

Solving Real-World Linear Programs: A Decade and More of Progress.

Operations Research 50 (2002)3-15.

Information about computational Mixed Integer Programming, see, e.g.,



This book appeared in MPS-SIAM "Series on Optimization", 2004,
One particularly interesting article on MIP:

Which LP solvers are used in practice?

- Fourier-Motzkin: hopeless
- Ellipsoid Method: total failure
- primal Simplex Method: good
- dual Simplex Method: better
- Barrier Method: for LPs frequently even better
- For LP relaxations of IPs: dual Simplex Method

Summary

- We can solve today explicit LPs with
  - up to **500,000** of variables and
  - up to **5,000,000** of constraints routinely

  in relatively short running times.
- We can solve today structured implicit LPs (employing column generation and cutting plane techniques) in special cases with hundreds of million (and more) variables and almost infinitly many constraints in acceptable running times.
  (Examples: TSP, bus circulation in Berlin)

State of the art in LP and IP

- This will be covered in another lecture
- Operations Research Management Science Today - June 2005
- OR-MS Today - LINEAR PROGRAMMING SOFTWARE SURVEY

01M1 Lecture
Linear and Integer Programming: an Introduction

The End

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