

01M1 Lecture

Linear and Integer Programming: an Introduction

Martin Grötschel

Beijing Block Course

"Combinatorial Optimization at Work"

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Martin Grötschel

- Institut für Mathematik, Technische Universität Berlin (TUB)
- DFG-Forschungszentrum "Mathematik für Schlüsseltechnologien" (MATHEON)
- Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)

groetschel@zib.de

<http://www.zib.de/groetschel>

Contents

1. Linear, integer, nonlinear programming, optimization: What's that?
2. Historic roots
3. LP-Theory
4. Algorithms for the solution of linear programs
 - 1) Fourier-Motzkin Elimination
 - 2) The Simplex Method
 - 3) The Ellipsoid Method
 - 4) Interior-Point/Barrier Methods
5. Algorithms for the solution of integer programs
 - 1) Branch&Bound
 - 2) Cutting Planes
6. Where are we today?
 - 1) State of the art in LP
 - 2) State of the art in IP
 - 3) Examples



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typical optimization problems

$$\max f(x) \text{ or } \min f(x)$$

$$g_i(x) = 0, \quad i = 1, 2, \dots, k$$

$$h_j(x) \leq 0, \quad j = 1, 2, \dots, m$$

$$x \in \mathbb{R}^n \text{ (and } x \in S)$$

$$\min c^T x$$

$$Ax = a$$

$$Bx \leq b$$

$$x \geq 0$$

$$(x \in \mathbb{R}^n)$$

$$(x \in \mathbb{k}^n)$$

$$\min c^T x$$

$$Ax = a$$

$$Bx \leq b$$

$$x \geq 0$$

$$x \in \mathbb{Z}^n$$

$$(x \in \{0, 1\}^n)$$

„general“
(nonlinear)
program
NLP

linear
program
LP

(linear)
integer
program
IP, MIP

program = optimization problem



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will be covered by Bob Bixby
on Saturday



Linear Programming

$$\max c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2$$

.

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

linear program
in standard form

Linear Programming

- 1939 L. V. Kantorovitch: Foundations of linear programming (Nobel Prize 1975)
- 1947 G. B. Dantzig: Invention of the simplex algorithm

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

- Today: In my opinion and from an economic point of view, **linear programming is the most important development of mathematics in the 20th century.**



Optimal use of scarce resources

foundation and economic interpretation of LP



Leonid V. Kantorovich Tjalling C. Koopmans
Nobel Prize for Economics 1975

Stiglers „Diet Problem“: „The first linear program“

$$\text{Min } x_1 + x_2$$

$$2x_1 + x_2 \geq 3$$

$$x_1 + 2x_2 \geq 3$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

costs

protein

carbohydrates

potatoes

beans

minimizing the
cost of food



George J. Stigler
Nobel Prize in
economics 1982

Sets of nutrients / calorie thousands , protein grams , calcium grams , iron milligrams vitamin-a thousand ius, vitamin-b1 milligrams, vitamin-b2 milligrams, niacin milligrams , vitamin-c milligrams /

of foods / wheat , cornmeal , cannedmilk, margarine , cheese , peanut-b , lard liver , porkroast, salmon , greenbeans, cabbage , onions , potatoes spinach, sweet-pot, peaches , prunes , limabeans, navybeans /

Parameter b(n) required daily allowances of nutrients / calorie 3, protein 70 , calcium .8 , iron 12 vitamin-a 5, vitamin-b1 1.8, vitamin-b2 2.7, niacin 18, vitamin-c 75 /

Table a(f,n) nutritive value of foods (per dollar spent)

	calorie (1000)	protein (g)	calcium (g)	iron (mg)	vitamin-a (1000iu)	vitamin-b1 (mg)	vitamin-b2 (mg)	niacin (mg)	vitamin-c (mg)
wheat	44.7	1411	2.0	365		55.4	33.3	441	
cornmeal	36	897	1.7	99	30.9	17.4	7.9	106	
cannedmilk	8.4	422	15.1	9	26	3	23.5	11	60
margarine	20.6	17	.6	6	55.8	.2			
cheese	7.4	448	16.4	19	28.1	.8	10.3	4	
peanut-b	15.7	661	1	48		9.6	8.1	471	
lard	41.7				.2		.5	5	
liver	2.2	333	.2	139	169.2	6.4	50.8	316	525
porkroast	4.4	249	.3	37		18.2	3.6	79	
salmon	5.8	705	6.8	45	3.5	1	4.9	209	
greenbeans	2.4	138	3.7	80	69	4.3	5.8	37	862
cabbage	2.6	125	4	36	7.2	9	4.5	26	5369
onions	5.8	166	3.8	59	16.6	4.7	5.9	21	1184
potatoes	14.3	336	1.8	118	6.7	29.4	7.1	198	2522
spinach	1.1	106		138	918.4	5.7	13.8	33	2755
sweet-pot	9.6	138	2.7	54	290.7	8.4	5.4	83	1912
peaches	8.5	87	1.7	173	86.8	1.2	4.3	55	57
prunes	12.8	99	2.5	154	85.7	3.9	4.3	65	257
limabeans	17.4	1055	3.7	459	5.1	26.9	38.2	93	
navybeans	26.9	1691	11.4	792		38.4	24.6	217	

Positive Variable x(f) dollars of food f to be purchased daily (dollars)

Free Variable cost total food bill (dollars)

Equations nb(n) nutrient balance (units), cb cost balance (dollars) ;

$nb(n).. \sum(f, a(f,n)*x(f)) =g= b(n); cb.. cost=e= \sum(f, x(f));$

Model diet stigers diet problem / nb,cb /;

Solution of the Diet Problem

Goal: find the cheapest combination of foods that will satisfy the daily requirements of a person

motivated by the army's desire to meet nutritional requirements of the soldiers at minimum cost

Army's problem had 77 unknowns and 9 constraints.

Stigler solved problem using a heuristic: \$39.93/year (1939)

Laderman (1947) used simplex: \$39.69/year (1939 prices)



first "large-scale computation"
took 120 man days on hand operated
desk calculators (10 human "computers")

<http://www.mcs.anl.gov/home/otc/Guide/CaseStudies/diet/index.html>



Milton Friedman on George J. Stigler

An early example of the latter is an article on "The Cost of Subsistence" (1945), which starts, "Elaborate investigations have been made of the adequacy of diets at various income levels, and a considerable number of 'low-cost,' 'moderate,' and 'expensive' diets have been recommended to consumers. Yet, so far as I know, no one has determined the minimum cost of obtaining the amounts of calories, proteins, minerals, and vitamins which these studies accept as adequate or optimum." George then set himself to determine the minimum cost diet, in the process **producing one of the earliest formulations of a linear programming problem in economics**, for which he found an approximate solution, explaining that **"there does not appear to be any direct method of finding the minimum of a linear function subject to linear constraints."** Two years later **George Dantzig** provided such a direct method, the **simplex method**, now widely used in many economic and industrial applications.



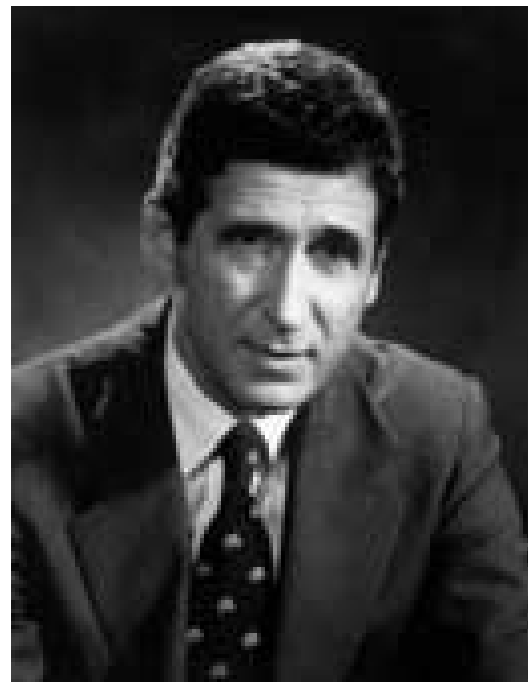
George Dantzig and Ralph Gomory



„founding fathers“

~1950

linear programming



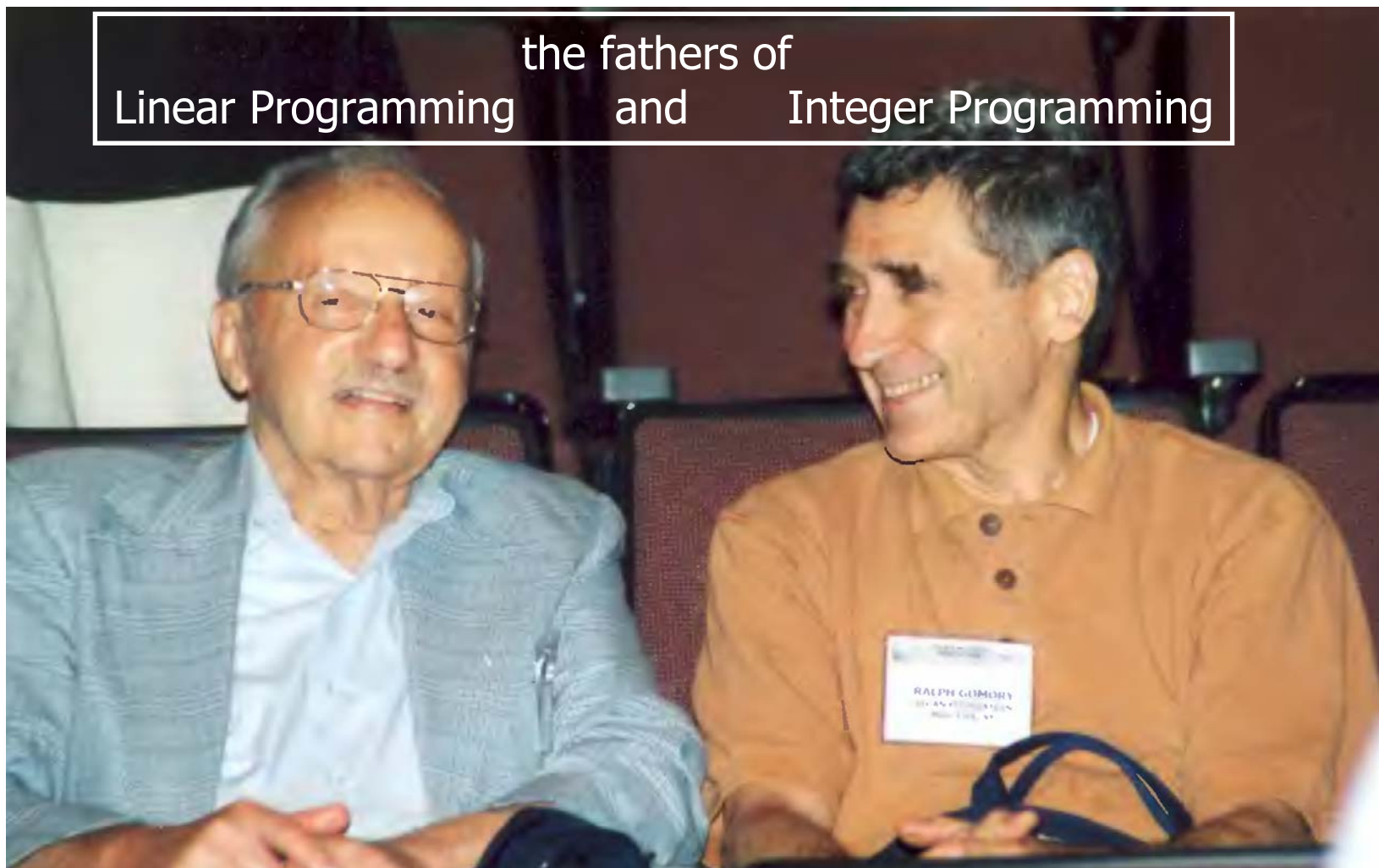
~1960

integer programming

George Dantzig and Ralph Gomory

ISMP Atlanta 2000

the fathers of
Linear Programming and Integer Programming



Dantzig and Bixby



George Dantzig and
Bob Bixby

at the International
Symposium on Mathematical
Programming,

Atlanta, August 2000

Commercial software

William Orchard-Hayes (in the period 1953-1954)

The first commercial LP-Code was on the market in 1954 (i.e., **~50 years ago**) and available on an IBM CPC (card programmable calculator):

Code: Simplex Algorithm with explicit basis inverse, that was recomputed in each step.

Shortly after, Orchard-Hayes implemented a version with product form of the inverse (idea of A. Orden),

Record: 71 variables, 26 constraints, 8 h running time



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Optimizers' dream: Duality theorems

- Max-Flow Min-Cut Theorem

The maximal (s,t)-flow in a capacitated network is equal to the minimal capacity of an (s,t)-cut.

- The Duality Theorem of linear programming

$$\begin{array}{l} \max c^T x \\ Ax \leq b \\ x \geq 0 \end{array} = \begin{array}{l} \min y^T b \\ y^T A \geq c^T \\ y \geq 0 \end{array}$$



Optimizers' dream: Duality theorems for integer programming

- The **Max-Flow Min-Cut Theorem**
does not hold if several source-sink relations are given
(multicommodity flow).
- The **Duality Theorem of linear programming**
does not hold if integrality conditions are added

$$\begin{array}{ccc} \max c^T x & \leq & \min y^T b \\ Ax \leq b & & y^T A \geq c^T \\ x \geq 0 & & y \geq 0 \\ x \in \mathbb{Z}^n & & y \in \mathbb{Z}^m \end{array}$$

Important theorems

- complementary slackness theorems
- redundancy characterizations



LP Solvability

- Linear programs can be solved in polynomial time with
 - the Ellipsoid Method (Khachiyan, 1979)
 - Interior Points Methods (Karmarkar, 1984, and others)
- **Open**: is there a strongly polynomial time algorithm for the solution of LPs?
- Certain variants of the Simplex Algorithm run – under certain conditions – in expected polynomial time (Borgwardt, 1977...)
- **Open**: Is there a polynomial time variant of the Simplex Algorithm?



LP Solvability: Generalizations

Theorem (GLS 1979, 1988) (modulo technical details) :

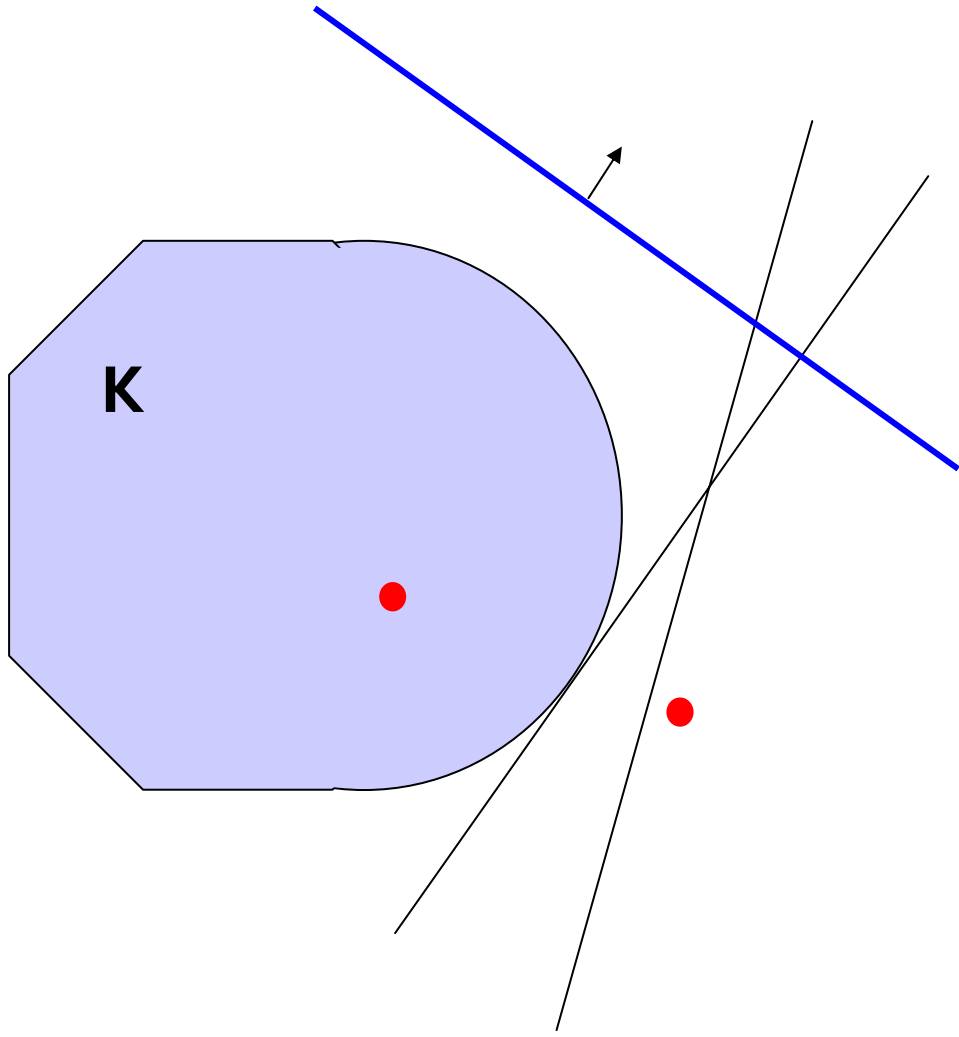
There exists a polynomial time algorithm to minimize convex functions (e.g., linear functions) over the elements of a class of convex bodies \mathbf{K} (e. g. polyhedra) if and only if, there exists a polynomial time algorithm that decides, for any given point \mathbf{x} , whether \mathbf{x} is in \mathbf{K} , and that, when \mathbf{x} is not in \mathbf{K} , finds a hyperplane that separates \mathbf{x} from \mathbf{K} .

Short version: Optimization and Separation are polynomial-time equivalent.

Consequence: Theoretical Foundation of cutting plane algorithms.



Separation



IP Solvability

Theorem

Integer, 0/1, and mixed integer programming are NP-hard.

Consequence

- special cases
- special purposes
- heuristics



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Algorithms for nonlinear programming

- **Iterative methods** that solve the equation and inequality systems representing the **necessary local optimality conditions**.

$$x_{i+1} = x_i + \lambda_i d_i$$

$d_i \sim$ "descent direction"

$\lambda_i \sim$ "steplength"

$$d_i = -\nabla f(x_i) \quad \text{Steepest descent}$$

$$d_i = -(H(x_i))^{-1} \nabla f(x_i) \quad \text{Newton}$$

(Quasi-Newton, conjugate-gradient-, SQP-, subgradient...methods)

- **Sufficient conditions** are rarely checked.

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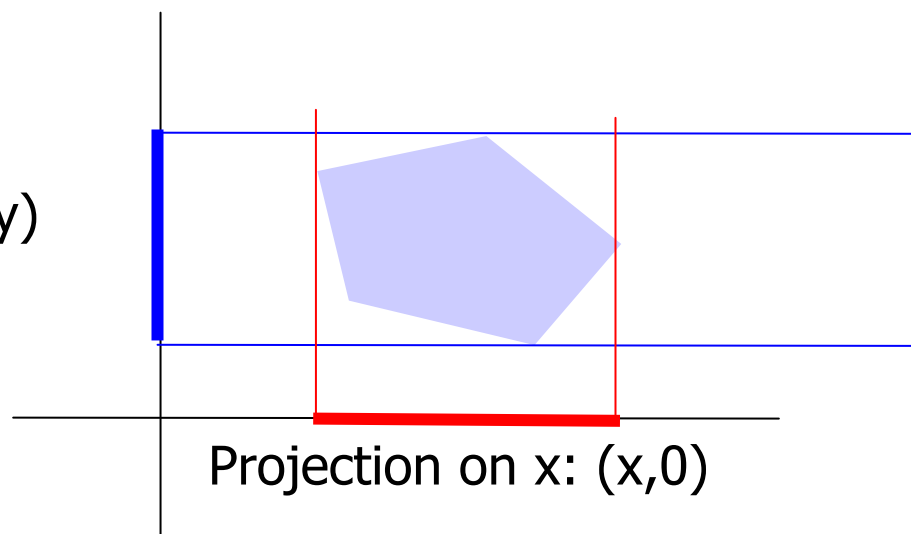
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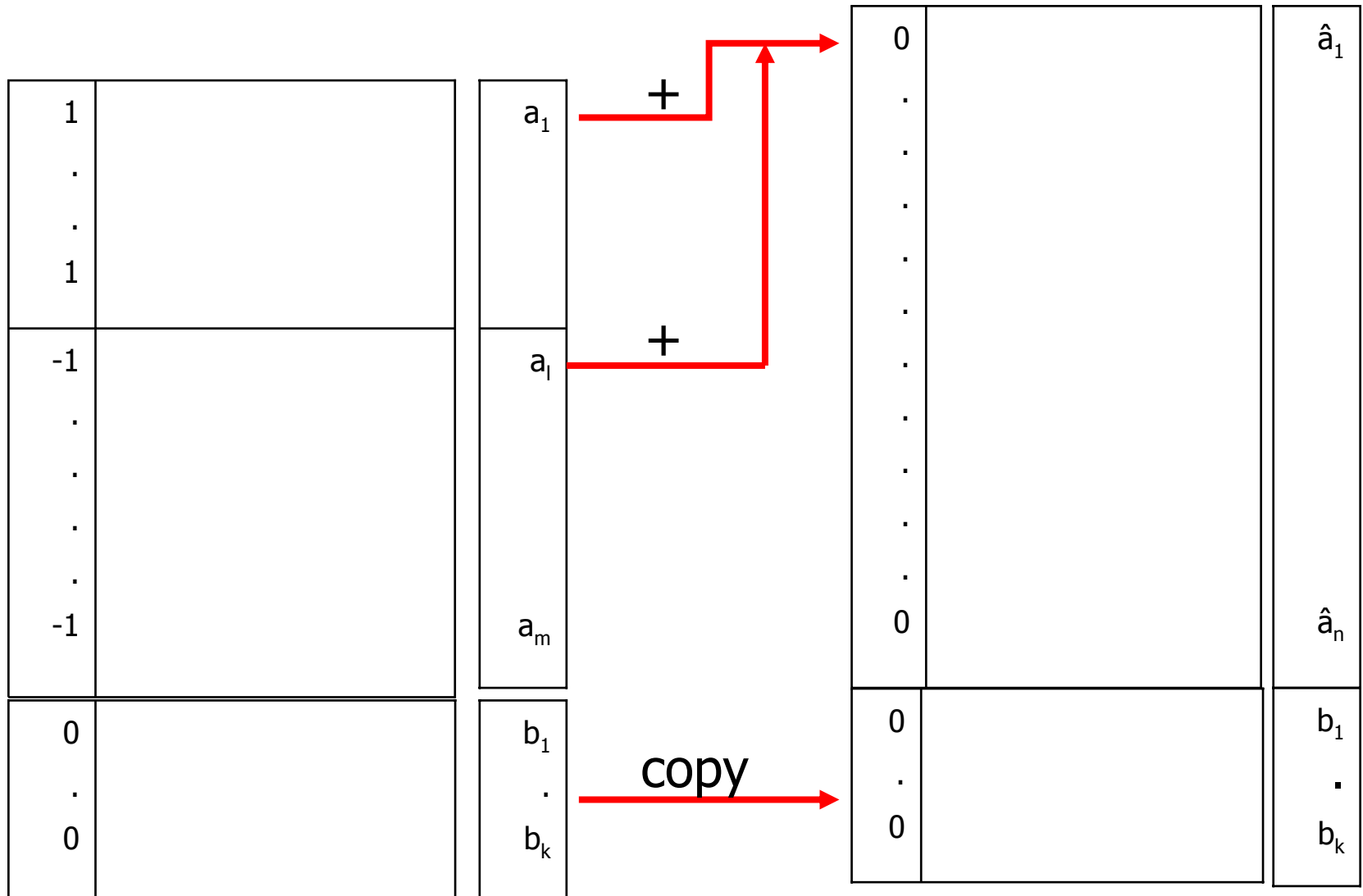
Fourier-Motzkin Elimination

- Fourier, 1847
- Motzkin, 1938
- **Method:** successive projection of a polyhedron in n -dimensional space into a vector space of dimension $n-1$ by elimination of one variable.

Projection on y : $(0,y)$



A Fourier-Motzkin step



Fourier-Motzkin Elimination: an example

min/max $+ x_1 + 3x_2$

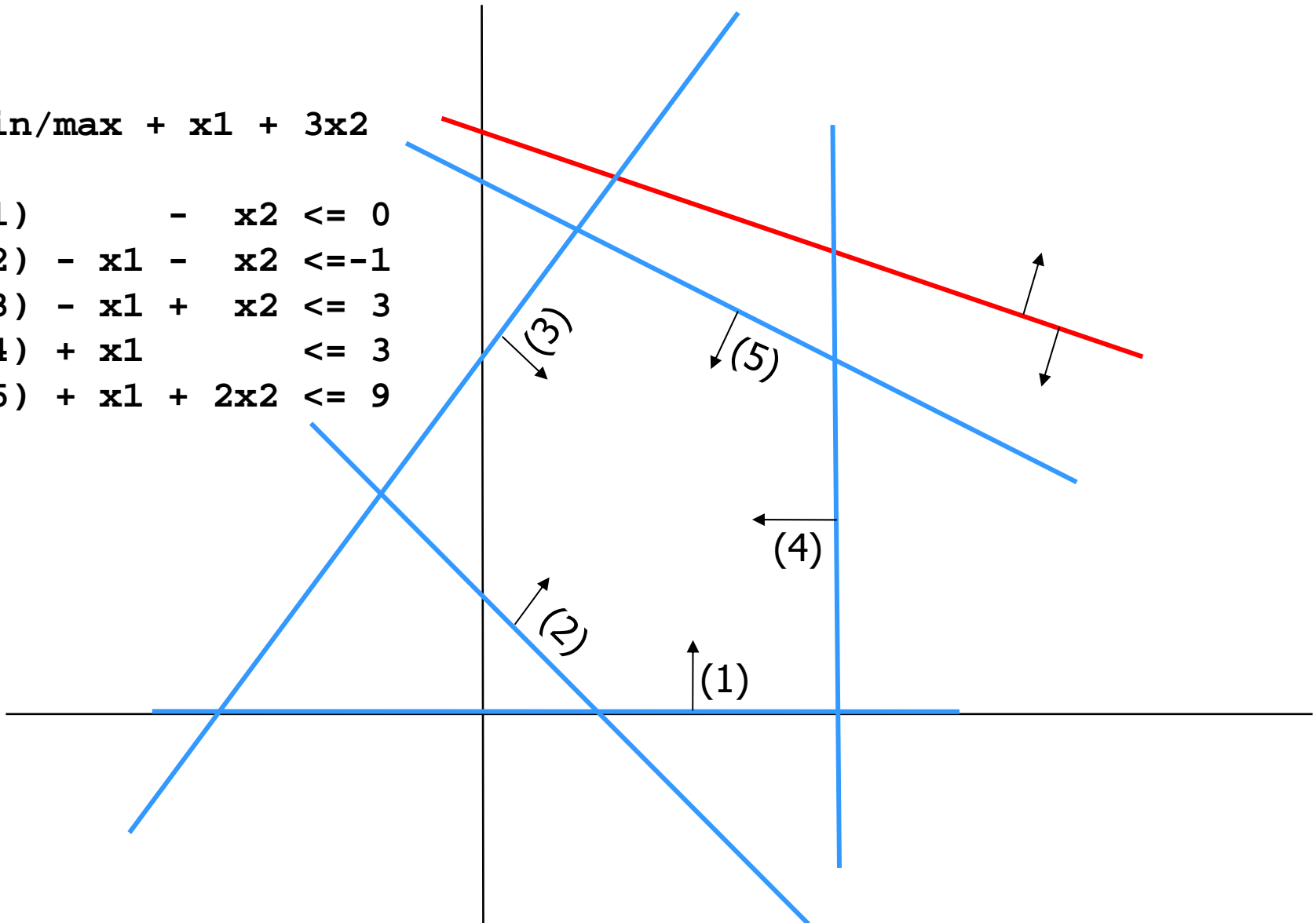
$$(1) \quad -x_2 \leq 0$$

$$(2) \quad -x_1 - x_2 \leq -1$$

$$(3) \quad -x_1 + x_2 \leq 3$$

$$(4) \quad +x_1 \leq 3$$

$$(5) \quad +x_1 + 2x_2 \leq 9$$



Fourier-Motzkin Elimination: an example, call of PORTA

DIM = 3

min/max + x1 + 3x2

- (1) - x2 ≤ 0
- (2) - x1 - x2 ≤ -1
- (3) - x1 + x2 ≤ 3
- (4) + x1 ≤ 3
- (5) + x1 + 2x2 ≤ 9

INEQUALITIES_SECTION

- (1) - x2 ≤ 0
- (2) - x1 - x2 ≤ -1
- (3) - x1 + x2 ≤ 3
- (4) + x1 ≤ 3
- (5) + x1 + 2x2 ≤ 9
- (6) + x1 + 3x2 - x3 ≤ 0
- (7) - x1 - 3x2 + x3 ≤ 0



ELIMINATION_ORDER

1 0 0

Fourier-Motzkin Elimination: an example

DIM = 3



DIM = 3

INEQUALITIES_SECTION

```
(1) (1) - x2 <= 0
(2,4) (2) - x2 <= 2
(2,5) (3) + x2 <= 8
(2,6) (4) +2x2 - x3 <= -1
(3,4) (5) + x2 <= 6
(3,5) (6) + x2 <= 4
(3,6) (7) +4x2 - x3 <= 3
(7,4) (8) -3x2 + x3 <= 3
(7,5) (9) - x2 + x3 <= 9
(7,6)
```

INEQUALITIES_SECTION

```
(1) - x2 <= 0
(2) - x1 - x2 <= -1
(3) - x1 + x2 <= 3
(4) + x1 <= 3
(5) + x1 + 2x2 <= 9
(6) + x1 + 3x2 - x3 <= 0
(7) - x1 - 3x2 + x3 <= 0
```

ELIMINATION_ORDER

1 0 0

Fourier-Motzkin Elimination: an example

DIM = 3



INEQUALITIES_SECTION

(1)	(1)	- x2	<=	0	(1,4)	(1)	-x3	<=	-1
(2,4)	(2)	- x2	<=	2	(1,7)	(2)	-x3	<=	3
(2,5)	(3)	+ x2	<=	8	(2,4)	(3)	-x3	<=	3
(2,6)	(4)	+2x2 - x3	<=	-1	(2,7)	(4)	-x3	<=	11
(3,4)	(5)	+ x2	<=	6	(8,3)	(5)	+x3	<=	27
(3,5)	(6)	+ x2	<=	4	(8,4)	(6)	-x3	<=	3
(3,6)	(7)	+4x2 - x3	<=	3	(8,5)	(7)	+x3	<=	21
(7,4)	(8)	-3x2 + x3	<=	3	(8,6)	(8)	+x3	<=	15
(7,5)	(9)	- x2 + x3	<=	9	(8,7)	(9)	+x3	<=	21
(7,6)					(9,3)	(10)	+x3	<=	17
					(9,4)	(11)	+x3	<=	17
					(9,5)	(12)	+x3	<=	15
					(9,6)	(13)	+x3	<=	13
					(9,7)	(14)	+3x3	<=	39

ELIMINATION_ORDER

0 1 0

min = 1 <= x3 <= 13 = max

x1 = 1

x2 = 0

x1 = 1

x2 = 4

Fourier-Motzkin Elimination: an example

min/max $+ x_1 + 3x_2$

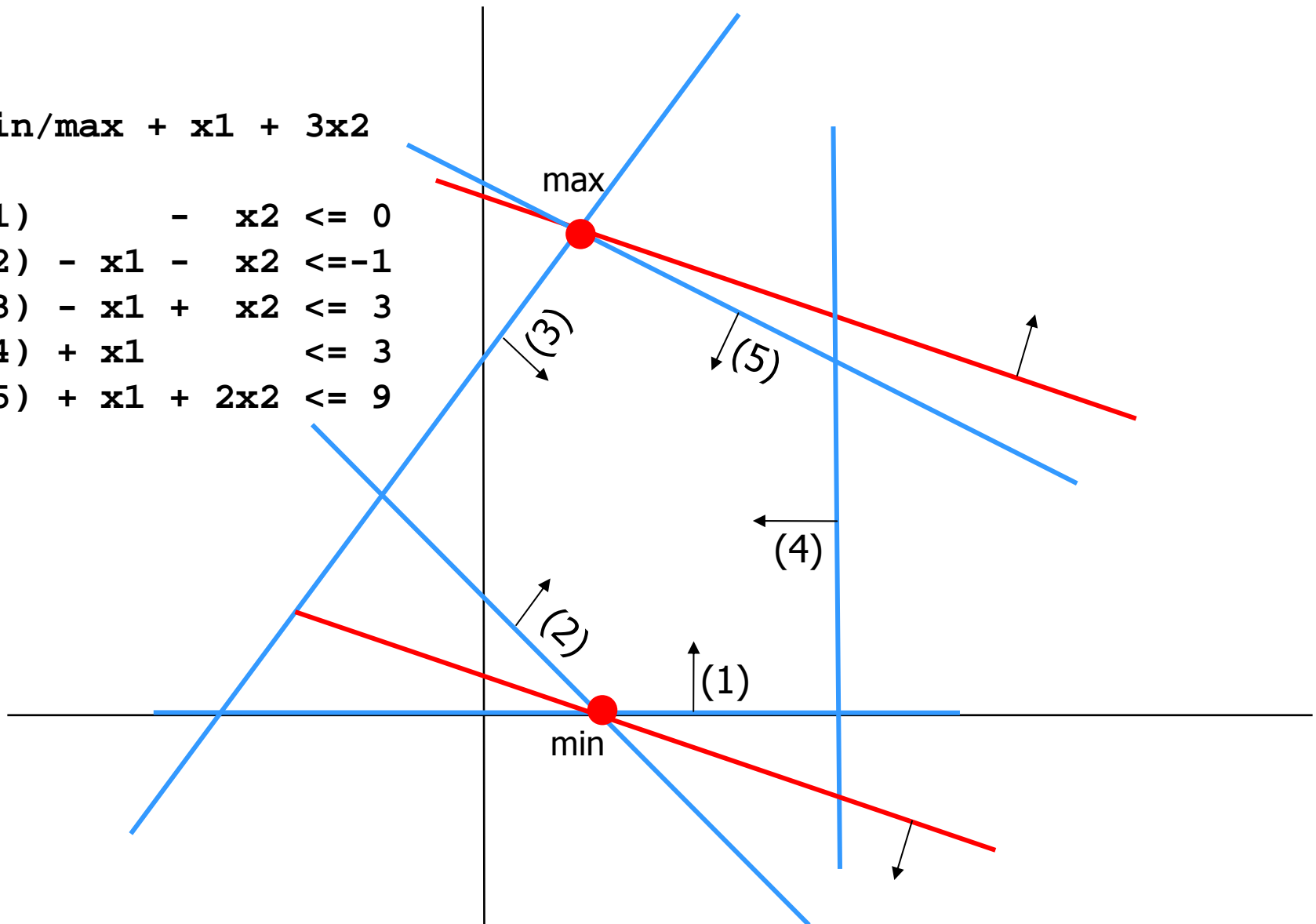
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The Simplex Method

- Dantzig, 1947: primal Simplex Method
- Lemke, 1954; Beale, 1954: dual Simplex Method
- Dantzig, 1953: revised Simplex Method
-
- **Underlying Idea:** Find a vertex of the set of feasible LP solutions (polyhedron) and move to a better neighbouring vertex, if possible.



The Simplex Method: an example

min/max $+ x_1 + 3x_2$

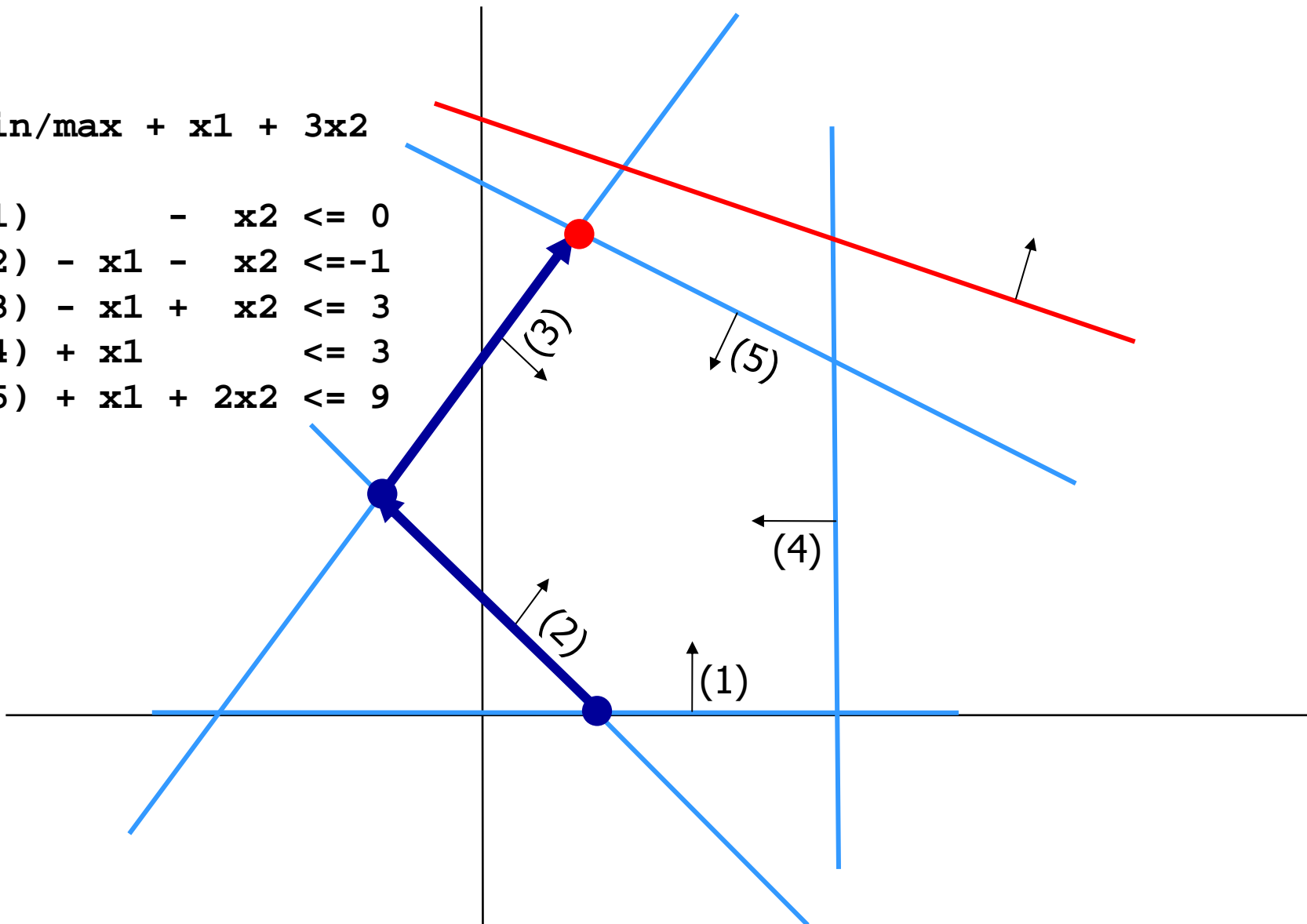
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The Simplex Method: an example

min/max $+ x_1 + 3x_2$

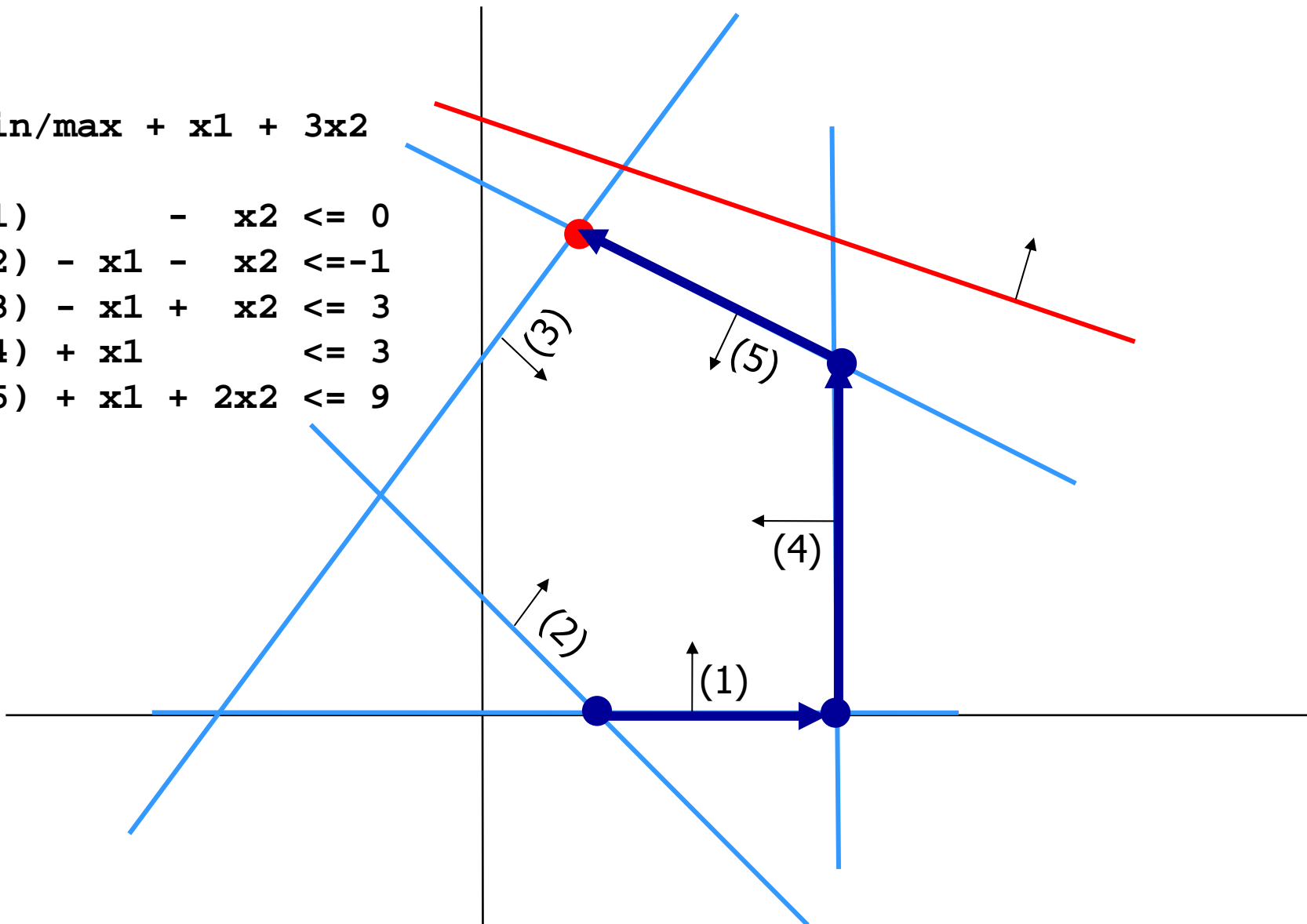
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Hirsch Conjecture

If P is a polytope of dimension n with m facets then every vertex of P can be reached from any other vertex of P on a path of length at most $m-n$.

In the example before: $m=5$, $n=2$ and $m-n=3$, conjecture is true.

At present, not even a polynomial bound on the path length is known.

Best upper bound: Kalai, Kleitman (1992): The diameter of the graph of an n -dimensional polyhedron with m facets is at most $m^{\log n + 1}$.

Lower bound: Holt, Klee (1997): at least $m-n$ (m, n large enough).



computationally important idea of the Simplex Method

Let a (m,n) -Matrix A with full row rank m , an m -vector b and an n -vector c with $m < n$ be given. For every vertex y of the polyhedron of feasible solutions of the LP,

$$\max c^T x$$

$$Ax = b$$

$$x \geq 0$$

$$A =$$

B	N
---	---

there is a non-singular (m,m) -submatrix B (called basis) of A representing the vertex y (basic solution) as follows

$$y_B = B^{-1}b, \quad y_N = 0$$

Many computational consequences:

Update-formulas, reduced cost calculations,

number of non-zeros of a vertex,...

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The Ellipsoid Method

- Shor, 1970 - 1979
- Yudin & Nemirovskii, 1976
- Khachiyan, 1979
- M. Grötschel, L. Lovász, A. Schrijver,
Geometric Algorithms and Combinatorial Optimization
Algorithms and Combinatorics 2, Springer, 1988





Algorithms and Combinatorics 2

Martin Grötschel
László Lovász
Alexander Schrijver

Geometric Algorithms and Combinatorial Optimization

Second Corrected Edition

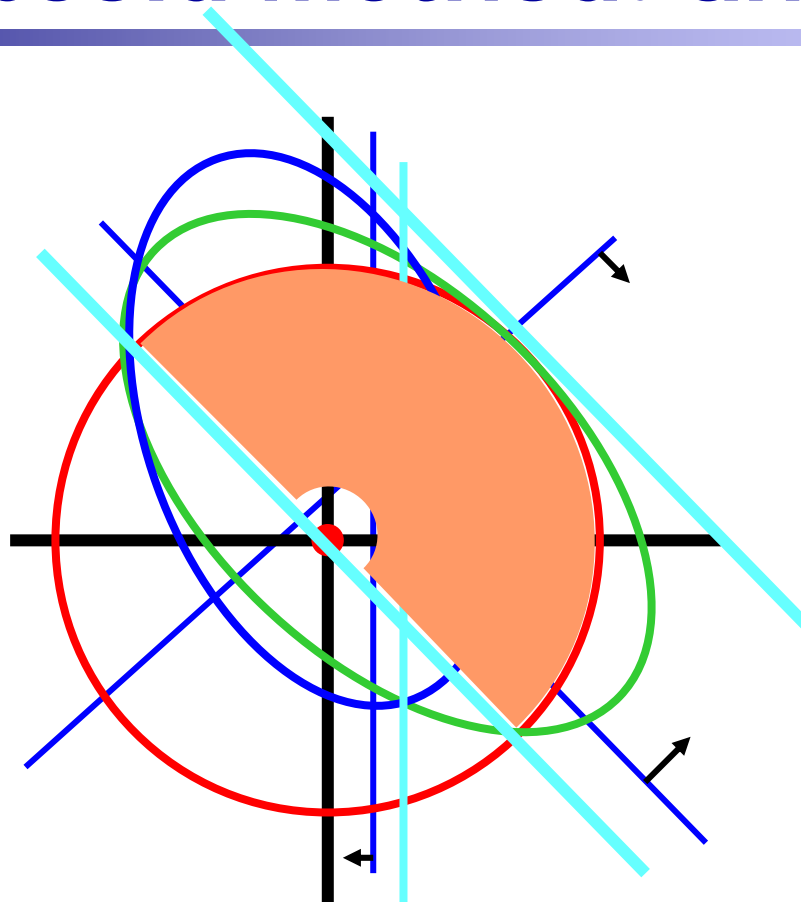


Springer-Verlag

There is a Chinese version



The Ellipsoid Method: an example



$$k := 0,$$

$$N := 2n((2n + 1)\langle C \rangle + n\langle d \rangle - n^3)$$

$$A_0 := R^2 I \text{ with } R := \sqrt{n} 2^{\langle C, d \rangle - n^2}$$

$$P := \{x \mid Cx \leq d\}$$

Initialization

$$a_0 := 0$$

If $k = N$, *STOP!* (Declare P empty.)

Stopping criterion

If $a_k \in P$, *STOP!* (A feasible solution is found.)

Feasibility check

If $a_k \notin P$, then choose an inequality, say $c^T x \leq \gamma$, of the system $Cx \leq d$ that is violated by a_k .

Cutting plane choice

$$b := \frac{1}{\sqrt{c^T A_k c}} A_k c$$

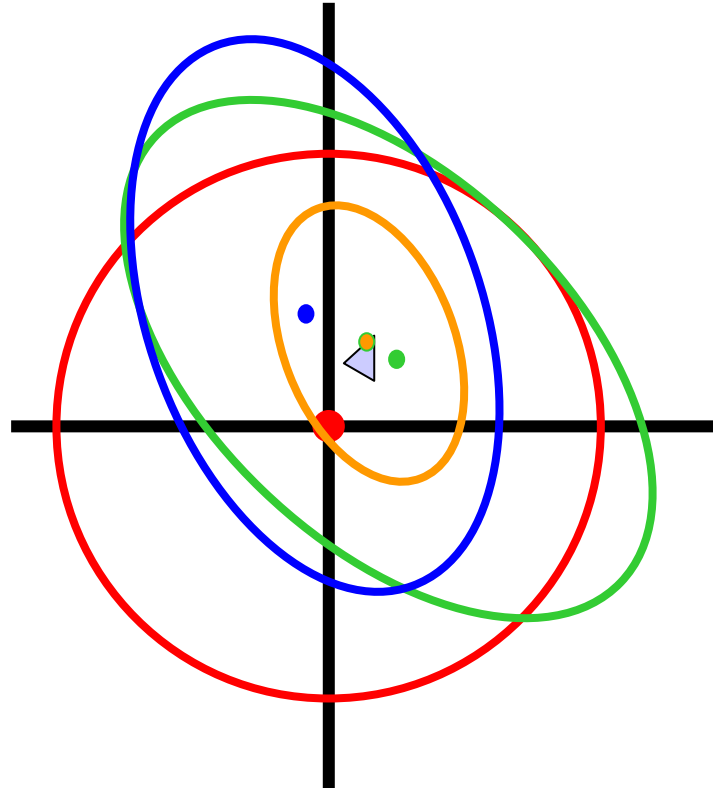
$$a_{k+1} := a_k - \frac{1}{n+1} b$$

Update

$$A_{k+1} := \frac{n^2}{n^2 - 1} \left(A_k - \frac{2}{n+1} b b^T \right)$$

The Ellipsoid Method

Ellipsoid Method



$a(0)$

$a(1)$

$a(2)$

$a(7)$

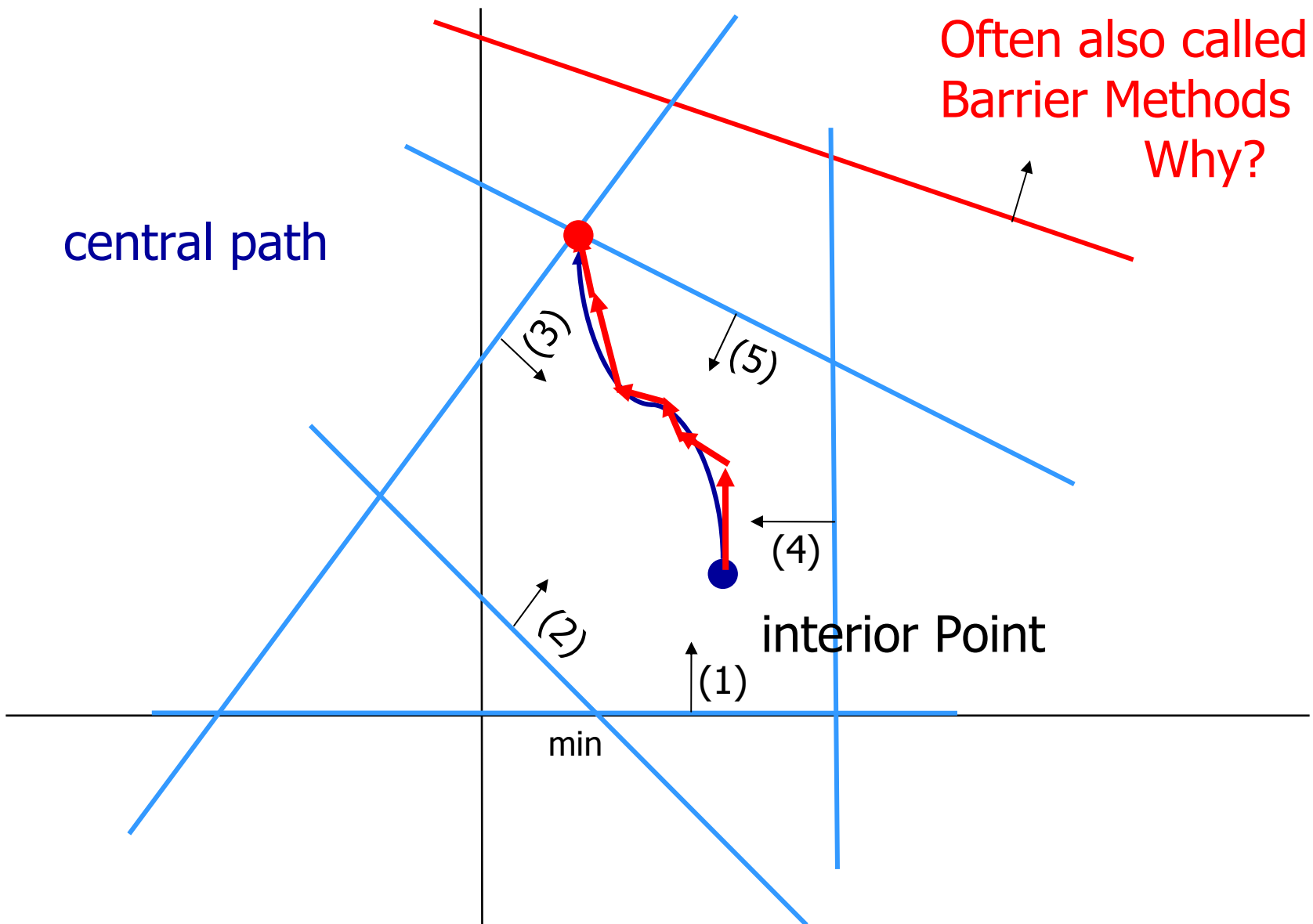
feasible
solution
found

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1. Linear, integer, nonlinear programming, optimization: What's that?
2. Historic roots
3. LP-Theory
4. Algorithms for the solution of linear programs
 - 1) Fourier-Motzkin Elimination
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Interior-Point Methods: an example



Another approach

- Lagrangean Relaxation
(for very large scale and structured LPs)
- plus
 - subgradient
 - bundle
 - bundle trust region

or any other nondifferentiable NLP method that looks promising



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special „simple“ combinatorial optimization problems

Finding a

- minimum spanning tree
- shortest path
- maximum matching
- maximal flow through a network
- cost-minimal flow
- ...

solvable in polynomial time by special purpose algorithms



special „hard“ combinatorial optimization problems

- travelling salesman problem
- location und routing
- set-packing, partitioning, -covering
- max-cut
- linear ordering
- scheduling (with a few exceptions)
- node and edge colouring
- ...

NP-hard (in the sense of complexity theory)

The most successful solution techniques employ linear programming.



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The Branch&Bound Technique: An Example

$$\min c^T x$$

$$Ax = a$$

$$Bx \leq b$$

$$x \geq 0$$

$$x \in \{0,1\}^n$$

0/1-
program

$$\min c^T x$$

$$Ax = a$$

$$Bx \leq b$$

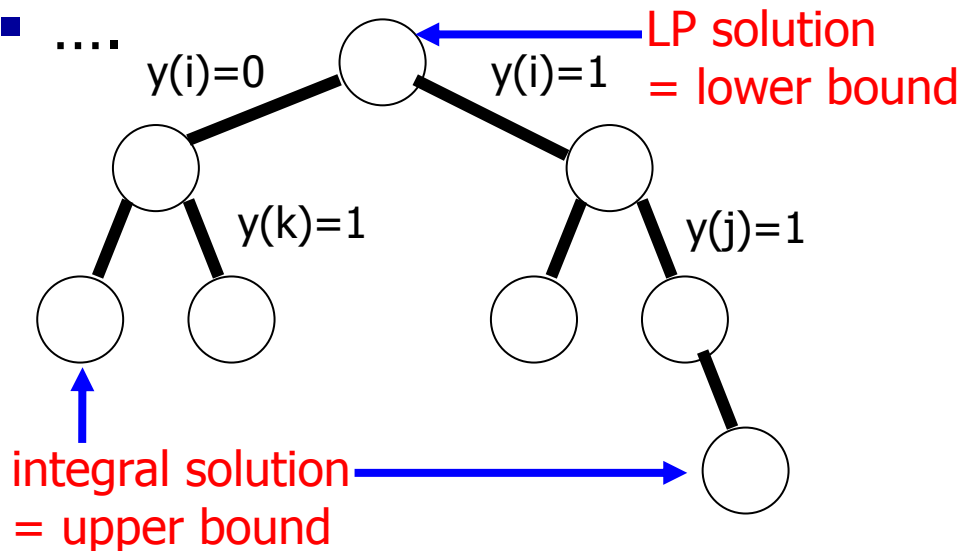
$$x \geq 0$$

~~$$x \in \{0,1\}^n$$~~

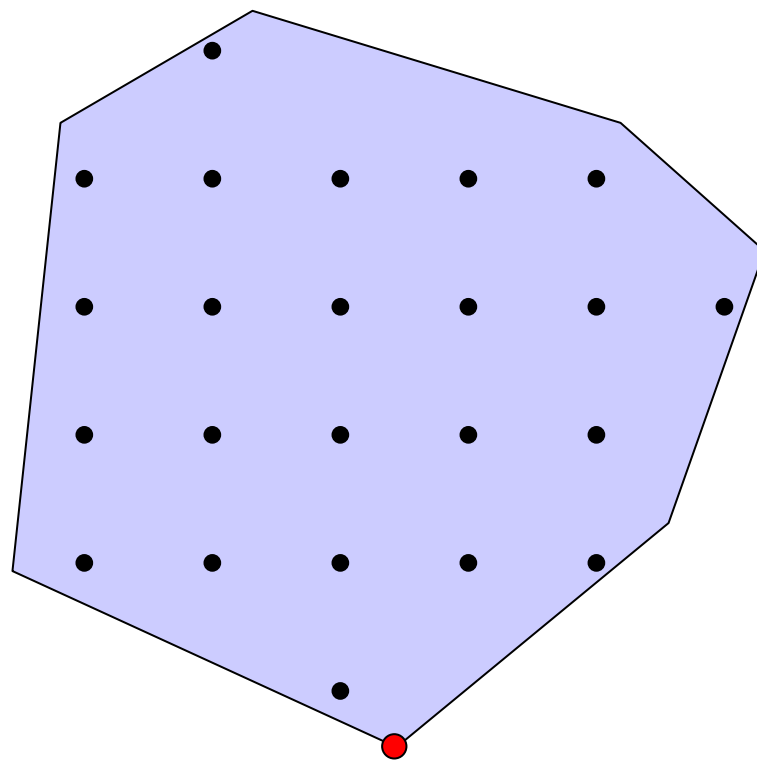
$$x \leq 1$$

LP-
relaxation

- Solve the LP-relaxation and get optimal solution y . (lower bound)
- If y integral, DONE!
- Otherwise pick fractional component $y(i)$.
- Create two new subproblems by adding $y(i)=1$ and $y(i)=0$, resp.
-



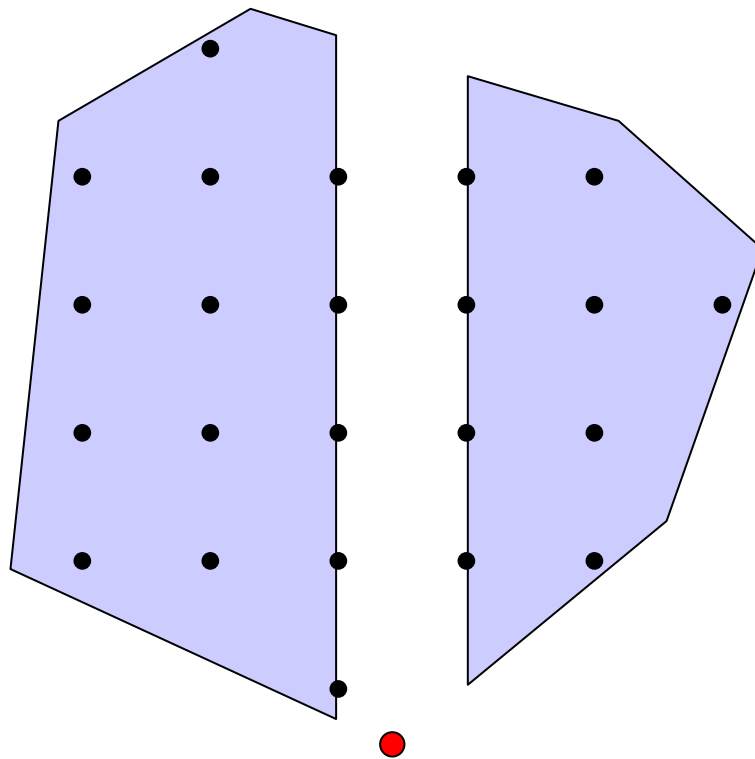
Branching (in general)



- Current solution is infeasible

Branching (in general)

- Rounding a fractional component up and down



- Decomposition into subproblems removes infeasible solution

A Branching Tree

Applegate

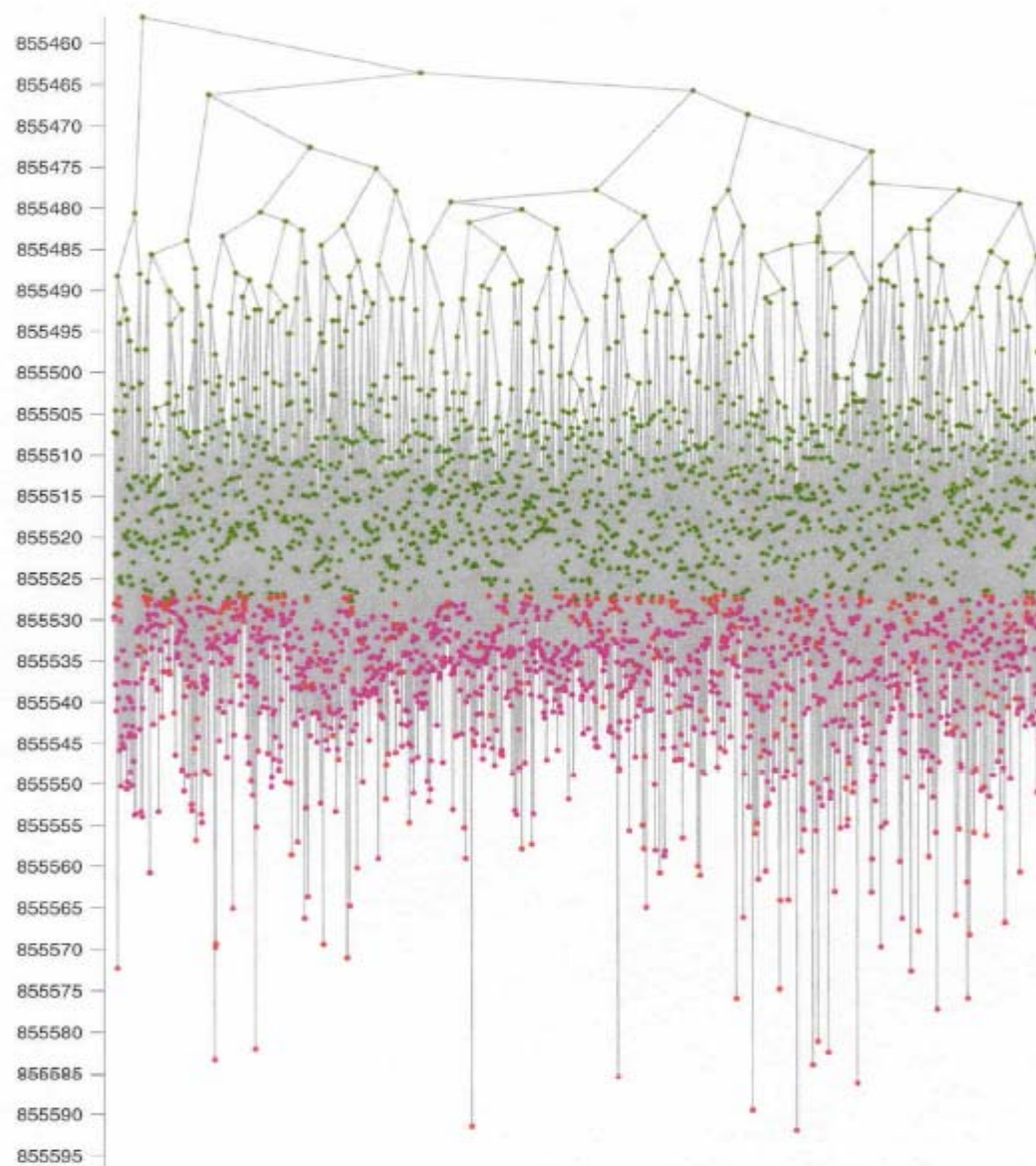
Bixby

Chvátal

Cook

sw24978 Branching Tree

Computation Carried out in Parallel at Georgia Tech, Princeton, Rice



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cutting plane technique for integer and mixed-integer programming

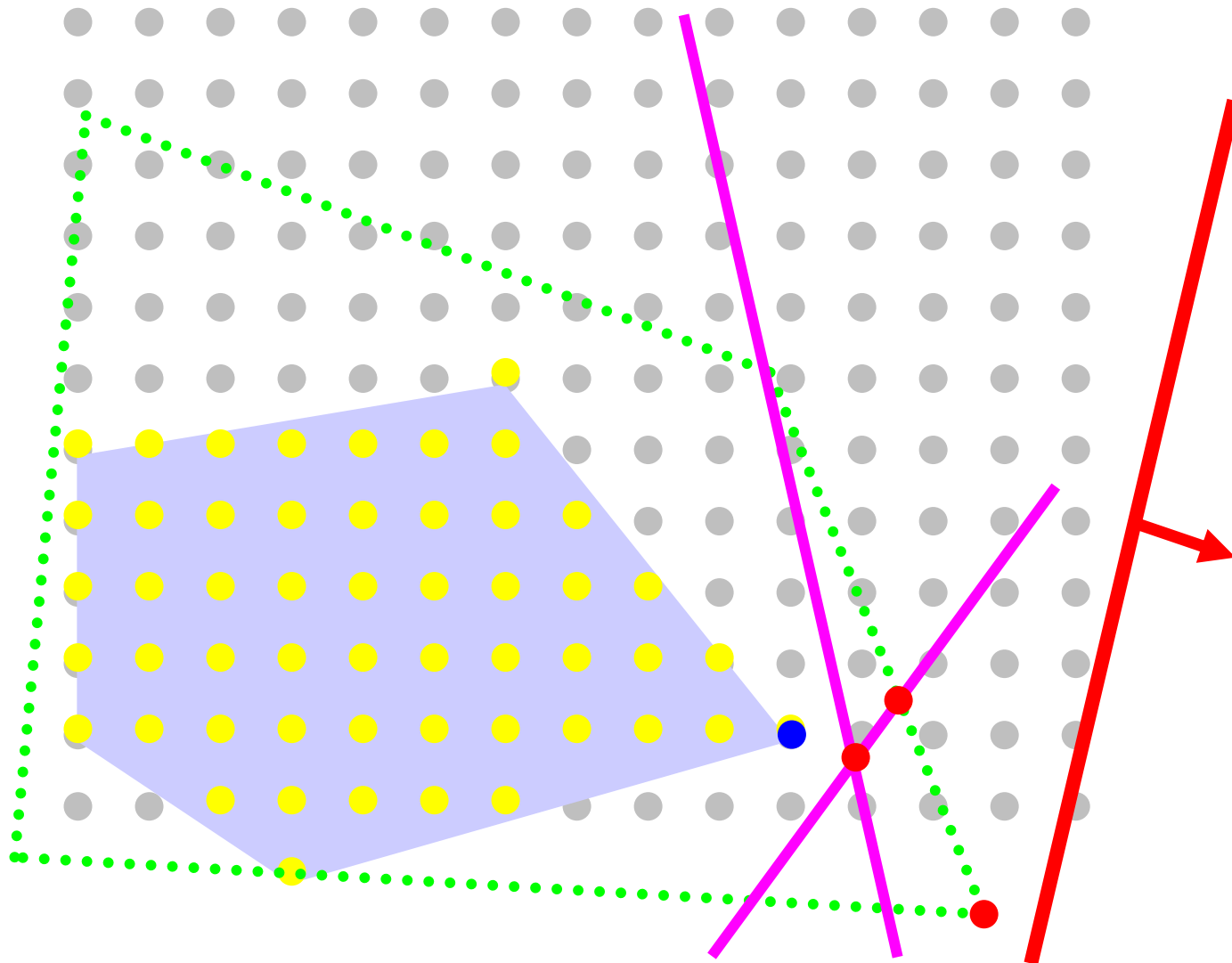
Feasible
integer
solutions

Objective
function

Convex
hull

LP-based
relaxation

Cutting
planes



Other Names

- Branch & Cut
- Branch & Price
- Branch & Cut & Price
- etc.



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The most recent LP survey

Robert E. Bixby,

Solving Real-World Linear Programs: A Decade and More
of Progress.

Operations Research 50 (2002)3-15.



Which LP solvers are used in practice?

- Fourier-Motzkin: hopeless
- Ellipsoid Method: total failure
- primal Simplex Method: good
- dual Simplex Method: better
- Barrier Method: for LPs frequently even better
- For LP relaxations of IPs: dual Simplex Method



Summary

- We can solve today explicit LPs with
 - up to **500,000** of variables and
 - up to **5,000,000** of constraints routinelyin relatively short running times.
- We can solve today structured implicit LPs (employing column generation and cutting plane techniques) in special cases with hundreds of million (and more) variables and almost infinitely many constraints in acceptable running times.
(Examples: TSP, bus circulation in Berlin)



State of the art in LP and IP

- This will be covered in another lecture
- [Operations Research Management Science Today - June 2005](#)
- [OR-MS Today - LINEAR PROGRAMMING SOFTWARE SURVEY](#)



01M1 Lecture

Linear and Integer Programming: an Introduction

The End



Martin Grötschel

- Institut für Mathematik, Technische Universität Berlin (TUB)
- DFG-Forschungszentrum "Mathematik für Schlüsseltechnologien" (MATHEON)
- Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)

groetschel@zib.de

<http://www.zib.de/groetschel>